Logarithmic corrections to a^2 scaling in lattice QCD

Nikolai Husung

Zeuthen, Germany, May 25th, 2020

based on

NH, P. Marquard and R. Sommer. Asymptotic behaviour of cutoff effects in Yang-Mills theory and in Wilson's lattice QCD. Eur. Phys. J. C, 80(3):200, 2020.

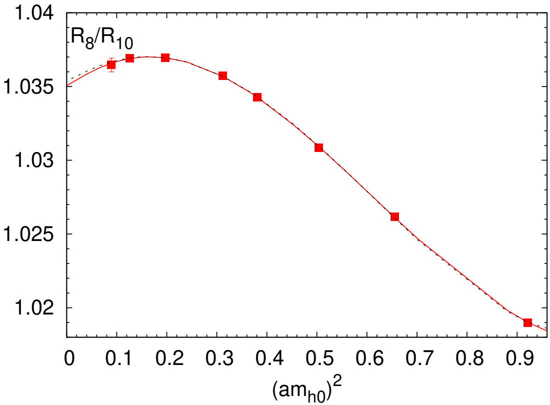
and new work.

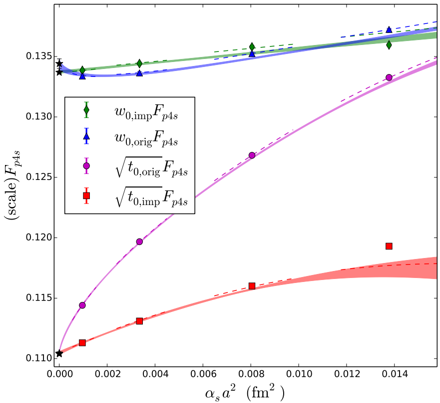
HELMHOLTZ RESEARCH FOR GRAND CHALLENGES

DESY.

Motivation.

> Results obtained on the lattice need to be continuum extrapolated.





Extrapolation of $\frac{R_8}{R_{10}}$, $R_n = (G_n/G_n^{(0)})^{1/(n-4)}$ with nth moment of the pseudo-scalar quarkonium correlator $G_n(t)$ [Petreczky and Weber, 2019].

Extrapolation of the decay constant F_{p4s} of a pseudo-scalar meson [Bazavov et al., 2016].

Motivation.

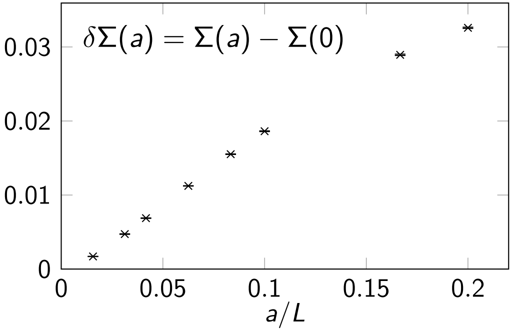
- Results obtained on the lattice need to be continuum extrapolated.
- To do so in a controlled way the leading lattice spacing dependence must be understood.
- Usually naive power corrections of the form a^n with $n \in \mathbb{N}$ are assumed as for a classical field theory. However, quantum corrections spoil this behaviour.

- True asymptotic behaviour in an asymptotically free theory should be of the form

$$a^n[\alpha(1/a)]^{\hat{\gamma}} \sim a^n[-\ln(a\Lambda)]^{-\hat{\gamma}},$$

where $\hat{\gamma}$ can be extracted perturbatively from 1-loop anomalous dimensions of higher dimensional operators.

Motivation.
O(3) model



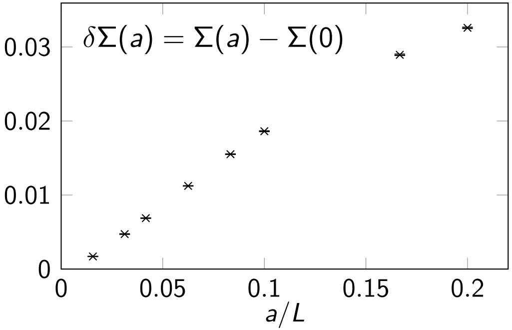
Deviation of the step scaling function from its continuum counterpart, as an example, in the 2-dimensional O(3) model [Balog et al., 2009, 2010].

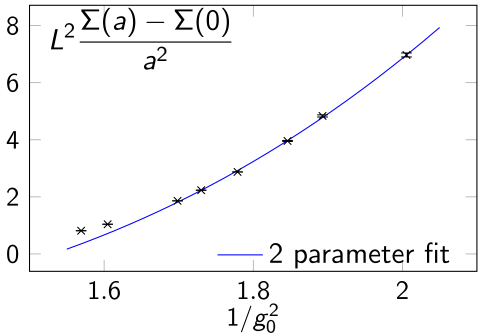
Motivation.

O(3) model

$$\delta\Sigma = \text{const.}\,a^2(g_0^2)^{-3}\left[1 - 1.1386 g_0^2 + \mathrm{O}((g_0^2)^2)\right]$$

$$g_0^2 \sim -1/\ln(a\Lambda)$$





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Symanzik effective theory.

Idea: Parametrise lattice artifacts originating from the lattice action [and for a field Φ] by a minimal basis of operators living in continuum Symanzik effective theory [Symanzik, 1980, 1981, 1983a,b]

$$\mathscr{L}_{\mathrm{eff}} = \mathscr{L} + a^{n_{\mathrm{min}}} \delta\mathscr{L} + \mathrm{O}(a^{n_{\mathrm{min}}+1})\,, \qquad \Phi_{\mathrm{eff}} = \Phi + a^{n_{\mathrm{min}}} \delta\Phi + \mathrm{O}(a^{n_{\mathrm{min}}+1})\,,$$

where a is the lattice spacing and

$$\delta\mathscr{L} = \sum_i c_i \mathcal{O}_i\,, \qquad \delta\Phi = \sum_i d_i \Phi_i\,,$$

with free coefficients c_i and d_i.

Symanzik effective theory.

Consider the connected 2-point function in the effective theory

$$\langle \Phi_{\mathrm{eff}}(x)\Phi_{\mathrm{eff}}(0)\rangle^{\mathrm{con}}_{\mathrm{eff}} = \frac{1}{\mathcal{Z}_{\mathrm{eff}}}\int \mathcal{D}A\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, \Phi_{\mathrm{eff}}(x)\Phi_{\mathrm{eff}}(0)e^{-S_{\mathrm{eff}}[A,\bar{\Psi},\Psi]} - \left[\frac{1}{\mathcal{Z}_{\mathrm{eff}}}\int \mathcal{D}A\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, \Phi_{\mathrm{eff}}(0)e^{-S_{\mathrm{eff}}[A,\bar{\Psi},\Psi]}\right]^2 .$$

This is not well defined but an expansion in a yields (notice that a is **not** the regulator)

$$\langle \Phi_{\mathrm{eff}}(x)\Phi_{\mathrm{eff}}(0)\rangle^{\mathrm{con}}_{\mathrm{eff}} = \langle \Phi(x)\Phi(0)\rangle^{\mathrm{con}}_{\mathrm{cont}} + a^{n_{\mathrm{min}}}\langle \Phi(x)\delta\Phi(0) + \delta\Phi(x)\Phi(0)\rangle^{\mathrm{con}}_{\mathrm{cont}} - a^{n_{\mathrm{min}}}\int \mathrm{d}^4y\,\langle \Phi(x)\Phi(0)\delta\mathscr{L}(y)\rangle^{\mathrm{con}}_{\mathrm{cont}} + \mathrm{O}(a^{n_{\mathrm{min}}+1}),$$

where each term can be evaluated in the continuum theory, i.e. here QCD.

Symanzik effective theory.

Matching

To give statements about the lattice theory we must match the coefficients c_i [and d_i], e.g.,

$$\langle \Phi(x)\Phi(0)\rangle_{\substack{\mathrm{con}\\ \mathrm{lattice}}} \overset{!}{=} \langle \Phi_{\mathrm{eff}}(x)\Phi_{\mathrm{eff}}(0)\rangle_{\substack{\mathrm{con}\\ \mathrm{eff}}}\Big|_{\mu=1/a} + \mathrm{O}(a^{n_{\min}+1})$$

$$= \langle \Phi(x)\Phi(0)\rangle_{\substack{\mathrm{con}\\ \mathrm{cont}}} + a^{n_{\min}} \sum_i d_i \, \langle \Phi(x)\Phi_i(0) + \Phi_i(x)\Phi(0)\rangle_{\substack{\mathrm{con}\\ \mathrm{cont}}}\Big|_{\mu=1/a}$$

$$- a^{n_{\min}} \sum_j \int \mathrm{d}^4 y \, c_j \, \langle \Phi(x)\Phi(0)\delta\mathcal{O}_j(y)\rangle_{\substack{\mathrm{con}\\ \mathrm{cont}}}\Big|_{\mu=1/a} + \mathrm{O}(a^{n_{\min}+1}).$$

We choose for simplicity a RGI (e.g. a conserved vector current) and the renormalisation scale as $\mu = 1/a$ as this is the relevant scale for lattice artifacts.

Remark: Tree-level coefficients

$$c_i(g^2) = \bar{c}_i + \mathrm{O}(g^2)\,, \qquad d_i(g^2) = \bar{d}_i + \mathrm{O}(g^2)$$

can be obtained from classical expansion in the lattice spacing a.

Symanzik effective theory.

Operator basis

Occurring operators $\mathcal{O}_i$ and Φ_i must comply with symmetries of the lattice formulation, e.g. for Wilson's lattice QCD [Wilson, 1974, 1975]

$$S_{\mathrm{W}} = \frac{1}{g_0^2} \sum_{x,\mu\neq\nu} \mathrm{Re}\,\mathrm{tr}(\mathbb{1} - U_{\mu\nu}(x)) + a^4 \sum_x \bar{\Psi}(x) \left[\frac{1}{2} \left\{ \gamma_\mu (\nabla_\mu^* + \nabla_\mu) - a\nabla_\mu^* \nabla_\mu \right\} + M \right] \Psi(x)$$

- Local SU(N) gauge symmetry,
- $\mathcal{C}$-, $\mathcal{P}$- and $\mathcal{T}$-symmetry,
- discrete rotation and translation invariance,
- flavour symmetries,
- manifolds with boundaries necessitate additional surface terms (e.g. Schrödinger functional).

Symanzik effective theory.

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- manifolds with boundaries necessitate additional surface terms (e.g. Schrödinger functional).

In contrast to continuum theory

- broken O(4) symmetry due to reduced rotation symmetry,
- no $\mathrm{SU}(N_\mathrm{f})_\mathrm{L} \times \mathrm{SU}(N_\mathrm{f})_\mathrm{R} \times \mathrm{U}(1)_\mathrm{V}$ flavour symmetry for massless QCD.

Symanzik effective theory.

Operator basis

Remarks:

- Require minimal basis for physical matrix elements ("on-shell")
  $\Rightarrow$ use EOMs to reduce set of operators [Lüscher and Weisz, 1985; Georgi, 1991], e.g.

$$-\frac{2}{g_0^2}\mathrm{tr}(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}) \overset{\mathrm{EOM}}{=} \sum_f \bar{f}\gamma_\mu D_\nu F_{\nu\mu} f \overset{\mathrm{EOM}}{=} g_0^2 \sum_{f,f'} (\bar{f}\gamma_\mu T^a f)(\bar{f}'\gamma_\mu T^a f')$$

- Chosen lattice discretisation determines realised symmetries and may affect n_{min}
  $\Rightarrow$ require different bases.

Symanzik effective theory.

Operator basis

Flavour symmetries:

fermion action	massless	mass-degenerate	massive	$n_{\min}$
Continuum				–
Domain wall	$\mathrm{SU}(N_\mathrm{f})_\mathrm{L} \times \mathrm{SU}(N_\mathrm{f})_\mathrm{R} \times \mathrm{U}(1)_\mathrm{V}$	$\mathrm{SU}(N_\mathrm{f})_\mathrm{V} \times \mathrm{U}(1)_\mathrm{V}$	$\bigtimes_{f=1}^{N_\mathrm{f}} \mathrm{U}(1)_\mathrm{V}$	2
Ginsparg-Wilson				2
Wilson	$\mathrm{SU}(N_\mathrm{f})_\mathrm{V} \times \mathrm{U}(1)_\mathrm{V}$	$\mathrm{SU}(N_\mathrm{f})_\mathrm{V} \times \mathrm{U}(1)_\mathrm{V}$	$\bigtimes_{f=1}^{N_\mathrm{f}} \mathrm{U}(1)_\mathrm{V}$	1
tmQCD ($N_\mathrm{f} = 2$)	$\mathrm{SU}(N_\mathrm{f})_\mathrm{tw} \times \mathrm{U}(1)$	$\mathrm{SU}(N_\mathrm{f})_\mathrm{tw} \times \mathrm{U}(1)$	–	1
		T_1 @ maximal twist		2
staggered	$\mathrm{U}(1)_\mathrm{V} \times \mathrm{U}(1)_{\tilde{\mathrm{A}}}$	$\mathrm{U}(1)_\mathrm{V}$	$\mathrm{U}(1)_\mathrm{V}$	2

staggered has flavour changing interactions.

discrete T_1: $\psi \to i\gamma_5\tau^1\psi$, $\bar\psi \to \bar\psi i\gamma_5\tau^1$

Symanzik effective theory.

Operator basis

Minimal basis at mass-dimensions 5 and 6

$n_{\mathrm{min}} = 1$	$n_{\mathrm{min}} = 2$	
$i\bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi$	$\frac{1}{g_0^2}\mathrm{tr}(D_\mu F_{\nu\rho} D_\mu F_{\nu\rho})$	$g_0^2\bar{\psi}\Gamma\psi\bar{\chi}\Gamma\chi$
	$\frac{1}{g_0^2}\sum_\mu \mathrm{tr}(D_\mu F_{\mu\nu} D_\mu F_{\mu\nu})$	$g_0^2\bar{\psi}\Gamma T^a\psi\bar{\chi}\Gamma T^a\chi$
	$\sum_\mu \bar{\psi}\gamma_\mu D_\mu^3\psi$	$\Gamma \in \{\mathbb{1}, \gamma_5, \gamma_\mu, i\gamma_5\gamma_\mu, \sigma_{\mu\nu}\}$

with flavours $\psi, \chi \in \{u, d, \ldots\}$ + explicitly mass-dependent operators such as $\frac{m_f}{g_0^2}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$.

Symanzik effective theory.

Operator basis

Wilson-like [Sheikholeslami and Wohlert, 1985]
pure gauge [Lüscher and Weisz, 1985]
$\mathrm{O}(a)$ improved [Sheikholeslami and Wohlert, 1985]

Minimal basis at mass-dimensions 5 and 6

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	$\frac{1}{g_0^2}\sum_\mu \mathrm{tr}(D_\mu F_{\mu\nu} D_\mu F_{\mu\nu})$	$g_0^2\bar{\psi}\Gamma T^a\psi\bar{\chi}\Gamma T^a\chi$
	$\sum_\mu \bar{\psi}\gamma_\mu D_\mu^3\psi$	$\Gamma \in \{\mathbb{1}, \gamma_5, \gamma_\mu, i\gamma_5\gamma_\mu, \sigma_{\mu\nu}\}$

with flavours $\psi, \chi \in \{u, d, \ldots\}$ + explicitly mass-dependent operators such as $\frac{m_f}{g_0^2}\mathrm{tr}(F_{\mu\nu}F_{\mu\nu})$.

$\Rightarrow$ e.g. $\mathrm{O}(a)$ improved massless Wilson with $N_\mathrm{f} > 1$: 18 operators

Symanzik effective theory.

Spectral quantity

Consider as an example the mass

$$am^{\Phi}_{\mathrm{lattice}} = -\lim_{x_0\to\infty} \ln \frac{\sum_{\mathbf{x}} \langle \Phi(x_0+a,\mathbf{x})\Phi(0)\rangle^{\mathrm{con}}_{\mathrm{lattice}}}{\sum_{\mathbf{x}} \langle \Phi(x_0,\mathbf{x})\Phi(0)\rangle^{\mathrm{con}}_{\mathrm{lattice}}} \quad \text{with scale setting } a = \frac{am^{\mathrm{ref}}_{\mathrm{lattice}}}{m^{\mathrm{ref}}}.$$

Symanzik effective theory.

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Reminder: We found for the connected 2-point function

$$\frac{\langle \Phi(x)\Phi(0)\rangle^{\text{con}}_{\text{lattice}}}{\langle \Phi(x)\Phi(0)\rangle^{\text{con}}_{\text{cont}}} = 1 + a^{n_{\min}}\left(\sum_i d_i\delta^{\Phi}_i(x;a) - \sum_j c_j\delta^{\mathcal{O}}_j(x;a)\right) + \mathrm{O}(a^{n_{\min}+1}),$$

$$\delta^{\mathcal{O}}_j(x;a) = \int \mathrm{d}^4y\, \frac{\langle \Phi(x)\Phi(0)\mathcal{O}_j(y)\rangle^{\text{con}}_{\text{cont}}}{\langle \Phi(x)\Phi(0)\rangle^{\text{con}}_{\text{cont}}}\Bigg|_{\mu=1/a}, \quad \delta^{\Phi}_i(x;a) = \frac{\langle \Phi_i(x)\Phi(0) + \Phi(x)\Phi_i(0)\rangle^{\text{con}}_{\text{cont}}}{\langle \Phi(x)\Phi(0)\rangle^{\text{con}}_{\text{cont}}}\Bigg|_{\mu=1/a}.$$

Symanzik effective theory.

Spectral quantity

Leading lattice artifacts of spectral quantity

$$\frac{m^{\Phi}_{\mathrm{lattice}}}{m^{\Phi}_{\mathrm{cont}}} = 1 - a^{n_{\min}} \sum_j \bar{c}_j \mathcal{M}_j(1/a) \times [1 + \mathrm{O}(\alpha(1/a))] + \mathrm{O}(a^{n_{\min}+1}), \quad \mathcal{M}_j(\mu) = \frac{1}{2} \langle \Phi_0 | \mathcal{O}_j(0;\mu) | \Phi_0 \rangle$$

with tree-level coefficients $\bar{c}_j$ of the action and ground state $|\Phi_0\rangle$, $\langle \Phi_0 | \Phi_0 \rangle = 2L^3$.

Symanzik effective theory.

Spectral quantity

Leading lattice artifacts of spectral quantity

$$\frac{m^\Phi_{\rm lattice}}{m^\Phi_{\rm cont}} = 1 - a^{n_{\rm min}} \sum_j \bar{c}_j \mathcal{M}_j(1/a) \times [1 + \mathrm{O}(\alpha(1/a))] + \mathrm{O}(a^{n_{\rm min}+1}), \quad \mathcal{M}_j(\mu) = \frac{1}{2} \langle \Phi_0 | \mathcal{O}_j(0;\mu) | \Phi_0 \rangle$$

with tree-level coefficients $\bar{c}_j$ of the action and ground state $|\Phi_0\rangle$, $\langle \Phi_0 | \Phi_0 \rangle = 2L^3$.

Remarks:

- Lattice artifacts of spectral quantities only depend on the chosen lattice action.
- Tree-level coefficients $\bar{c}_j$ can be obtained from classical expansion of the action in the lattice spacing.
- We limit ourselves to the leading behaviour as $a \searrow 0$, i.e. we do not require 1-loop coefficients. However, if tree-level coefficient is zero 1-loop might be needed to obtain leading logarithms.
- Strictly speaking $\mathcal{M} \to \mathcal{M} - \mathcal{M}^{\rm scale}$.

Renormalisation Group.

Use Renormalisation Group Equations (RGEs) to determine renormalisation scale dependence

$$\mu^2 \frac{\mathrm{d}\mathcal{M}_i(\mu)}{\mathrm{d}\mu^2} = \gamma_{ik}\mathcal{M}_k(\mu)\,, \qquad \beta(\alpha) = \mu^2 \frac{\mathrm{d}\alpha(\mu)}{\mathrm{d}\mu^2} = -\alpha^2 \sum_{n\geq 0} \beta_n \alpha^n,$$

where γ is the anomalous dimension matrix

$$\gamma_{ik} = \mu^2 \frac{\mathrm{d}Z_{ij}}{\mathrm{d}\mu^2}(Z^{-1})_{jk} = -\,(\gamma_0)_{ik}\,\alpha + \mathrm{O}(\alpha^2)\,, \qquad \mathcal{O}_{i;\mathrm{R}} = Z_{ij}\mathcal{O}_j\,.$$

renormalisation scheme independent

Renormalisation Group.

We choose a basis such that $\gamma_0 = \mathrm{diag}\{(\gamma_0)_1, \ldots, (\gamma_0)_n\}$ and introduce the Renormalisation Group Invariant (RGI)

$$\mathcal{M}_{i;\mathrm{RGI}} = \lim_{\mu\to\infty} [2\beta_0\alpha(\mu)]^{-\hat{\gamma}_i}\,\mathcal{M}_i(\mu)\,, \qquad \hat{\gamma}_i = \frac{(\gamma_0)_i}{\beta_0}\,,$$

with RGI scale Λ. This allows us to rewrite

$$\mathcal{M}_i(\mu) \quad = \quad [2\beta_0\alpha(\mu)]^{\hat{\gamma}_i}\,\mathrm{Pexp}\left[\int_0^{\alpha(\mu)} \mathrm{d}x \left\{\frac{\gamma(x)}{\beta(x)} - \frac{\gamma_0}{\beta_0 x}\right\}\right]_{ij} \mathcal{M}_{j;\mathrm{RGI}}$$

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with RGI scale Λ. This allows us to rewrite

$$\begin{aligned} \mathcal{M}_i(\mu) &= [2\beta_0\alpha(\mu)]^{\hat{\gamma}_i}\,\mathrm{Pexp}\left[\int_0^{\alpha(\mu)} \mathrm{d}x \left\{\frac{\gamma(x)}{\beta(x)} - \frac{\gamma_0}{\beta_0 x}\right\}\right]_{ij} \mathcal{M}_{j;\mathrm{RGI}} \\ &= [2\beta_0\alpha(\mu)]^{\hat{\gamma}_i}\,\mathcal{M}_{i;\mathrm{RGI}} + \mathrm{O}\left([\alpha(\mu)]^{1+\hat{\gamma}_i}\right). \end{aligned}$$

Note: The renormalisation scale dependence is only in the prefactor of the RGI with leading power determined by $\hat{\gamma}_i$.

Renormalisation Group.

Plugging

$$\mathcal{M}_j(1/a) = [2\beta_0\alpha(1/a)]^{\hat{\gamma}_j}\,\mathcal{M}_{j;\mathrm{RGI}} + \mathrm{O}\left([\alpha(1/a)]^{1+\hat{\gamma}_j}\right)$$

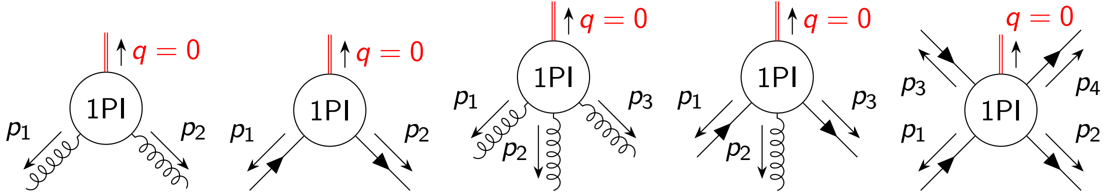
back into the formula of lattice artifacts for m^Φ yields

$$\frac{m^\Phi_{\mathrm{lattice}}}{m^\Phi_{\mathrm{cont}}} = 1 - \frac{a^{n_{\min}}}{2}\sum_j \bar{c}_j\,[2\beta_0\alpha(1/a)]^{\hat{\gamma}_j}\,\mathcal{M}_{j;\mathrm{RGI}} \times [1 + \mathrm{O}(\alpha(1/a))] + \mathrm{O}(a^{n_{\min}+1}).$$

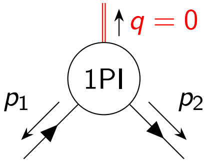
$\Rightarrow$ Need to compute all relevant $\hat{\gamma}_j$ to determine leading behaviour (given by smallest $\hat{\gamma}_j$).

Computing leading anomalous dimensions.

Renormalise operator basis at 1-loop by computing 1PI graphs with operator insertion $\tilde{\mathcal{O}}(q)$ in background field gauge ['t Hooft, 1975; Abbott, 1981, 1982; Lüscher and Weisz, 1995].



e.g.



$$\hat{=} \quad \langle \bar{\psi}_{\mathrm{R}}(p_1)\psi_{\mathrm{R}}(p_2)\tilde{\mathcal{O}}_{i;\mathrm{R}}(q)\rangle_{\mathrm{1PI}} = Z_{\psi}^{-2} Z_{ij} \langle \bar{\psi}(p_1)\psi(p_2)\tilde{\mathcal{O}}_j(q)\rangle_{\mathrm{1PI}}$$

Computing leading anomalous dimensions.

Obtain relevant part of mixing matrix via

needed additionally

$$\begin{pmatrix} \mathcal{O} \\ \mathcal{E} \end{pmatrix}_{\mathrm{R}} = \begin{pmatrix} Z_{\mathcal{O}\mathcal{O}} & Z_{\mathcal{O}\mathcal{E}} \\ 0 & Z_{\mathcal{E}\mathcal{E}} \end{pmatrix} \begin{pmatrix} \mathcal{O} \\ \mathcal{E} \end{pmatrix}$$

with class of EOM-vanishing operators $\mathcal{E}$.

- We use dimensional regularisation combined with the $\overline{\mathrm{MS}}$ renormalisation scheme.
- To obtain the anomalous dimensions we extract only the UV-pole contributions following e.g. the procedure from [Misiak and Münz, 1995; Chetyrkin et al., 1998].
- Checked results for massless QCD through renormalisation of connected on-shell graphs and against literature [Narison and Tarrach, 1983; Alonso et al., 2014; Boito et al., 2015]. Found disagreement of [Narison and Tarrach, 1983] with [Boito et al., 2015] and our results for 4-fermion operators.

Tools: QGRAF [Nogueira, 1993, 2006], FORM [Vermaseren, 2000]

Lattice artifacts originating from the action.

Pure gauge

For pure gauge actions build from plaquette and/or rectangle terms e.g. Wilson plaquette, Iwasaki or DBW2 action one finds ($\bar{c}_2 = 4\bar{c}_1$)

$$\frac{m^{\Phi}_{\mathrm{lattice}}}{m^{\Phi}_{\mathrm{cont}}} = 1 - a^2\bar{c}_1[2\beta_0\alpha(1/a)]^{\hat{\gamma}_1}\left\{\mathcal{M}_{1;\mathrm{RGI}} + 4[2\beta_0\alpha(1/a)]^{\hat{\gamma}_2-\hat{\gamma}_1}\mathcal{M}_{2;\mathrm{RGI}}\right\}$$
$$\times[1+\mathrm{O}(\alpha(1/a))] + \mathrm{O}(a^4)\,, \qquad n_{\mathrm{min}} = 2$$

with ratio of leading cutoff effects Wilson : Iwasaki : DBW2 $\approx$ 1 : (−3) : (−16)
and minimal **diagonalised** basis

$$\mathcal{O}_1 = \frac{1}{g_0^2}\mathrm{tr}(D_\mu F_{\nu\rho} D_\mu F_{\nu\rho})\,, \qquad \hat{\gamma}_1 = \frac{7}{11} \approx 0.636\,,$$

$$\mathcal{O}_2 = \frac{1}{g_0^2}\sum_\mu \mathrm{tr}(D_\mu F_{\mu\nu} D_\mu F_{\mu\nu}) - \frac{1}{4g_0^2}\mathrm{tr}(D_\mu F_{\nu\rho} D_\mu F_{\nu\rho})\,, \qquad \hat{\gamma}_2 = \frac{63}{55} \approx 1.145$$

independent of the number of colours.

Lattice artifacts originating from the action.

TL improved short distance observables

Consider as an example the coupling (in pure gauge)

$$\alpha_{\rm qq}(1/r;a/r) = \frac{4\pi}{C_{\rm F}} r^2 F(r;a/r) = r^2 \partial_r \lim_{T\to\infty} \partial_T \ln \mathcal{W}(r,T;a/r)$$

with $r \times T$ Wilson loop $\mathcal{W}(r,T)$ and assume ∂_r to be correct up to $\mathrm{O}(a^4)$. Fixed order lattice PT e.g. in the MS-lat scheme yields

$$\alpha_{\rm qq}(1/r;a/r) = \alpha_{\rm qq}(1/r;0)\left\{1 + \delta_0(a/r) + \mathrm{O}\left(\frac{a^2}{r^2}\alpha(1/r)\right)\right\}, \quad \delta_0(a/r) = \frac{a^2}{r^2}\sum_{k\geq 0}\frac{a^{2k}}{r^{2k}} p_{k0}\,,$$

with TL coefficients $p_{k0} = \text{const.}$

Lattice artifacts originating from the action.

TL improved short distance observables

TL improvement can then be achieved through

$$\alpha_{\rm qq}^{\rm impr}(1/r;a/r)=\frac{\alpha_{\rm qq}(1/r;a/r)}{1+\delta_0(a/r)}\overset{\rm SET}{=}\alpha_{\rm qq}(1/r;0)\{1+a^2\bar{c}_2[\partial_r\mathcal{M}(r)|_{\rm TL}-[2\beta_0\alpha(1/a)]^{\hat{\gamma}_2}\partial_r\mathcal{M}_{2;\rm RGI}(r)]+{\rm O}(a^2\alpha(1/r))\}$$

since $\partial_r\mathcal{M}_1(r;\mu)={\rm O}(\alpha(\mu))$ and thus $p_{00}=-\bar{c}_2r^2\partial_r\mathcal{M}_2(r)|_{\rm TL}\underset{\text{on-axis}}{\overset{\text{Wilson}}{=}}-\frac{3}{4}$.

$$\Rightarrow\frac{\alpha_{\rm qq}^{\rm impr}(1/r;a/r)}{\alpha_{\rm qq}(1/r;0)}=1+a^2\bar{c}_2\left\{1-\left[\frac{\alpha(1/a)}{\alpha(1/r)}\right]^{\hat{\gamma}_2}\right\}\partial_r\mathcal{M}_2(r)|_{\rm TL}+{\rm O}\left(\frac{a^2}{r^2}\alpha(1/r)\right),$$

which is TL but not RG improved and thus carries a large $\log(a/r)\alpha(1/r)$ as $r=$ fixed and $a\searrow 0$. Full TL and RG improvement at ${\rm O}(a^2)$ is achieved through

$$\alpha_{\rm qq}^{\text{TL,RG impr}}(1/r;a/r)=\frac{\alpha_{\rm qq}^{\rm impr}(1/r;a/r)}{1+a^2\bar{c}_2\left\{1-\left[\frac{\alpha(1/a)}{\alpha(1/r)}\right]^{\hat{\gamma}_2}\right\}\partial_r\mathcal{M}_2(r)|_{\rm TL}}.$$

Lattice artifacts originating from the action.

Unimproved massless Wilson

$$\frac{m^{\Phi}_{\mathrm{lattice}}}{m^{\Phi}_{\mathrm{cont}}} = 1 - a\bar{c}_{\mathrm{SW}}[2\beta_0\alpha(1/a)]^{\hat{\gamma}_{\mathrm{SW}}}\mathcal{M}_{\mathrm{SW;RGI}} \times [1 + \mathrm{O}(\alpha(1/a))] + \mathrm{O}(a^2)\,, \quad n_{\min} = 1$$

with only one operator [Sheikholeslami and Wohlert, 1985]

$$\mathcal{O}_{\mathrm{SW}} = \frac{i}{4}\bar{\Psi}\sigma_{\mu\nu}F_{\mu\nu}\Psi\,,$$

$$\hat{\gamma}_{\mathrm{SW}} = \frac{15C_{\mathrm{F}} - 5C_{\mathrm{A}}}{11C_{\mathrm{A}} - 2N_{\mathrm{f}}} \overset{N=3}{=} \frac{5}{33 - 2N_{\mathrm{f}}} \ll 1 \text{ unless close to conformal window,}$$

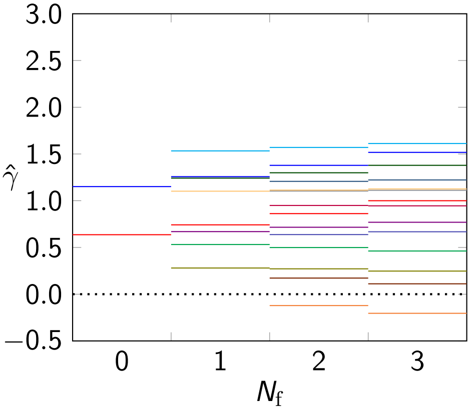
which was known for $N_{\mathrm{f}} = 1$ in the literature [Narison and Tarrach, 1983].

Lattice artifacts originating from the action.

Preliminary

O(*a*) improved massless lattice QCD w/o flavour violating interactions

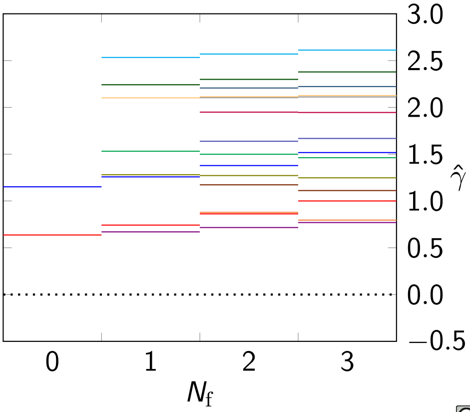
$$\frac{m^{\Phi}_{\mathrm{lattice}}}{m^{\Phi}_{\mathrm{cont}}} = 1 - a^2 \sum_j \bar{c}_j [2\beta_0 \alpha(1/a)]^{\hat{\gamma}_j} \mathcal{M}_{j;\mathrm{RGI}} \times [1 + \mathrm{O}(\alpha(1/a))] + \mathrm{O}(a^3)\,, \qquad n_{\min} = 2$$



IF $\bar{c}_i^{\text{4-ferm}} = 0$ $\Longrightarrow$ (to be confirmed)

Remember:

$\mathcal{O}^{\text{4-ferm}} = g_0^2 \bar{\psi}\Gamma\psi\bar{\chi}\Gamma\chi$



Yang-Mills Gradient flow.

The Yang-Mills Gradient flow is defined as

$$\partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x)\,, \quad B_\mu(0,x) = A_\mu(x)\,, \quad t \geq 0\,.$$

To describe lattice artifacts in terms of a local effective Lagrangian one introduces the Lagrange multiplier $L_\mu(t,x)$ to rewrite the action for a $4+1$-dimensional theory with fifth dimension $t \in [0,\infty[$ [Lüscher and Weisz, 2011]

$$S_{\mathrm{GF}} = S_{\mathrm{QCD}} + S_{\mathrm{flow}}\,, \quad S_{\mathrm{flow}} = -2\int_0^\infty \mathrm{d}t \int \mathrm{d}^4x\,\mathrm{tr}\left(L_\mu(t,x)\left[\partial_t B_\mu(t,x) - D_\nu G_{\nu\mu}(t,x)\right]\right).$$

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$\Rightarrow$ Modification of the gluonic EOM at the QCD boundary [Ramos and Sint, 2016]:

$$\frac{1}{g_0^2} D_\mu F^a_{\mu\nu}(x) = \bar{\Psi}\gamma_\nu T^a \Psi(x) - L^a_\nu(0,x).$$

Yang-Mills Gradient flow.

Pure gauge

Due to modified EOMs one additional pure gauge operator must be included on the $t = 0$ boundary [Ramos and Sint, 2016], we choose

$$\mathcal{O}_3 = \frac{1}{g_0^2}\mathrm{tr}(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu})\,.$$

To get the additional mixing contributions we computed $E(t', x) = -\frac{1}{2}\mathrm{tr}[G_{\mu\nu} G_{\mu\nu}](t', x)$

$$\int_p \left\langle \tilde{E}(t', p)\frac{1}{g_0^2}\mathrm{tr}[D_\mu \widetilde{F_{\nu\rho} D_\mu} F_{\nu\rho}](0)\right\rangle e^{ipx}, \quad \int_p \left\langle \tilde{E}(t', p)\frac{1}{g_0^2}\sum_\mu \mathrm{tr}[D_\mu \widetilde{F_{\mu\rho} D_\mu} F_{\mu\rho}](0)\right\rangle e^{ipx}$$

$$\int_p \left\langle \tilde{E}(t', p)\tilde{\mathcal{O}}_3(0)\right\rangle e^{ipx},$$

which gives the full 1-loop contribution at the $t = 0$ boundary and also the mixing Z_{i3} by reusing mixing of the pure gauge on-shell basis. $Z_{3i} = 0$ due to vanishing of $\mathcal{O}_3$ at $t = 0$.

Yang-Mills Gradient flow.

Preliminary

Pure gauge

All lattice artifacts originating from $t > 0$, e.g. flow action and flowed fields, can be described classically.

We find for the **diagonalised** basis

$$\mathcal{O}_1 = \frac{1}{g_0^2}\mathrm{tr}(D_\mu F_{\nu\rho} D_\mu F_{\nu\rho}) - \frac{23}{7}\mathcal{O}_3\,, \qquad \hat{\gamma}_1 = \frac{7}{11}\,,$$

$$\mathcal{O}_2 = \frac{1}{g_0^2}\sum_\mu \mathrm{tr}(D_\mu F_{\mu\rho} D_\mu F_{\mu\rho}) - \frac{1}{4g_0^2}\mathrm{tr}(D_\mu F_{\nu\rho} D_\mu F_{\nu\rho}) - \frac{1}{6}\mathcal{O}_3\,, \qquad \hat{\gamma}_2 = \frac{63}{55}\,,$$

$$\mathcal{O}_3 = \frac{1}{g_0^2}\mathrm{tr}(D_\mu F_{\mu\nu} D_\rho F_{\rho\nu})\,, \qquad \hat{\gamma}_3 = 0\,.$$

Conclusion.

- No bad behaviour as in the O(3) model.
- Leading anomalous dimensions of contributions from typical lattice QCD actions at $N_\mathrm{f} \leq 1$ improve convergence as $a \searrow 0$.
- Depending on the coefficients $\bar{c}_i$ the 4-fermion operators might spoil this general behaviour for $N_\mathrm{f} > 1$.
- The presence of 4-fermion operators gives a dense spectrum for $\hat{\gamma}$, i.e. no clearly dominating contributions. This can lead to complicated lattice artifacts with cancellations and pile ups.
- Anomalous dimensions for Yang-Mills Gradient flow do not explain sizeable lattice artifacts for pure gauge theory - classical a^2 scaling is leading contribution.
- We cannot make statements where leading powers in a dominate.

Outlook.

- Leading asymptotic behaviour is now known and should be incorporated into continuum extrapolations.
- **To be done:**
  - corrections to electro-weak flavour currents,
  - heavy quark correlator moments (relevant for α_{s}, m_{c}),
  - Gradient flow:
    - full QCD for flowed gauge fields (1 new fermionic operator),
    - flowed fermion fields?
  - leading matching coefficients of some actions still need to be worked out,
  - ...

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