Perturbative quantities on the lattice QCD and the
tensor renormalization group

[D. Kadoh, K.N. arXiv:1803.07960]
[D. Kadoh, K.N. arXiv:1812.10642]
[D. Kadoh, K.N. arXiv:1912.xxxxx]

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Contents

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Why Lattice QFT?

What is the role of Lattice QFT?

◆ Non-perturbative calculation:
  → Precise determination of physical quantities.
    (e.g. hadron mass, quark mass, form factor, g-2…)

◆ Theoretical formulation of QFT without perturbation:
  → Fundermental interest for the definition.
    (e.g. SU(N) chiral gauge fermion still unknown)

◆ Interdisciplinary study:
  → cond-mat, quant-ph, and any discrete systems.
    (e.g. Domain wall fermion ↔ Topological insulator)
Contents

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(4): Summary
$q^2 \Pi(q^2) = i \int dx e^{iqx} \langle j_5(x) j_5(0) \rangle$

\[ g^2 n = \frac{1}{n!} \partial^n q^2 \uparrow (q^2) = 0 \]

1.0 × 10^{-16}

1.0 × 10^{-14}

1.0 × 10^{-12}

1.0 × 10^{-10}

1.0 × 10^{-8}

1.0 × 10^{-6}

1.0 × 10^{-4}

1.0 × 10^{-2}

1.0 × 10^{0}

1.0 × 10^{2}

1.0 × 10^{4}

1.0 × 10^{6}

0 20 40 60 80 100 120

t

\[ e^{iHt} \rightarrow e^{-m_{\text{eff}} t} \]

\[ m_{\text{lattice}} = 2.9928(5) \text{GeV} \]

\[ m_{\text{experiment}} = 2.9834(5) \text{GeV} \]
Perturbative information in Lattice calculations

Any other information?

Non-perturbative value (meson mass) from Low energy

Perturbative value (quark mass) from High energy

→ Moments (Derivative of correlator, or Adler function)

\[ g_{2n} = \frac{1}{n!} \left( \frac{\partial}{\partial q^2} \right)^n (q^2 \Pi(q^2))_{q^2=0} \]
Moment method

Derivative of the correlators

Calculable perturbatively as a function of $\mu_{\text{MS}}$.

Current correlator

$G(t) = a^6 \sum_x (am_{0c})^2 \langle j_5(x)j_5(0) \rangle$

Diagonal line

High energy

$g_{2n} = \frac{1}{n!} \left( \frac{\partial}{\partial q^2} \right)^n (q^2 \Pi(q^2))_{q^2=0}$

$G_n = \sum_t \left( \frac{t}{a} \right)^n G(t)$

$\diamond$ Moment

$\rightarrow$ Differential
What’s the Moment?

- Moment is defined by $G(t)$ at typical energy scale

$$G_n = \sum_t \left( \frac{t}{a} \right)^n G(t)$$

$am_{\eta_c} \sim 0.6636$
\[ g_{2n}^{\text{lattice}} = G_{2n}^{\text{perturbative}}(\alpha_s^{\overline{\text{MS}}}, m_c^{\overline{\text{MS}}}; \mu) \]

\[ G_{2n}^{\text{perturbative}}(\alpha_s^{\overline{\text{MS}}}, m_c^{\overline{\text{MS}}}; \mu) = 1 + \left(3.9 + 2.0 \log \frac{m_c(\mu)^2}{\mu^2}\right) \frac{\alpha_s}{\pi} \]

\[ + \left(13.6 + 3.0 \log \frac{m_c(\mu)^2}{\mu^2} - 0.08 \left(\log \frac{m_c(\mu)^2}{\mu^2}\right)^2\right) (\frac{\alpha_s}{\pi})^2 \]

\[ + \left(13.2 + 14.2 \log \frac{m_c(\mu)^2}{\mu^2} + 1.03 \left(\log \frac{m_c(\mu)^2}{\mu^2}\right)^2 + 0.06 \left(\log \frac{m_c(\mu)^2}{\mu^2}\right)^3\right) (\frac{\alpha_s}{\pi})^3 \]
\[ R_n = \frac{m_{\eta_c}^{\text{exp}}}{2m_{c}^{\text{MS}}} r(n)(m_{c}^{\text{MS}}, \alpha_{\overline{\text{MS}}}) \]

### Possible systematic errors

1. **Truncation error from perturbative expansion of**
2. **Input meson mass error**
3. **Gluon condensate contribution**

**After correcting for...**

- (a) Electromagnetic effect,
- (b) Disconnected diagram contributions.

**Result with three inputs.**

\[ \langle \text{Gluon condensate is} \rangle \]

**Possible systematic errors**

- (1): Truncation error from perturbative expansion of
- (2): Input meson mass error

**After correcting for...**

- (a) Electromagnetic effect,
- (b) Disconnected diagram contributions.

**Gluon condensate contribution**

\[ m_{c}^{\text{exp}} \gamma_c \] (see Backup slides)

<table>
<thead>
<tr>
<th>( m_c (3\text{GeV}) ) [GeV]</th>
<th>pert</th>
<th>( t_0^{1/2} )</th>
<th>stat</th>
<th>( O(a^4) )</th>
<th>vol</th>
<th>( m_{\eta_c}^{\text{exp}} )</th>
<th>disc</th>
<th>EM</th>
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</thead>
<tbody>
<tr>
<td>1.0033(96)</td>
<td>(77)</td>
<td>(49)</td>
<td>(4)</td>
<td>(30)</td>
<td>(4)</td>
<td>(3)</td>
<td>(4)</td>
<td>(6)</td>
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<tr>
<td>0.2528(127)</td>
<td>(120)</td>
<td>(32)</td>
<td>(2)</td>
<td>(26)</td>
<td>(1)</td>
<td>(0)</td>
<td>(0)</td>
<td>(1)</td>
</tr>
</tbody>
</table>
Another example: Dirac eigenvalue density

\[ \rho(\lambda) \]  

\[ \lambda_k [\text{GeV}] \]

\[ \rho(\lambda_k) [\text{GeV}^3] \]

\[ 0.0 \quad 1.0 \quad 2.0 \quad 3.0 \]

\[ 0.0 \quad 1.0 \quad 2.0 \quad 3.0 \]

\[ 4.47 \]

Low energy:
→ Chiral condensate

[\text{G.Cossu et al. arXiv:1607.01099}]

Higher energy:
→ Strong coupling

Perturbative calculation up to $O(\alpha_s^3)$

Eigenvalue density from chiral condensate

\[ \rho(\lambda) = -\frac{1}{2\pi} \langle \bar{q}q(m = i\lambda + \epsilon) \rangle - \langle \bar{q}q(m = i\lambda - \epsilon) \rangle \]

Known up to $O(\alpha_s^3)(n_f = 3)$

[J.Kneur, A.Neveu arXiv:1506.07506]

\[ \rho(\mu = \lambda) = \frac{3\lambda^3}{4\pi^2} \left(1 + 1.06\alpha_s - 2.14\alpha_s^2 - 5.98\alpha_s^3 + O(\alpha_s^4) \right) \]

Exponent \[ \frac{d \log \rho(\lambda)}{d \log \lambda} \] can be predicted up to $O(\alpha_s^4)$.
$$-\frac{\rho_5}{\pi^5} = c_5$$

$$-\frac{4}{15} L_\lambda^5 \gamma_0 \left[ 3 \beta_0^4 - 25 \beta_0^3 \gamma_0 + 70 \beta_0^2 \gamma_0^2 - 80 \beta_0 \gamma_0^3 + 32 \gamma_0^4 \right]$$

$$+ \frac{1}{3} L_\lambda^4 \left[ 3 \beta_0^4 c_1 + 4 \beta_0 \gamma_0^2 (13 \beta_1 - 20 c_1 \gamma_0 + 60 \gamma_0^2 - 48 \gamma_1) \right]$$

$$+ \frac{1}{3} L_\lambda^4 \left[ \beta_0^3 (-25 c_1 \gamma_0 + 25 \gamma_0^2 - 12 \gamma_1) - 16 \gamma_0^3 (3 \beta_1 - 2 c_1 \gamma_0 + 8 \gamma_0^2 - 8 \gamma_1) \right]$$

$$+ \frac{1}{3} L_\lambda^4 \left[ \beta_0^2 (12 c_2 - 13 c_1 \gamma_0) + \beta_0^2 (13 \beta_1 c_1 - 52 c_2 \gamma_0 + 88 c_1 \gamma_0^2 - 70 \gamma_0^3 - 36 c_1 \gamma_1 + 88 \gamma_0 \gamma_1 - 24 \gamma_2) \right]$$

$$+ \frac{1}{3} L_\lambda^4 \left[ -4 \beta_0 (3 \beta_2 \gamma_0 + \beta_1 (13 c_1 \gamma_0 - 13 \gamma_0^2 + 7 \gamma_1)) \right]$$

$$+ \frac{1}{3} L_\lambda^4 \left[ -2 \gamma_0 (3 \beta_1^2 - 12 \beta_1 (2 c_1 \gamma_0 - 4 \gamma_0^2 + 3 \gamma_1)) \right]$$

$$+ \frac{1}{3} L_\lambda^4 \left[ -2 \gamma_0 (4(-3 \beta_2 \gamma_0 + 4(c_2 \gamma_0^2 + 3(-c_1 \gamma_0^3 + 2 \gamma_0^4 + c_1 \gamma_0 \gamma_1 - 4 \gamma_0^2 \gamma_1 + 2 \gamma_1^2 + 4 \gamma_0 \gamma_2))) \right]$$

$$+ L_\lambda^2 \left[ \frac{3 \beta_0^2 c_1}{2} + \beta_0^2 (6 c_3 - 7 c_2 \gamma_0 + \frac{9 c_1 \gamma_0^2}{2} - 3 c_1 \gamma_1) \right]$$

$$+ L_\lambda^2 \left[ \beta_1 (\beta_0 (7 c_2 - 6 c_1 \gamma_0) - 2(5 c_2 \gamma_0 - 7 c_1 \gamma_0^2 + 6 \gamma_0^3 + 4 c_1 \gamma_1 - 9 \gamma_0 \gamma_1 + 3 \gamma_2)) \right]$$

$$+ L_\lambda^2 \left[ \beta_0 (3 \beta_2 c_1 - 2(7 c_3 \gamma_0 - 11 c_2 \gamma_0^2)) \right]$$

$$+ L_\lambda^2 \left[ -2 \beta_0 (+12 c_1 \gamma_0^3 - 10 \gamma_0^4 + 6 c_2 \gamma_1 - 17 c_1 \gamma_0 \gamma_1 + 24 \gamma_0^2 \gamma_1 - 6 \gamma_1^2 + 5 c_1 \gamma_2 - 12 \gamma_0 \gamma_2 + 4 \gamma_3) \right]$$

$$+ L_\lambda^2 \left[ -2(\beta_3 \gamma_0 + \beta_2 (3 c_1 \gamma_0 - 3 \gamma_0^2 + 2 \gamma_1) - 4(c_3 \gamma_0^2 - 2 c_2 \gamma_0^3 + 3 c_1 \gamma_0^4 - 4 \gamma_0^5 + 2 c_2 \gamma_0 \gamma_1)) \right]$$

$$+ L_\lambda^2 \left[ -2(-4(-6 c_1 \gamma_0^2 \gamma_1 + 12 \gamma_0^3 \gamma_1 + c_1 \gamma_1^2 - 6 \gamma_0 \gamma_1^2 + 2 c_1 \gamma_0 \gamma_2 - 6 \gamma_0^2 \gamma_2 + 2 \gamma_1 \gamma_2 + 2 \gamma_0 \gamma_3)) \right]$$

$$+ L_\lambda \left[ \beta_3 c_1 + 2 \beta_2 c_2 + 3 \beta_1 c_3 + 4 \beta_0 c_4 - \beta_2 c_2 \gamma_0 - 2 \beta_1 c_2 \gamma_0 - 3 \beta_0 c_3 \gamma_0 - 4 c_4 \gamma_0 + \beta_1 c_1 \gamma_0^2 + 2 \beta_0 c_2 \gamma_0^2 \right]$$

$$+ L_\lambda \left[ 4 c_2 \gamma_0^2 - \beta_0 c_1 \gamma_0^3 - 4 c_2 \gamma_0^3 + 4 c_1 \gamma_0^4 - 4 \gamma_0^5 - \beta_1 c_1 \gamma_1 - 2 \beta_0 c_2 \gamma_1 - 4 c_3 \gamma_1 + 2 \beta_0 c_1 \gamma_0 \gamma_1 \right]$$

$$+ L_\lambda \left[ 8 c_2 \gamma_0 \gamma_1 - 12 c_1 \gamma_0^2 \gamma_1 + 16 \gamma_0^3 \gamma_1 + 4 c_1 \gamma_1^2 - 12 \gamma_0 \gamma_1^2 - \beta_0 c_1 \gamma_2 - 4 \gamma_0 \gamma_2 + 8 c_1 \gamma_0 \gamma_2 - 12 \gamma_0^2 \gamma_2 \right]$$

$$+ L_\lambda \left[ 8 \gamma_1 \gamma_2 - 4 c_1 \gamma_3 + 8 \gamma_0 \gamma_3 - 4 \gamma_4 \right]$$
Pauli-Villars mass generalization

Overlap operator construction

\[ aD_{OV}(m_f = 0) \equiv \left[ \mathcal{P} D_{DW}^{-1}(1) D_{DW}(m_f = 0) \mathcal{P} \right] \]

\[ = 1 + \gamma_5 \text{sign}_{L_5} \left[ \gamma_5 aD_{Mobius} \right] \]

General Pauli–Villars mass

\[ aD_{OV}(m_f = 0) \equiv \left[ \mathcal{P} D_{DW}^{-1}(m_p) D_{DW}(m_f = 0) \mathcal{P} \right] \]

\[ = m_p \frac{1 + \gamma_5 \text{sign}_{L_5} \left[ \gamma_5 aD_{Mobius} \right]}{(1 + m_p) + (1 - m_p) \gamma_5 \text{sign}_{L_5} \left[ \gamma_5 aD_{Mobius} \right]} \]
Pauli-Villars mass as an eigenvalue cutoff (Tree)

\[ m_p = 1 \]

\[ m_p = 3.14 \]

\[ |aD_{OV}| \leq m_p \]
Lattice power of eigenvalue density (Tree level)

\[
\frac{4}{1 + \gamma m} - 1
\]

\[m_p = 1\]
\[m_p = 3\]
\[m_p \rightarrow \infty\]
Wilson
Continuum

\[d \log \rho(\lambda)\]
\[d \log \lambda\]
\[a \lambda_k\]
Reducing discretization effect

\( a \) dependence is reduced by \( m_p = 1 \rightarrow 3 \)

\[
a^2 \lambda^2(m_p) = \frac{a^2 \lambda^2(m_p = 1)}{1 - (1 - 1/m_p^2) a^2 \lambda^2(m_p = 1)}
\]
◇ Eigenvalue

\[ a^2 \lambda^2 (m_p) = \frac{a^2 \lambda^2 (m_p = 1)}{1 - (1 - 1/m_p^2) a^2 \lambda^2 (m_p = 1)} \]

◇ Chirality (Ginsparg-Wilson)

\[
\{ D_{OV}^{-1}, \gamma_5 \} = 2a \gamma_5 \quad \text{[P.H. Ginsparg, K.G. Wilson (1982)]}
\]

General Pauli–Villars mass

\[
\{ D_{OV}^{-1}, \gamma_5 \} = \frac{2a \gamma_5}{m_p}
\]

◇ Locality → Exponential locality is maintained

(Exponential locality of sign function) …… [P.Hernandez et.al.(1999)]
### Strong coupling constant determination

<table>
<thead>
<tr>
<th>$\lambda(\mu = 6 \text{ GeV})$ [GeV]</th>
<th>$\alpha_s(\mu = 6 \text{ GeV})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$'s</td>
<td>0.204(10)</td>
</tr>
<tr>
<td>$2 \beta$'s</td>
<td></td>
</tr>
<tr>
<td>$1 \beta$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>Error</th>
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<tbody>
<tr>
<td>statistical</td>
<td>±0.002</td>
</tr>
<tr>
<td>discretization</td>
<td>±0.007</td>
</tr>
<tr>
<td>perturbative</td>
<td>±0.007</td>
</tr>
<tr>
<td>lattice scale</td>
<td>±0.001</td>
</tr>
<tr>
<td>renormalization</td>
<td>±0.001</td>
</tr>
<tr>
<td>total</td>
<td>±0.010</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha_s(\mu = 6 \text{ GeV})$</th>
<th>(stat.) (pert.) ($O(a^4)$)</th>
<th>(δ$t_0$)</th>
<th>(δ$Z_m$)</th>
<th>PDG</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.204(10)</td>
<td>(2) (7) (7) (2) (2)</td>
<td></td>
<td></td>
<td>0.191(3)</td>
</tr>
</tbody>
</table>
Others

Non-factorizable charmonium contribution in B→K II

Finite Volume (Casimir) effects in the model of nucleon

Summary for QCD part

- Since lattice calculations are a (first principle) numerical calculation, it provides physical quantities without perturbative method.

- There are many source of uncertainty, but under control.

- The error from the truncation of perturbative expansion is often dominant.

- The systematic error of the fermion formulation is also important.
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Theoretical aspects of Lattice QFT

What is the Lattice QFT (again)?

Classical field theory on the discrete spacetime

+ Path integral quantization

\[ Z = \int D\phi e^{-S_{\text{lattice}}} \]

\[ \int dx \rightarrow \sum_x \]

Lattice quantum field theory
Differential $\partial$ and symmetric difference $\nabla$

$$a\partial_\mu \psi(x) \rightarrow \frac{\psi(x + a) - \psi(x - a)}{2} = (\sinh a \partial_\mu) \psi(x) = \nabla_\mu \psi(x)$$

Leibniz rule violation

$$\partial(fg) = \partial fg + f \partial g \rightarrow \nabla(fg) = \nabla f M[g] + M[f] \nabla g$$

Propagator

$$\frac{1}{i\gamma_\mu ap_\mu + M} \rightarrow \frac{1}{i\gamma_\mu \sin ap_\mu + M}$$
Classical correspondence

Lattice QFT \( \leftrightarrow \) Condensed matter

- Lagrangian \( L_{\text{lattice}} \)
  - Mass \( m_{\text{eff}} \)
  - Gamma matrix \( \gamma_\mu \)

- Hamiltonian \( H_{\text{cond}} \)
  - Magnetic field \( M \)
  - Pauli operator \( \sigma_i \)

→ No need to consider path integral (explicitly)
Classical correspondence

Lattice QFT ↔ Condensed matter

- Staggard
- Minimal doubling
- Domain-wall

- Quantum simulation
- Honeycomb (graphen)
- Topological Insulator

→ No need to consider path integral (explicitly)
Quantum correspondence

Is there Quantum correspondence?

Tensor Renormalization group (for Fermion, Boson), World sheet representation (for Gauge)

\[
\int D\bar{\psi}D\psi \exp \left[ -\sum_x \mathcal{L}(\bar{\psi}_x, \psi_x) \right] \leftrightarrow \sum_{\sigma_i} \exp \left[ -\beta \left( J \sum_i \sigma_i \sigma_{i+1} + h \sum_i \sigma_i \right) \right]
\]

\[\{\bar{\psi}, \psi\} \leftrightarrow \{\sigma_i\}\]

Need to consider path integral measure
Fermionic field/Spin variables

Field variable integration (Fermion part)

\[ \langle \psi_N \bar{\psi}_k \rangle = \int D\psi D\bar{\psi} \psi_N \bar{\psi}_k e^{\sum_{t=1}^{N} \bar{\psi}_t A_t \psi_t + \bar{\psi}_t B_t \psi_{t-1}} \]

\[ = \int D\psi D\bar{\psi} \psi_N \bar{\psi}_k e^{\sum_{t=1}^{N} \bar{\psi}_t A_t \psi_t} \prod_{t=1}^{N} \sum_{b_t=0}^{1} (\bar{\psi}_t B_t \psi_{t-1})^{b_t} \]

To spin variable by field integration

\[ \{ \bar{\psi}_t, \psi_t \} \rightarrow \{ b_t \} \]

→ Efficient numerical integration using ‘locality’ of action, and Grassmann property
Scalar field/Spin variables

Loop formulation

\[
S^B_\Lambda = \sum_x \{ -w \cdot \phi_x \phi_{x-1} + V(\phi_x) \}
\]

\[
e^{-S^B_\Lambda} = \prod_x \left( \sum_{n^b(x)=0}^{\infty} \frac{(w \cdot \phi_{x-1}\phi_x)^{n^b(x)}}{n^b(x)!} \right) \prod_x e^{-V(\phi_x)}
\]

\{\phi_x\} \rightarrow \{n^b(x)\}

→ Straightforward extension to bosonic field with cutoff

Scalar field/Spin variables

- Multiple integration
  - Cost grows exponentially
    - (e.g.) Simpson rule $O \left( n^{\frac{d}{8}} \right)$

- Tensor (matrix) rep.

\[
\int d\phi_1 \ldots d\phi_N \prod_{t=1}^{N} e^{-\frac{1}{2} (\phi_t - \phi_{t-1})^2} \rightarrow \text{tr} \left[ \hat{F}^N_{\phi_t \phi_{t-1}} \right]
\]

\[ O \left( n^{\frac{N}{8}} \right) \rightarrow O(\log N) \]

- Dramatic cost reduction

[D.Kadoh, K.N. arXiv:1803.07960]
Gauge field/flux tubes

World sheet representation

\[ S_G[U] = -\frac{\beta}{3} \sum_{x,\mu<\nu} \text{Re Tr} \ U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu} U_{x,\nu} \]

\[
\int D[U] \prod_{x,\mu<\nu} \prod_{a,b,c,d} \sum_{n_{abcd}^{x,\mu\nu}=0} \sum_{\bar{n}_{abcd}^{x,\mu\nu}=0} \frac{(\beta/6) n_{abcd}^{x,\mu\nu} + \bar{n}_{abcd}^{x,\mu\nu}}{n_{abcd}^{x,\mu\nu} \bar{n}_{abcd}^{x,\mu\nu}} \left( U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu} U_{x,\nu} \right)^* n_{abcd}^{x,\mu\nu}! (c.c.) \bar{n}_{abcd}^{x,\mu\nu}!
\]

\[ \{ U_{x,\mu\nu}^{abcd}, U_{x,\mu\nu}^{abcd*} \} \rightarrow \{ n_{x,\mu\nu}^{abcd}, \bar{n}_{x,\mu\nu}^{abcd} \} \]

→ Integrate out can be done at least SU(3) case.

Interest

What are the points?

Advantages and disadvantages

- Sign problem
- Another representation
- Low cost (low dim)
- Partition function

× High cost (high dim)
△ Systematic error

→ We study SUSYQM as a good example.

[D.Kadoh, K.N. arXiv:1803.07960]
[D.Kadoh, K.N. arXiv:1812.10642]
1-dim SUSYQM

\[ S = \sum_{t=1}^{N} \left[ \frac{1}{2} (\nabla \phi_t + W(\phi_t))^2 + \bar{\psi}_t (\nabla + W'(\phi_t)) \psi_t \right] \]

\[ Q_1 \phi = \psi \]
\[ Q_1 \psi = -\partial_t - W \]
\[ Q_1 \bar{\psi} = 0 \]

Q-exact form

\[ Q_1 S = 0 \quad \text{Latticize} \quad Q_1 S = 0 \]
\[ Q_2 S = 0 \quad \rightarrow \quad Q_2 S \neq 0 \]

\[ \{ Q_1, Q_2 \} = 2 \partial_t \quad \{ Q_1, Q_2 \} = 2 \nabla_t \]
1-dim SUSYQM

\[ S = \sum_{t=1}^{N} \left[ \frac{1}{2} (\nabla \phi_t + W(\phi_t))^2 + \bar{\psi}_t (\nabla + W'(\phi_t)) \psi_t \right] \]

\[ W(\phi) = m\phi + \lambda m^{3/2} \phi^2 \]
Comparing to Monte-Carlo

\[ N = 20 \]

[Fig. 3. Linear extrapolations of \( \langle I.K.Kanamori,F.Sugino,\text{and H.Suzuki, Annals Phys.}(2008) \rangle \)]

\[ am \geq 0.05 \]

\[ N = 1500 \sim 30000 \]

\[ am \geq 0.001 \]

→ Great improvement
Spectral degeneration

\[ W(\phi) = m\phi + \lambda m^2 \phi^3 \]

\[ m_{\text{fermi}}/m = 1.6678215773620(2) \]
\[ m_{\text{boson}}/m = 1.667821577(1) \]

→ Degenerate with large volume
Hamiltonian (extrapolation)

\[ W(\phi) = m\phi + \lambda m^{3/2}\phi^2 \]

Our result

\[ [I.\text{Kanamori}, F.\text{Sugino}, \text{and H.}\text{Suzuki, PTEP(2008)}] \]
Hamiltonian (Weak coupling regime)

\[ W(\phi) = m\phi + \lambda m^{3/2}\phi^2 \]

\[ \lambda^2 \]
<table>
<thead>
<tr>
<th>MPS</th>
<th>TRG</th>
<th>Warm algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deterministic</td>
<td>Deterministic</td>
<td>Sampling</td>
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<tr>
<td>usual Vol.</td>
<td>exponentially large</td>
<td>usual Vol</td>
</tr>
<tr>
<td>w/o sign-prob.</td>
<td>w/o sign-prob.</td>
<td>It depends.</td>
</tr>
<tr>
<td>~2 or 3 dim</td>
<td>~2 or 3 dim</td>
<td>any-dim</td>
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<tr>
<td>No rigorous error</td>
<td>No rigorous error</td>
<td>statistical error</td>
</tr>
<tr>
<td>State base</td>
<td>Measurement base</td>
<td>Measurement base</td>
</tr>
</tbody>
</table>

- **MPS**
  - Deterministic
  - usual Vol.
  - w/o sign-prob.
  - ~2 or 3 dim
  - No rigorous error
  - State base

- **TRG**
  - Deterministic
  - exponentially large
  - w/o sign-prob.
  - ~2 or 3 dim
  - No rigorous error
  - Measurement base

- **Warm algorithm**
  - Sampling
  - usual Vol
  - It depends.
  - any-dim
  - statistical error
  - Measurement base
Tensor representation for Lattice

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<td>Deterministic</td>
<td>Deterministic</td>
<td>Sampling</td>
</tr>
<tr>
<td>usual Vol.</td>
<td>exponentially large</td>
<td>usual Vol</td>
</tr>
<tr>
<td>w/o sign-prob.</td>
<td>w/o sign-prob.</td>
<td>It depends.</td>
</tr>
<tr>
<td>~2 or 3 dim</td>
<td>~2 or 3 dim</td>
<td>any-dim</td>
</tr>
<tr>
<td>No rigorous error</td>
<td>No rigorous error</td>
<td>statistical error</td>
</tr>
<tr>
<td>State base</td>
<td>Measurement base</td>
<td>Measurement base</td>
</tr>
</tbody>
</table>
Examples: Variational TRG

(Simple) Tensor renormalization group (TRG)

[M. Levin and C. P. Nave arXiv:cond-mat/0611687]

Higher-Order TRG (HOTRG)


Anisotropic TRG (ATRG)


Triad RG (Tri-RG)

[D. Kadoh and K.N. arXiv:1911.xxxxx]
(Simple) Tensor renormalization group (TRG)

[Z = \sum \prod T_{a_x,y,a_{x+1},y,b_x,y,b_{x,y+1}}]

Starting point: network representation

Naive contraction costs \( \propto \{\text{dim}(a)\text{dim}(b)\}^{\text{Volume}}\)

How can we calculate (approximate) this contractions?
Singular Value Decomposition (SVD)

$$T_{abcd} = \sum_{k}^{D} A_{ab}^{k} \lambda^{k} B_{cd}^{k}$$

Larger singular values $\lambda$ have much “information” of $T$ → (Frobenius norm)

→ We can approximate the matrix by the cutoff of index $k$

$$\dim(k) = \dim(a) \dim(b) \rightarrow D$$
SVD for a coarse graining (e.g. Image)

SVD for a coarse graining (e.g. Ising)

One step

\[
T_{abcd} = \sum_k A_{ac}^k B_{bd}^k
\]

\[
T_{abcd} = \sum_k C_{ad}^k D_{bc}^k
\]

\[
T_{klnm}^{(n+1)} = \sum_{a,b} A_{a xy+1 b xy+1}^k B_{a xy b x-1y+1}^l C_{a xy b xy+1}^m D_{a xy+1 b x-1y+1}^n
\]

→ Exponential Volume reduction \[ V \rightarrow \frac{V}{2} \]
SVD for a coarse graining (e.g. Ising)

ALL step (graphical)

\[ Z = \sum_{a,b} \prod_{x,y} T_{a_x, y, a_{x+1}, y, b_x, y, b_{x+1}, y+1} \]
\[ \rightarrow \sum_{a, b} T'_{a a b b} \]
Numerical costs for simple TRG

How fast?

\[ T_{abcd} = \sum_{k} A_{ac}^{k} B_{bd}^{k} \]

SVD: \( O(D^6) \)

Contraction:

\[ T_{klmn}^{(n+1)} = \sum_{a,b} A_{a_{xy+1}b_{xy+1}}^{k} B_{a_{xy}b_{x-1y+1}}^{l} C_{a_{xy}b_{xy+1}}^{m} D_{a_{xy+1}b_{x-1y+1}}^{n} \]

\( O(D^6) \)

\[ 2^{2V} \rightarrow O((\log V) \times D^6) \]

Large D is still difficult.
(Higher dimension, complicated system...)

→ We need more sophisticated algorithm.
Cost for Higher-dimensional system

Can we generalize (simple) TRG for higher dimension?

→ Formally we can, but bad approximation (and Cost)

Cutoff: \( D^3 \rightarrow D \)

Cost: \( O(D^{12}) \)

We need more faster method (for more interesting system)
Examples: Variational TRG

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Higher-Order TRG (HOTRG)


Contraction by projection operator $U$

\[ M_{x_1 x_2 x'_1 x'_2 y y'} = \sum_k T_{x_1 x'_1 y k} T_{x_2 x'_2 k y'} \]

\[ [M^t M]_{[x_1 x_2][\chi_2 \chi_2]} = \sum_{abcd} M^t_{[x_1 x_2][abcd]} M_{[abcd][\chi_1 \chi_2]} \]

SVD Cutoff: $D^2 \rightarrow D$

Cost: $O(D^{11})$

\[ [M^t M]_{[x_1 x_2][\chi_2 \chi_2]} = \sum_k U^k_{[x_1 x_2]} \lambda^k U^k_{[\chi_1 \chi_2]} \]

Contraction Cost: $O(D^{4 \dim - 1})$
Numerical costs for HOTRG

More higher dimension? → still difficult.

Cost: \( O(D^{4\text{dim}-1}) \)

4-dim Ising \( O(D^{15}) \) [S. Akiyama, Y. Kuramashi et al. arXiv:1906.06060]

\[
D = 13 @ Oakforest-PACS
\]

\[
L^4 \leq 1024^4
\]

\[
T_c(D = 13) = 6.650365(5)
\]

Monte-Carlo

\[
L^4 \leq 80^4
\]

\[
T_c = 6.68026(2)
\]

[ P.H. Lundow and K. Markstrom arXiv:1202.3031 ]
**Examples: Variational TRG**

(Simple) Tensor renormalization group (TRG)

[M. Levin and C. P. Nave arXiv:cond-mat/0611687]

Higher-Order TRG (HOTRG)


Anisotropic TRG (ATRG)


\[O(D^{4\text{dim}-1}) \rightarrow O(D^{2\text{dim}+1})\]

Triad RG (Tri-RG)

[D. Kadoh and K. N. arXiv:1911.xxxxx]
Anisotropic TRG (ATRG)


◊ Auxiliary SVD before SVD (In order to reduce indices)

$O(D^{4\text{dim}-1}) \rightarrow O(D^{2\text{dim}+1})$

◊ Key-point: Low rank rep. reduces the leading cost.
Examples: Variational TRG

(Simple) Tensor renormalization group (TRG)

[M. Levin and C. P. Nave arXiv:cond-mat/0611687]

Higher-Order TRG (HOTRG)


Anisotropic TRG (ATRG)


\[ O(D^{4\text{dim}-1}) \rightarrow O(D^{2\text{dim}+1}) \]

Triad RG (Tri-RG)

[D. Kadoh and K. N. arXiv:1912.xxxxx]

\[ O(D^{4\text{dim}-1}) \rightarrow O(D^{\text{dim}+3}) \]
**Triad RG**

- Using the Triad (Rank-3) tensor as a fundamental tensor

→ We apply HOTRG-like procedure to Triad tensor rep.

- projection operator $U$

- contraction part

[D. Kadoh and K.N. arXiv:1912.xxxxx]
**Triad RG**

- Triad rep. reduces the cost without loss of precision
  
  \[ O(D^{4\text{dim}-1}) \rightarrow O(D^{\text{dim}+3}) \]

**3-dim Ising**

---

[D. Kadoh and K.N. arXiv:1912.xxxxx]
Triad RG

[D. Kadoh and K.N. arXiv:1912.xxxxx]

Numerical results are compared to those obtained from the HOTRG and the ATRG. We employ an improved algorithm of HOTRG with the RSVD, which is shown in appendix ??, and the leading cost does not change. The ATRG is implemented with RSVD and twice a larger index size $2^D$ only for swapping step. Although several improvements which do not change the leading cost are studied, we only consider the ATRG without any other improvement.

In Tri-RG, an $O(D^5)$ isometry is prepared with the RSVD. The computational cost strongly depends on $D$, and those of HOTRG, ATRG, and Tri-RG methods are theoretically $O(D^{11})$, $O(D^7)$, and $O(D^6)$ in three dimensions, respectively.

Figure 5 shows the $D$-dependence of the free energy. The accessible $D$ is different among the three methods. Roughly speaking, three results approach almost the same converged value, which is expected to be the true value, as $D$ increases. This figure implies that the error of Tri-RG is well-controlled for larger $D$s, and the results of Tri-RG are closest to the true value. We try to extrapolate our result of Tri-RG by the function $a + bD^{-c}$ with the fitting variables $a$, $b$, and $c$. The result in the range $10 \leq D \leq 56$ can be extrapolated to the $D \to \infty$ limit as $a = -1.7546(1)$. Note that we confirm the stability of the fit by the difference from the fit in the range $20 \leq D \leq 56$ for Tri-RG.

Figure 6 shows the free energy against the computational time. Since the true value at the large $D$ limit lies on near $-1.7546$, as Figure 5 implies, one can conclude that the Tri-RG converges faster than the other methods.

In Figure 7, the running time is shown as a function of $D$. The theoretical $D^d$-dependence, which is $D^2d+1$ for ATRG and $D^d+3$ for Tri-RG, is reproduced at a practical level in three dimensions ($d=3$). Thus we can consider that the Tri-RG method will open a door to studying a wide class of higher dimensional field theory with tensor networks.

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Summary for TRG part

- How to construct the tensor network rep. is one important step to apply TRG method to the statistical system.
- For 1 or 2 dimension, TRG method provide the precise determination of the physical quantities without sign-problem.
- Now TRG methods are improved to apply to the higher dim. Low-rank tensor is a key-idea.
- We need theory of the systematic error of cutoff parameter. For more realistic calculation, (maybe) we should introduce some stochastic method such as Monte-Carlo simulation.