Pseudoscalar distribution amplitudes from lattice QCD

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with

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Outline

- DAs: definition and moments of DAs
- Lattice calculation of the 2nd moment of the pion DA
- Calculation of the pion DA in X-space
- First X-space results on leading and higher twist DAs
- Outlook

1) RQCD: GB, VM Braun, M Göckeler, M Gruber, F Hutzler, P Korcyl, B Lang, A Schäfer, 1705.10236 + in preparation
2) RQCD: GB, VM Braun, B Gläßle, M Göckeler, M Gruber, F Hutzler, P Korcyl, B Lang, A Schäfer, P Wein, J-H Zhang, 1709.04325;

Not covered:

ρ DAs [RQCD: VM Braun et al, 1612.02955]
octet baryon DAs [RQCD: GB et al, 1512.02050 + in preparation]

Attention: Some results shown are still preliminary.
What are distribution amplitudes?

Wavefunction of a hadron (here pion) near the infinite momentum frame, written as a superposition of different Fock states:

$$|\pi\rangle = c_1|\bar{q}q\rangle + c_2|\bar{q}gq\rangle + c_3|\bar{q}q\bar{q}q\rangle + \ldots$$

Light front wavefunction (Distribution amplitude, DA) describes the distribution of the longitudinal momentum among the partons.

Momentum fractions $0 \leq u_i \leq 1$, $\sum_{i \in \{q, \bar{q}, g\}} u_i = 1$.

At leading twist (twist 2) only the valence quarks contribute:

$$u = u_q = 1 - u_{\bar{q}}, \quad \xi = u_q - u_{\bar{q}} = 2u - 1 \in [-1, 1].$$

In hard processes higher Fock states are power suppressed.

PDFs are (within the parton model) single particle probability densities and can **directly be extracted from fits to DIS and SIDIS data**.

DAs are wavefunctions and only appear within convolutions in hard exclusive processes. Due to other hadronic uncertainties (and experimental techniques), it is much **harder to extract these reliably from experimental data**.
Distribution amplitudes II

DAs are needed for the theoretical description of hard exclusive processes. Example: collinear factorization of the $\gamma \gamma^* \rightarrow \pi^0$ photoproduction formfactor ($Q \gtrsim \mu \gg \Lambda$) [Belle, 1205.3249]

\[
F_{\pi\gamma}(Q^2) = \frac{2F_\pi}{3} \int_0^1 du \ H_{\bar{q}q\gamma}(u, \mu_F, Q^2) \cdot \phi_\pi(u, \mu_F) + \text{higher twist}.
\]

$\mu_F$ is the factorization scale and we renormalize the hard coefficient function $H$ at the scale $\mu_R^2 = Q^2$, $\mu_F^2 \sim Q^2/4$. $F_\pi \approx 92$ MeV.
Definition of DAs

Non-local light front matrix element at a separation $n\ (n^2 = 0)$:

$$
\langle 0 | \bar{d} \left( \frac{n}{2} \right) \not{n} \gamma_5 \left[ \frac{n}{2}, -\frac{n}{2} \right] u \left( -\frac{n}{2} \right) | \pi^+(p) \rangle
$$

$$
= i F_\pi \, n \cdot p \int_0^1 du \exp \left\{ i \left[ u - (1 - u) \right] \frac{(n \cdot p)}{2} \right\} \phi_\pi(u, \mu)
$$

$[n/2, -n/2]$ above denotes a gauge covariant connection.

The DA is not accessible in Euclidean spacetime but moments of DAs are:

$$
\langle \xi^n \rangle = \int_0^1 du \, (2u - 1)^n \phi_\pi(u, \mu) , \quad \langle \xi^0 \rangle = 1 , \quad \langle \xi^1 \rangle = 0 .
$$

$\langle \xi^{0,2} \rangle$ can be extracted from local matrix elements $\langle 0 | O_{\mu\nu\rho}^\pm | \pi^+(p) \rangle$ with

$$
O_{\mu\nu\rho}^\pm = \bar{d} \left\{ \left[ \not{D}_\mu \not{D}_\nu \pm 2 \not{D}_\mu \not{D}_\nu + \not{D}_\mu \not{D}_\nu \right] \gamma_\rho \right\} u ,
$$

where $(\cdots)$ gives a traceless, symmetrized expression.
Gegenbauer expansion:

\[ \phi_{\pi}(u, \mu) = 6u(1 - u) \left[ 1 + \sum_{n \in \mathbb{N}} a^{\pi}_{2n}(\mu) C_{2n}^{3/2}(2u - 1) \right] \]

Collinear conformal symmetry: \( C_{n}^{3/2}(\xi) \) in SL(2, \( \mathbb{R} \)) analogous to \( Y_{\ell m}(\theta, \phi) \) in SO(3). \( \langle \xi^{2n} \rangle \) and \( a^{\pi}_{2n} \) are related by simple algebraic expressions (\( n = 1 \) example):

\[ a^{\pi}_{2}(\mu) = \frac{7}{12} \left( 5\langle \xi^{2} \rangle - 1 \right) = \frac{7}{12} \left( 5\langle \xi^{2} \rangle - \langle \xi^{0} \rangle \right) \]

\( a^{\pi}_{2n}(\mu) \to 0 \) as \( \mu \to \infty \): At large scales the lower moments will dominate.

Note the difference in the counting: 2nd DA-moment \( \sim \) 3rd PDF-moment.
Previous results (\(\overline{\text{MS}}\) scheme at \(\mu = 2\ \text{GeV}\))

\(N_f = 2,\ M_\pi = 150 - 490\ \text{MeV},\ LM_\pi = 3.4 - 6.7.\)

\[
\overline{a}_2^{\overline{\text{MS}}}(2\ \text{GeV}) = 0.136(15)(15)(?)
\]

[RQCD: VM Braun et al, 1503.03656],

\[
\overline{a}_2^{\overline{\text{MS}}}(2\ \text{GeV}) = 0.233(30)(60)\ (N_f = 2 + 1)
\]

[RBC/UKQCD: R Arthur et al, 1011.5906]

\[
\overline{a}_2^{\overline{\text{MS}}}(2\ \text{GeV}) = 0.211(114)
\]

[QCDSF/UKQCD: VM Braun et al, hep-lat/0606012]
Challenge: statistical errors

- Continuum extrapolation could not be carried out:

- Second moment of pion DA requires at least two non-vanishing momentum components, e.g., $\vec{p} = (1, 1, 0)2\pi/L$.
- Employing two derivatives considerably deteriorates the signal-to-noise ratio.
Momentum smearing

Intuition: interpolating wavefunction used to create $|\pi^+(p)\rangle$ should acquire a phase for $\vec{p} \neq \vec{0}$ $\Rightarrow$ momentum smearing.

[RQCD: GB, B Lang, B Musch, A Schäfer, 1602.05525]
Note that in one loop ChPT does not have chiral logs in this DA moment.

\[ N_f = 2 + 1, \text{ where } m_s + 2m_{ud} = \text{Tr } M = \text{const} \]

[RQCD: GB, VM Braun, M Göckeler, M Gruber, F Hutzler, P Korcyl, B Lang, A Schäfer, 1705.10236]

NLO matching between RI’/SMOM and \( \overline{\text{MS}} \). NNLO matching to \( \overline{\text{MS}} \)? Above was at \( a \approx 0.086 \text{ fm} \). **New**: the continuum limit.
CLS simulation strategy

Simulate along $m_s + 2m_\ell = \text{const}$ [QCDSF+UKQCD: W Bietenholz et al, 1003.1114], and $\tilde{m}_s \approx \text{const}$ [G Bali et al, 1606.09039; 1702.01035], enabling Gell-Mann–Okubo/SU(3) and SU(2) $\chi$PT extrapolations.

(Only linear unconstrained baryon mass fits are shown.)
### CLS ensemble overview

![Graph showing ensemble overview](image)

**Physical**

\[ m_s + 2m_\ell = \text{const} \]

\[ \hat{m}_s \approx \text{const} \]

- **E**: 192 · 96³
- **J**: 192 · 64³
- **D**: 128 · 64³
- **N**: 128 · 48³
- **C**: 96 · 48³
- **S**: 128 · 32³
- **H**: 96 · 32³
- **B**: 64 · 32³
- **U**: 128 · 24³

\[ \exists \] additional ensembles with \( m_s = m_\ell \).
Results I: $\langle \xi^0 \rangle \rightarrow 1 \quad (a \rightarrow 0)$? (preliminary)

Check of renormalization: $\langle \xi^0 \rangle = 1 + \mathcal{O}(a) \sim (\zeta_{22} - 5\zeta_{12}) \mathcal{O}^+$

$\zeta_{ij}$: renormalization from lattice to $\overline{\text{MS}}$. 
Results II: chiral extrapolation $\text{Tr } M = \text{const}$ (preliminary)

SU(3) NLO ChPT and linear lattice spacing effects (8 parameters):

$$\xi_X^2(M_\pi, M_K, a) = \left[ \xi_0^2 + \Delta M^2 + A_\chi \delta M^2 \right] \cdot \left[ 1 + a \left( c_0 + \bar{c} M^2 + c_X \delta M^2 \right) \right], \quad X \in \{\pi, K, \eta_8\},$$

$$A_\pi = -2A_K = -A_{\eta_8}, \quad \delta M^2 = M_K^2 - M_\pi^2, \quad \overline{M}^2 = \frac{1}{3} \left( 2M_K^2 + M_\pi^2 \right).$$
Results III: chiral extrapolation $\hat{m}_s \approx \text{const}$ (preliminary)

$m_s = \text{phys. } \beta = 3.4 \ a \approx 0.0854 \ \text{fm}$

$m_s = \text{phys. } \beta = 3.46 \ a \approx 0.076 \ \text{fm}$

$m_s = \text{phys. } \beta = 3.55 \ a \approx 0.0644 \ \text{fm}$

$m_s = \text{phys. } \beta = 3.7 \ a \approx 0.05 \ \text{fm}$
Results IV: chiral extrapolation $m_s = m_\ell$ (preliminary)

\begin{align*}
\text{physical point} & = m_\ell = \text{const.} \\
m_s & = \text{ms} = \text{const.} \\
\text{Tm} & = \text{const.}
\end{align*}

\begin{align*}
\langle \xi^2 \rangle & = \text{Symmetric, } \beta = 3.4 \ a \approx 0.0854 \ \text{fm} \\
\langle \xi^2 \rangle & = \text{Symmetric, } \beta = 3.55 \ a \approx 0.0644 \ \text{fm} \\
\langle \xi^2 \rangle & = \text{Symmetric, } \beta = 3.7 \ a \approx 0.05 \ \text{fm} \\
\langle \xi^2 \rangle & = \text{Symmetric, } \beta = 3.85 \ a \approx 0.039 \ \text{fm}
\end{align*}
Continuum limit, at the physical point:

\[ a_2^{\pi,\overline{\text{MS}}} (2 \text{ GeV}) \approx 0.116(19), \quad \langle \xi^2 \rangle_\pi \approx 0.2399(64), \quad (a_2 \propto 5 \langle \xi^2 \rangle - 1) \]

\[ a_2^{K,\overline{\text{MS}}} (2 \text{ GeV}) \approx 0.100(13), \quad \langle \xi^2 \rangle_K \approx 0.2343(43). \quad \text{PRELIMINARY!} \]

More statistics is in preparation. Renormalization is not finalized.

**Errors are statistical only!!** Systematics are under investigation.
Euclidean spacetime determination of the DA

Large momentum effective theory (LaMET) [X Ji,1305.1539]: compute “quasi-distribution”, in analogy to “quasi-PDF”.

\[ \tilde{\phi}_\pi(u, a^{-1}, p_z) = \frac{i}{F_\pi} \int_{-\infty}^{\infty} \frac{dZ}{2\pi} e^{-i(u-1)p_zZ} \langle \pi(p)|\bar{\psi}(0)\gamma_z\gamma_5[0, z]|\psi(z)|0\rangle, \]

with quark fields separated along the spatial \( z \) direction \((z^\mu) = (0, 0, 0, Z))\). Then match to pion DA (like [X Ji, 1506.00248] for PDFs):

\[ \tilde{\phi}_\pi(u, a^{-1}, p_z) = \int_u^{1} \frac{dv}{v} Z_\phi \left( \frac{u}{v}, a^{-1}, \mu, p_z \right) \phi_\pi(v, \mu) + \mathcal{O}(\Lambda_2^2/p_z^2, M_{\pi}^2/p_z^2). \]

Practical problems: \( Z_\phi \) has many arguments and is power divergent. However: [K Orginos, A Radyushkin, J Karpie, S Safeiropoulos, 1706.05373] Perturbative matching to \( \overline{\text{MS}} \) at large \( Z \)? Contribution suppressed for large \( p_z \)? Discard large \( Z \) data and integrate over \( d(p_zZ)/p_z \) to minimize corrections? Does \( \exists \) enough data in relevant \( p_zZ \) region to carry out Fourier transform? Window and statistics: \( \pi/a \gg p_z \gg m_N \) for nucleon PDF.

The pion DA is an interesting test case.
What do we do differently?

We will compute the DA in $X$-space. The method is related to “quasi-PDFs”, however, there are essential differences:

- We do not employ a Wilson line $[z/2, -z/2]$ and we throw away $2/|z| < 1$ GeV data. This simplifies renormalization and matching.
- Most importantly: we do not transform the DA to the longitudinal momentum fraction space.

We compute an object ($z = (0, \vec{z}) \Rightarrow z^2 = -\vec{z}^2 < 0$, $|p \cdot z| = | - \vec{p} \cdot \vec{z}|$)

$$\sqrt{2} T(p \cdot z, z^2) = \langle 0 | \bar{d}(z/2) \Gamma_A q(z/2) \bar{q}(-z/2) \Gamma_B u(-z/2) | \pi^+ (p) \rangle$$

$q$ is an auxiliary field. We use $J_{\Gamma_A} J_{\Gamma_B} = SP + PS, VA + AV, VV + AA$. $T(z^2)$ can then be factorized just like

$$F_{\pi\gamma}(Q^2) = \frac{2F_\pi}{3} \int_0^1 du H_{qq\gamma}(u, \mu, Q^2) \phi_\pi(u, \mu) + \text{higher twist}$$

into a hard, perturbative function and the DA, as long as $2/|z| \gg \Lambda$.

We set $\mu_R = 2/|z|$, which “plays” the role of $|Q|$. ($|z| = \sqrt{-z^2}$)
Following [VM Braun, D Müller, 0709.1348], we use a light quark propagator for \( q(z/2)\bar{q}(-z/2) \). Advantage: Renormalization factor of \( \bar{q}\Gamma u \) known. Other suggestions:

- “Static” propagator: Large Energy Effective Theory (LEET) [MJ Dugan, B Grinstein, PLB255(91)583], Large Momentum Effective Theory (LaMET) [X Ji, 1305.1539],
- Heavy quark propagator: [W Detmold, CJD Lin, hep-lat/0507007].

\[
T(p \cdot z, z^2) = F_\pi p \cdot z \frac{1}{2\pi^2 z^4} \Phi_\pi(p \cdot z),
\]

with the \( X \)-space (Ioffe time) DA

\[
\Phi_\pi(p \cdot z) = \int_0^1 du e^{i(u-1/2)z} \phi_\pi(u).
\]
The DA in momentum and in $X$-space

In practice we compute for large $t$ (example: $\Gamma_A \Gamma_B = \frac{1}{2} \{ \gamma_5, 1 \}$)

$$
\frac{T(p \cdot z, z^2)}{F_\pi} = \frac{Z_S Z_P}{Z_A} \frac{\langle 0 | [\bar{d} q](t, \bar{z}/2) [\bar{q} \gamma_5 u](t, -\bar{z}/2) O^\dagger_\pi (0, \bar{p}) | 0 \rangle}{\langle 0 | [\bar{d} \gamma_0 \gamma_5 u](t, 0) O^\dagger_\pi (0, \bar{p}) | 0 \rangle} E(\bar{p}),
$$

where the lattice currents are renormalized to the $\overline{\text{MS}}$ scheme via $Z_S(\mu_R a, g^2)$, $Z_P(\mu_R a, g^2)$ and $Z_A(g^2)$ and $\mu_R = \mu_F = 2/|z|$.

We tree-level correct for $\bar{z}$-dependent lattice artefacts:

$$
T(p \cdot z, z^2) \mapsto T(p \cdot z, z^2) \frac{\text{tr} [\bar{z} G^\text{tree}_{\text{cont}}(z)]}{\text{tr} [\bar{z} G^\text{tree}_{\text{latt}}(z, a)]} \quad \text{(for the chiral even part)}
$$

\[ p \cdot z = 0.39 \]
Three illustrative models of the DA (taken at a scale $\mu_0 = 1 \text{ GeV}$):

\[
\phi^{(1)}(u) = 6u(1 - u), \quad \phi^{(1/2)}(u) = \frac{8}{\pi} \sqrt{u(1 - u)}, \quad \phi^{(0)}(u) = 1
\]

\[
\phi(\alpha) = \frac{\Gamma[2(\alpha + 1)]}{[\Gamma(\alpha + 1)]^2} [u(1 - u)]^\alpha
\]

$|z| < 2/\text{GeV} \approx 0.4 \text{ fm}$ is necessary for factorization, $\overline{\text{MS}}$ scheme matching!

Large $|\vec{p} \cdot \vec{z}|/2$ values are desirable as this is conjugate to $\xi = 2u - 1$.

$|z|$ small $\Rightarrow$ large $|\vec{p}|$. No other reason why $|\vec{p}|$ has to be large!
Conformal partial waves: how large should $|p \cdot z|$ be?

Ioffe time DA: \[ \Phi_\pi(p \cdot z, \mu) = \sum_{n \geq 0} a_{2n}^{\pi} F_{2n} \left( \frac{1}{2} |p \cdot z| \right) \]

No sensitivity to $a_4$ for $|p \cdot z| < 4$!!!
$N_f = 2$ NP improved Wilson-clover quarks (QCDSF ensemble)

\[ a^{-1} \approx 2.76 \text{ GeV}, \quad M_\pi \approx 290 \text{ MeV}, \quad L = 32a \approx 3.4/M_\pi, \quad a_2^\pi \approx 0.136 \]

Large $|\vec{p}|a$, $a/|z| \to$ bigger lattice corrections. QCD factorization:

\[
\frac{T(p \cdot z, z^2)}{F_\pi} = \frac{p \cdot z}{2\pi^2 z^4} \int_0^1 du e^{i(u-1/2)p \cdot z} H(u, \mu, z^2)\phi_\pi(u, \mu) + T^{HT},
\]

We have computed $H(u, \mu, z^2)$ to order $\alpha_s$. $T^{HT} = \mathcal{O}(M_\pi^2 z^2, \Lambda^2 z^2)/z^4$

can be estimated from LCSR: $\delta_2^\pi \approx 0.17 \text{ GeV}^2$.

XDA Results II: SP+PS ($\mu = \mu_R = 2/|z|$) with fits

Dashed lines: higher twist effects subtracted from the fits.

The future: smaller $a$, larger $|\vec{p}|$. 
Higher twist effects (and data!) go in the different direction.
XDA Results IV: Ioffe time DA from VA+AV

All fits give \( a_2^\pi \sim 0.2 - 0.3 \) (instead of \( \sim 0.14 \) from the local method).

But higher twist \( 0.2 \text{ GeV}^2 \lesssim \delta_2^\pi \lesssim 0.25 \text{ GeV}^2 \) is well constrained.
Two fitted DAs at $\mu = 2$ GeV. Different shapes but similar $a_{2}^{\pi}$! Also discrimination from experimental data is difficult!

Dashed: $N_{f} \approx 2 + 1 + 1$, $M_{\pi} = 310$ MeV, $a = 0.12$ fm, $LM_{\pi} = 4.5$

From quasi-DA: [LP$^{3}$: J–W Chen et al, 1712.10025]
The future of Gegenbauer/Mellin moments:
- Continuum limit.
- So far NP matching to RI’/SMOM (RI’/MOM) scheme and then to \( \overline{MS} \) at NLO (NNLO). Exploring NNLO X-space matching.
- Also other meson and baryon DAs.

Euclidean X-space method:
- We presented a proof of concept.
- For 2/|z| \( \gtrsim 1 \text{ GeV} \) we need |\( \vec{p} \)| \( \gtrsim 3 \text{ GeV} \) to reach “Ioffe times” |\( p \cdot z \)| large enough to discriminate between DA parametrizations \( \Rightarrow \) small lattice spacings are necessary.
- Different current combinations depend differently on higher twist contributions: First lattice determination of higher twist DA!
- Ideal: smaller |z| to suppress higher twist effects \( \Rightarrow \) even higher |\( \vec{p} \)|.
- Soon: kaon X-space DA.
- Already at the limit of the state-of-the art for the pion DA: what about nucleon PDFs?
- Good news: matching to the \( \overline{MS} \) scheme is easier for currents without derivatives. The matching function requires “only” a continuum calculation \( \Rightarrow \) NNLO calculation ongoing.