The hadronic light-by-light scattering contribution to the muon $g - 2$ from lattice QCD

Antoine Gérardin

Institut für Kernphysik, Johannes Gutenberg-Universität Mainz

Zeuthen - May, 2018
Motivations

- The magnetic moment of the muon is proportional to the spin \( \vec{\mu} = g \left( \frac{Qe}{2m} \right) \vec{s} \)
The magnetic moment of the muon is proportional to the spin
\[ \bar{\mu} = g \left( \frac{Qe}{2m} \right) \bar{s} \]

Classical result: \( g = 2 \) for elementary fermions (Dirac equation)
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• Classical result : \( g = 2 \) for elementary fermions (Dirac equation)

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\begin{align*}
\text{Quantum field theory : } a_\mu &= \frac{g-2}{2} \neq 0 \\
\alpha_\mu^{(1)} &= \frac{\alpha}{2\pi} & \text{[Schwinger '48]}
\end{align*}
\]

• Theory estimate : \( a_\mu^{\text{th}} = (116\,591\,823 \pm 61) \times 10^{-11} \) [PDG '17]
  \( \rightarrow \) Precision of 0.5 ppm! Can be used to test the validity of the Standard Model of particle physics
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→ Precision of 0.5 ppm! Can be used to test the validity of the Standard Model of particle physics

• Experimental value : \( a_{\mu}^{exp} = (116\,592\,089 \pm 63) \times 10^{-11} \) [Bennett et al. '06]

→ 3 - 4 \( \sigma \) discrepancy.

Signal of new physics? Under-estimated errors?
(\(g - 2\))\(_\mu\) : present status

- **QED accounts for more than 99.99% of the final result**
  - \(\rightarrow\) 5-loop contributions are known!
  - \(\rightarrow\) very strong test of QED

- **Electroweak corrections**

  ![Electroweak diagrams]

  \(\rightarrow\) One-loop and also two-loop contributions are known

- **QCD corrections**
  - \(\rightarrow\) quarks and gluons do not directly couple to the muon : contribution via loop diagrams
  - \(\rightarrow\) The two main contributions are

  ![QCD diagrams]

  Hadronic Vacuum Polarisation (HVP, \(\alpha^2\))
  Hadronic Light-by-Light scattering (HLbL, \(\alpha^3\))
### $(g - 2)_{\mu}$: Current Status

<table>
<thead>
<tr>
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<tbody>
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<td>QED (leptons, 5th order)</td>
<td>116 584 718.846 ± 0.037</td>
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### Introduction

The pion-pole contribution

Forward light-by-light scattering amplitudes

Direct lattice calculation

Conclusion

\[ (g - 2)_\mu : \text{current status} \]

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- \( \Delta a_\mu = \frac{(g - 2)_\mu}{2} = a_\mu^{\text{exp}} - a_\mu^{\text{th}} = 278 \times 10^{11} \)

\[ \rightarrow \sim 3 - 4 \sigma \text{ discrepancy between experiment and theory} \]

- Future experiments at Fermilab and J-PARC: reduction of the error by a factor of 4
- Theory error is dominated by hadronic contributions
The hadronic light-by-light scattering contribution to the muon $g - 2$ from lattice QCD

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**Hadronic Vacuum Polarisation (HVP, $\alpha^2$)**

**Hadronic Light-by-Light scattering (HLbL, $\alpha^3$)**
Previous estimates: model calculations

\[ a_{\mu}^{\text{HLbL}} = \sum_{\pi^0, \eta, \eta'} + \sum_{\pi^+} + \ldots \]

[de Rafael '94]
1) Chiral counting
2) \( N_c \) counting

\[ \sum_{\pi^0, \eta, \eta'} + \sum_{\pi^+} + \ldots \] [extracted from A. Nyffeler's slide, units: \( a_\mu \times 10^{11} \)]

<table>
<thead>
<tr>
<th>Contribution</th>
<th>BPP</th>
<th>HKS, HK</th>
<th>KN</th>
<th>MV</th>
<th>BP, MdRR</th>
<th>PdRV</th>
<th>N, JN</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi^0, \eta, \eta' )</td>
<td>85\pm13</td>
<td>82.7\pm6.4</td>
<td>83\pm12</td>
<td>114\pm10</td>
<td>-</td>
<td>114\pm13</td>
<td>99 \pm 16</td>
</tr>
<tr>
<td>axial vectors</td>
<td>2.5\pm1.0</td>
<td>1.7\pm1.7</td>
<td>-</td>
<td>22\pm5</td>
<td>-</td>
<td>15 \pm 10</td>
<td>22 \pm 5</td>
</tr>
<tr>
<td>scalars</td>
<td>-6.8\pm2.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-7 \pm 7</td>
<td>-7 \pm 2</td>
</tr>
<tr>
<td>( \pi, K ) loops</td>
<td>-19\pm13</td>
<td>-4.5\pm8.1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-19 \pm 19</td>
<td>-19 \pm 13</td>
</tr>
<tr>
<td>( \pi, K ) loops + subl. ( N_C )</td>
<td>-</td>
<td>-</td>
<td>0\pm10</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>quark loops</td>
<td>21\pm3</td>
<td>9.7\pm11.1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2.3 (c-quark)</td>
<td>21\pm3</td>
</tr>
<tr>
<td>Total</td>
<td>83\pm32</td>
<td>89.6\pm15.4</td>
<td>80\pm40</td>
<td>136\pm25</td>
<td>110\pm40</td>
<td>105 \pm 26</td>
<td>116 \pm 39</td>
</tr>
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</table>

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, AN '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = AN '09, JN = Jegerlehner, AN '09

1) **Pseudoscalar contributions dominate numerically**: transition form factors as input

2) **Glasgow consensus**: \( a_{\mu}^{\text{HLbL}} = (105 \pm 26) \times 10^{-11} \)

3) Results are in good agreement but **errors are difficult to estimate** (model calculations)
**Outline**

- **Pion-pole contribution on the lattice**
  - Dominant contribution to the HLbL scattering in \((g - 2)_\mu\)
  - First-principle estimate

- **Hadronic Light-by-Light forward scattering amplitudes**
  - Full HLbL amplitudes contain more info than just \(a_\mu\)
  - Can be used to test the model
  - Extract information about single-meson transition form factor

- **Direct lattice QCD calculation**
  - Only one collaboration has published results so far [Blum et. al 14’, 16’]
  - Difficult calculation (4-pt correlation function)
The pion-pole contribution

In collaboration with Harvey Meyer and Andreas Nyffeler
The pion-pole contribution

\[ a^\text{HLbL;π} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \ w_1(Q_1, Q_2, \tau) \ F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1 + Q_2)^2) \ F_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) + \]
\[ w_2(Q_1, Q_2, \tau) \ F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \ F_{\pi^0\gamma^*\gamma^*}(-(Q_1 + Q_2)^2, 0) \]

→ Product of one single-virtual and one double-virtual transition form factors (spacelike virtualities)

→ \( w_{1,2}(Q_1, Q_2, \tau) \) are known model-independent weight functions

→ Weight functions are concentrated at small momenta below 1 GeV (here for \( \tau = -0.5 \))
The pion-pole contribution

Present status:

- Experimental results available for the single-virtual form factor
- And only for relatively large virtualities $Q^2 > 0.6$ GeV$^2$
- The theory imposes strong constraints for the normalisation and the asymptotic behavior of the TFF

\[
\mathcal{F}_{\pi^0\gamma^*\gamma^*}(0, 0) = 1/(4\pi^2 F_{\pi})
\]

\[
\mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2, 0) \sim 1/Q^2, \quad \mathcal{F}_{\pi^0\gamma^*\gamma^*}(Q^2, Q^2) \sim 1/Q^2
\]

- Most evaluations of the pion-pole contribution are therefore based on phenomenological models
- Systematic errors are difficult to estimate

Lattice QCD is particularly well suited to compute the form factor in the energy range relevant to $g - 2$
The pion-pole contribution

Introduction
Forward light-by-light scattering amplitudes
Direct lattice calculation
Conclusion

Lattice setup

- Lattice QCD is not a model: regularisation of the theory adapted to numerical simulations

- However there are systematic errors that we need to understand:
  1) We used $N_f = 2$ simulations (CLS ensembles)
     → we are currently analysing the $N_f = 2 + 1$ CLS ensembles with improved statistics
  2) Finite lattice spacing: discretisation errors
     → 3 lattice spacings ($a = 0.075, 0.065, 0.048$ fm): extrapolation to the continuum limit $a = 0$
  3) Unphysical quark masses
     → Different simulations with pion mass in the range [190-440] MeV: extrapolation to $m_\pi = m_{\pi}^{\text{exp}}$
  4) Finite volume
     → Discrete spatial momenta $\vec{q} = 2\pi / L\vec{n}$

  Pion at rest:
  
  \[
  \begin{align*}
  q_1 &= (\omega_1, \vec{q}_1) \\
  q_2 &= (m_\pi - \omega_1, \vec{q}_2)
  \end{align*}
  \]

  \[
  \begin{align*}
  q_1^2 &= \omega_1^2 - |\vec{q}_1|^2 \\
  q_2^2 &= (m_\pi - \omega_1)^2 - |\vec{q}_1|^2
  \end{align*}
  \]

  → we are currently adding a new frame with $\vec{p} \neq 0$
Results for one of the eight ensembles with $a = 0.048$ fm and $m_\pi = 270$ MeV

Extrapolation to the physical point

- Use phenomenological models to describe the lattice data
- e.g. VMD model, LMD model (Lowest Meson Dominance) [Moussallam ’94] [Knecht et al. ’99]

$$F^{\text{LMD}}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) = \frac{\alpha M_V^4 + \beta (q_1^2 + q_2^2)}{(M_V^2 - q_1^2)(M_V^2 - q_2^2)}$$

- Extrapolate the model parameters to the continuum and chiral limit: global fit
Results at the physical point

\[ Q^2 |F_{\pi^0\gamma^*\gamma^*}(-Q^2, -Q^2)| \]

\[ Q^2 |F_{\pi^0\gamma^*\gamma^*}(-Q^2, 0)| \]

\[ a_{\mu;LbL;\pi^0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \ w_1(Q_1, Q_2, \tau) \ F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -(Q_1 + Q_2)^2) \ F_{\pi^0\gamma^*\gamma^*}(-Q_2^2, 0) + w_2(Q_1, Q_2, \tau) \ F_{\pi^0\gamma^*\gamma^*}(-Q_1^2, -Q_2^2) \ F_{\pi^0\gamma^*\gamma^*}(-(Q_1 + Q_2)^2, 0) \]

\[ a_{\mu;LbL;\pi^0} = (65.0 \pm 8.3) \times 10^{-11} \]

\[ a_{\mu;LDB+V} = (62.6 \pm 3.0) \times 10^{-11} \]

→ most model calculations yield results in the range: \( a_{\mu;LbL;\pi^0} = (50 - 80) \times 10^{-11} \)

→ In the same ballpark as previous estimate: HLbL is unlikely to explain the 3-4 \( \sigma \) discrepancy

→ Recent result using the dispersive framework: \( a_{\mu;LbL;\pi^0} = (62.6 \pm 3.0) \times 10^{-11} \) [Kubis et al. '18]
The hadronic light-by-light scattering contribution to the muon $g - 2$ from lattice QCD

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$N_f = 2 + 1$ CLS ensembles

- Four lattice spacings
- One ensemble with physical pion mass
  (Correlators not yet computed)
- Several volumes

Full $O(a)$—improvement of the vector currents:

- Requires the improvement coefficient $c_V$ (for both the local and the conserved vector currents)
- Also $b_V$ and $\bar{b}_V$ for the local vector current

\[
J^{R,I}_\mu(x) = Z_V \left( 1 + 3\bar{b}_V a m + b_V a m_l \right) \left[ J_\mu(x) + a c_V \partial_\nu T_{\mu\nu}(x) \right]
\]

- New frame with $\vec{p} \neq 0$

  → We can probe much larger virtualities in the single-virtual case

Antoine Gérardin

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The hadronic light-by-light scattering contribution to the muon $g - 2$ from lattice QCD
In collaboration with J. Green, O. Gryniuk, G. von Hippel, H. Meyer, V. Pascalutsa and H. Wittig
Light-by-light forward scattering amplitudes

- Pion-pole contribution ✓. Other contributions: more difficult on the lattice (resonances)

- Forward scattering amplitudes $\mathcal{M}_{\lambda_3 \lambda_4 \lambda_1 \lambda_2}$

$$
\gamma^*(\lambda_1, q_1) \quad \gamma^*(\lambda_3, q_1) \\
\gamma^*(\lambda_2, q_2) \quad \gamma^*(\lambda_4, q_2)
$$

- 81 helicity amplitudes ($\lambda_i = 0, \pm 1$)

$$
\mathcal{M}_{\lambda'_1 \lambda'_2 \lambda_1 \lambda_2} = \mathcal{M}_{\mu \nu \rho \sigma} \ \epsilon^*(\lambda'_1) \ \epsilon^*(\lambda'_2) \ \epsilon^\rho(\lambda_1) \ \epsilon^\sigma(\lambda_2)
$$

- Photons virtualities: $Q_{1}^2 = -q_1^2 > 0$ and $Q_{2}^2 = -q_2^2 > 0$

- Crossing-symmetric variable: $\nu = q_1 \cdot q_2$

- Using parity and time invariance: only 8 independent amplitudes

$$(\mathcal{M}_{++;++} + \mathcal{M}_{+-;+-}) , \mathcal{M}_{++;--} , \mathcal{M}_{00;00} , \mathcal{M}_{+0;+0} , \mathcal{M}_{0+;0+} , (\mathcal{M}_{++;00} + \mathcal{M}_{0+;-0}) , \ (\mathcal{M}_{++;++} - \mathcal{M}_{+-;+-}) , (\mathcal{M}_{++;00} - \mathcal{M}_{0+;-0})$$

$\leftrightarrow$ Either even or odd with respect to $\nu$

$\leftrightarrow$ The eight amplitudes have been computed on the lattice for different values of $\nu, Q_1^2, Q_2^2$

Strategy:

1) Compute all the forward LbL scattering amplitudes on the lattice [Green et. al ’15]

2) Use a simple model to describe the lattice data (input: TFFs)

3) Extract information about TFFs by fitting the model parameters to lattice data
Dispersion relations

1) Optical theorem

\[ \gamma^*(\lambda_1, q_1) \rightarrow \text{Im} \, \gamma^*(\lambda_3, q_1) \]

\[ \gamma^*(\lambda_2, q_2) \rightarrow \text{Im} \, \gamma^*(\lambda_4, q_2) \]

\[ W_{\lambda_3\lambda_4,\lambda_1\lambda_2} = \text{Im} \, M_{\lambda_3\lambda_4,\lambda_1\lambda_2} = \frac{1}{2} \int d\Gamma (2\pi)^4 \delta (q_1 + p_X - q_2) \mathcal{M}_{\lambda_1\lambda_2}(q_1, q_2, p_X) \mathcal{M}^*_{\lambda_3\lambda_4}(q_1, q_2, p_X) \]

2) Dispersion relations [Pascalutsa et. al '12]

\[ \mathcal{M}_{\text{even}}(\nu) = \mathcal{M}_{\text{even}}(0) + \frac{2\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \frac{1}{\nu' (\nu'^2 - \nu^2 - i\epsilon)} W_{\text{even}}(\nu') \]

\[ \mathcal{M}_{\text{odd}}(\nu) = \nu \mathcal{M}_{\text{odd}}(\nu) + \frac{2\nu^3}{\pi} \int_{\nu_0}^{\infty} d\nu' \frac{1}{\nu' (\nu'^2 - \nu^2 - i\epsilon)} W_{\text{odd}}(\nu') \]

3) Higher mass singularities are suppressed with \( \nu^2 \):

\[ \xrightarrow{\nu^2} \text{Only a few states } X \text{ are necessary to saturate the sum rules and reproduce the lattice data} \]
Description of the lattice data using phenomenology

→ For each of the 8 amplitudes, we have a dispersion relation ($\sigma_\alpha/\tau_\alpha(\nu')$ : cross-section / interference term):

$$\overline{M}_\alpha(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^\infty d\nu' \frac{\sqrt{X'} \sigma_\alpha/\tau_\alpha(\nu')}{\nu'(\nu'^2 - \nu^2 - i\epsilon)}$$

Lattice calculation

$\gamma^*(\lambda_1, q_1) + \gamma^*(\lambda_2, q_2) \rightarrow X(p_X)$ fusion cross sections

$\gamma^*(\lambda_1, q_1)$

$\gamma^*(\lambda_2, q_2)$

$X(p)$

→ $X$ : any C-even states contribute

→ Main contribution is expected from mesons:
   - Pseudoscalars ($0^{-+}$)
   - Axial-vectors ($1^{++}$)
   - Scalar ($0^{++}$)
   - Tensors ($2^{++}$)

→ Described in terms of TFFs
The hadronic light-by-light scattering contribution to the muon $g - 2$ from lattice QCD

Contributions to the eight independent amplitudes

$$M(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \frac{\sqrt{X'}}{\nu'(\nu'^2 - \nu^2 - i\epsilon)}$$

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>Pseudoscalar</th>
<th>Scalar</th>
<th>Axial</th>
<th>Tensor</th>
<th>Scalar QED</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{TT}$</td>
<td>$\sigma_0/2$</td>
<td>$\sigma_0/2$</td>
<td>$\sigma_0/2$</td>
<td>$\frac{\sigma_0+\sigma_2}{2}$</td>
<td>$\sigma_{TT}$</td>
</tr>
<tr>
<td>$M_{TT}^t$</td>
<td>$-\sigma_0$</td>
<td>$\sigma_0$</td>
<td>$-\sigma_0$</td>
<td>$\sigma_0$</td>
<td>$\tau_{TT}$</td>
</tr>
<tr>
<td>$M_{TT}^a$</td>
<td>$\sigma_0/2$</td>
<td>$\sigma_0/2$</td>
<td>$\sigma_0/2$</td>
<td>$\frac{\sigma_0-\sigma_2}{2}$</td>
<td>$\tau_{TT}^a$</td>
</tr>
<tr>
<td>$M_{TL}$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\sigma_{TL}$</td>
<td>$\sigma_{TL}$</td>
<td>$\sigma_{TL}$</td>
</tr>
<tr>
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<td>$\times$</td>
<td>$\times$</td>
<td>$\sigma_{LT}$</td>
<td>$\sigma_{LT}$</td>
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</tr>
<tr>
<td>$M_{TL}^t$</td>
<td>$\times$</td>
<td>$\tau_{TL}$</td>
<td>$\tau_{TL}$</td>
<td>$\tau_{TL}$</td>
<td>$\tau_{TL}$</td>
</tr>
<tr>
<td>$M_{TL}^a$</td>
<td>$\times$</td>
<td>$\tau_{TL}$</td>
<td>$-\tau_{TL}$</td>
<td>$\tau_{TL}^a$</td>
<td>$\tau_{TL}^a$</td>
</tr>
<tr>
<td>$M_{LL}$</td>
<td>$\times$</td>
<td>$\sigma_{LL}$</td>
<td>$\times$</td>
<td>$\sigma_{LL}$</td>
<td>$\sigma_{LL}$</td>
</tr>
</tbody>
</table>

- $\sigma$: physical cross-section (positive contributions)
- $\tau$: interference terms
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Lattice calculation

- Four-point correlation function

\[ \Pi^{E}_{\mu \nu \rho \sigma}(Q_1, Q_2) = \sum_{X_1, X_2, X_3} \langle J^c_{\mu}(X_1) J^c_{\nu}(X_2) J^l_{\rho}(X_3) J^c_{\sigma}(0) \rangle e^{iQ_1(X_1-X_3)} e^{iQ_2X_2} + \text{contact terms} \]

\[ J_{\mu}(x) = \frac{2}{3} \bar{u}(x) \gamma_{\mu} u(x) - \frac{1}{3} \bar{d}(x) \gamma_{\mu} d(x) \quad (3 \text{ conserved} + 1 \text{ local vector currents}) \]

\[ \text{Method of sequential propagators} \quad \text{(two values of } Q_2^2, \text{ all } Q_2 \text{ such that } Q_2^2 < 4 \text{ GeV}^2) \]

\[ \text{The eight helicity amplitudes are obtained from} \]

\[ \mathcal{M}(q_1^2, q_2^2, \nu) = e^4 T^{E}_{\mu \mu' \nu' \nu'}(Q_1, Q_2) \Pi^{E}_{\mu \nu \rho \sigma}(Q_1, Q_2) \]

- Five CLS ensembles \((N_f = 2 \text{ dynamical quarks, } \mathcal{O}(a)\text{-improved Wilson-Clover})\)

\[ 4 \text{ ensembles at } a \approx 0.065 \text{ fm and } m_\pi \text{ down to 180 MeV} \]

\[ 1 \text{ ensemble at } a \approx 0.048 \text{ fm} \]

- Only the fully-connected diagrams are included for all the ensembles

- Leading 2+2 disconnected diagrams computed for two ensembles
Lattice results : connected contribution

- Results for a given ensemble at a given value of $Q_1^2 = 0.352 \text{ GeV}^2$
- Four of the eight lattice amplitudes
Connected and disconnected contributions: flavor structure

- There are five different topologies:

- Wick contractions lead to:

\[
\Pi^{\text{HLbL}} = \sum_f Q_f^4 \Pi^4 + \sum_{f_1,f_2} Q_{f_1}^2 Q_{f_2}^2 \Pi^{2+2} + \sum_{f_1,f_2} Q_{f_1}^3 Q_{f_2} \Pi^{3+1} + \sum_{f_1,f_2,f_3} Q_{f_1}^2 Q_{f_2} Q_{f_3} \Pi^{2+1+1} \\
+ \sum_{f_1,f_2,f_3,f_4} Q_{f_1} Q_{f_2} Q_{f_3} Q_{f_4} \Pi^{1+1+1+1}
\]
There are five different topologies:

\[
\Pi_{\text{HLbL}} = \sum_f Q_f^4 \Pi^4 + \sum_{f_1, f_2} Q_{f_1}^2 Q_{f_2}^2 \Pi^{2+2} + \sum_{f_1, f_2} Q_{f_1}^3 Q_{f_2} \Pi^{3+1} + \sum_{f_1, f_2, f_3} Q_{f_1}^2 Q_{f_2} Q_{f_3} \Pi^{2+1+1} + \sum_{f_1, f_2, f_3, f_4} Q_{f_1} Q_{f_2} Q_{f_3} Q_{f_4} \Pi^{1+1+1+1}
\]

The contribution to \( \Pi_{\text{HLbL}} \) of an isovector (isoscalar) resonance \( M_1 \) (\( M_0 \)) can be written as:

\[
\Pi_{\text{HLbL}}(M_1) = (Q_u^2 - Q_d^2)^2 \Pi_{M_1}
\]

\[
\Pi_{\text{HLbL}}(M_0) = (Q_u^2 + Q_d^2) \Pi_A + (Q_u + Q_d)^2 (Q_u^2 + Q_d^2) \Pi_B + (Q_u + Q_d)^4 \Pi_C
\]

\( \rightarrow \) consequence of the isospin decomposition of the electromagnetic current \( J^{\text{e.m.}}_\mu = J^1_\mu + J^0_\mu \):

\[
J^1_\mu = \frac{Q_u - Q_d}{2} (\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d), \quad J^0_\mu = \frac{Q_u + Q_d}{2} (\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d)
\]

\[
\mathcal{F}_{M_1 \gamma^+ \gamma^*} = (Q_u^2 - Q_d^2) \mathcal{F}, \quad \mathcal{F}_{M_0 \gamma^+ \gamma^*} = (Q_u^2 + Q_d^2) \mathcal{F}_C + (Q_u + Q_d)^2 \mathcal{F}_D
\]
Connected and disconnected contributions

- Identification of the polynomials in $Q_u$ and $Q_d$: leads to two sets of three equations

\[
\begin{align*}
\Pi_{M_1} &= \Pi_{M_1}^4 + \Pi_{M_1}^{2+2} + \Pi_{M_1}^{3+1} + \Pi_{M_1}^{2+1+1} + \Pi_{M_1}^{1+1+1+1} \\
-\Pi_{M_1} &= 0 + \Pi_{M_1}^{2+2} + 0 + \Pi_{M_1}^{2+1+1} + 3\Pi_{M_1}^{1+1+1+1} \\
0 &= 0 + 0 + \Pi_{M_1}^{3+1} + 2\Pi_{M_1}^{2+1+1} + 4\Pi_{M_1}^{1+1+1+1} \\
\Pi_A + \Pi_B + \Pi_C &= \Pi_{M_0}^4 + \Pi_{M_0}^{2+2} + \Pi_{M_0}^{3+1} + \Pi_{M_0}^{2+1+1} + \Pi_{M_0}^{1+1+1+1} \\
\Pi_A + \Pi_B + 3\Pi_C &= 0 + \Pi_{M_0}^{2+2} + 0 + \Pi_{M_0}^{2+1+1} + 3\Pi_{M_0}^{1+1+1+1} \\
3\Pi_B + 4\Pi_C &= 0 + 0 + \Pi_{M_0}^{3+1} + 2\Pi_{M_0}^{2+1+1} + 4\Pi_{M_0}^{1+1+1+1}
\end{align*}
\]

This was already noticed by [Bijnens '16] using large-$N_c$ arguments.
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\Pi_{M_1} &= \Pi_{M_1}^4 + \Pi_{M_1}^{2+2} + \Pi_{M_1}^{3+1} + \Pi_{M_1}^{2+1+1} + \Pi_{M_1}^{1+1+1+1} \\
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\end{align*}
\]

\[
\begin{align*}
\Pi_A + \Pi_B + \Pi_C &= \Pi_{M_0}^4 + \Pi_{M_0}^{2+2} + \Pi_{M_0}^{3+1} + \Pi_{M_0}^{2+1+1} + \Pi_{M_0}^{1+1+1+1} \\
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\end{align*}
\]

- Assume that all disconnected contributions with at least one isolated quark loop are negligible (then $\Pi_A \approx \Pi_{M_0}$)

\[
\begin{align*}
\Pi_{M_1} + \Pi_{M_0} &\approx (\Pi_{M_1}^4 + \Pi_{M_1}^4) + (\Pi_{M_1}^{2+2} + \Pi_{M_0}^{2+2}) \\
-\Pi_{M_1} + \Pi_{M_0} &\approx (\Pi_{M_1}^{2+2} + \Pi_{M_0}^{2+2})
\end{align*}
\]
Connected and disconnected contributions

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$$
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\end{align*}
$$

$$
\begin{align*}
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\begin{align*}
\Pi_{M_1} + \Pi_{M_0} &\approx (\Pi_{M_1}^4 + \Pi_{M_1}^{3+1}) + (\Pi_{M_1}^{2+2} + \Pi_{M_0}^{2+2}) \\
-\Pi_{M_1} + \Pi_{M_0} &\approx (\Pi_{M_1}^{2+2} + \Pi_{M_0}^{2+2})
\end{align*}
$$

• Then the contribution to the fully connected and $2+2$ disconnected contributions read

$$
(Q_u^4 + Q_d^4)\Pi_{M_1+M_0}^4 \approx 2 \left( Q_u^4 + Q_d^4 \right) \Pi_{M_1} \approx 2 \frac{Q_u^4 + Q_d^4}{(Q_u^2 - Q_d^2)^2} \Pi^{\text{HLbL}}(M_1) \approx \frac{34}{9} \Pi^{\text{HLbL}}(M_1)
$$

$$
(Q_u^2 + Q_d^2)^2 \Pi_{M_1+M_0}^{2+2} \approx -\frac{(Q_u^2 + Q_d^2)^2}{(Q_u^2 - Q_d^2)^2} \Pi^{\text{HLbL}}(M_1) + \Pi^{\text{HLbL}}(M_0) \approx -\frac{25}{9} \Pi^{\text{HLbL}}(M_1) + \Pi^{\text{HLbL}}(M_0)
$$

↩️ This was already noticed by [Bijnens '16] using large-$N_c$ arguments
Phenomenological model to describe lattice data

\[ \gamma^*(\lambda_1, q_1) \rightarrow X(p) \rightarrow \gamma^*(\lambda_2, q_2) \]
• We have computed only the fully connected contribution:
  \[ \gamma^*(\lambda_1, q_1) \rightarrow X(p) \rightarrow \gamma^*(\lambda_2, q_2) \]

\[ \gamma^*(\lambda_1, q_1) \rightarrow X(p) \rightarrow \gamma^*(\lambda_2, q_2) \]

  \rightarrow \text{we consider only isovector resonances but with a factor } \frac{34}{9} \]

• Higher mass resonances contributions are suppressed in the dispersion relations
  \[ \rightarrow \text{consider only the lightest state in each channel} \]

• We work with \( N_f = 2 \) lattice simulations: no \( \eta \) !

\[
\begin{array}{cccc}
\text{State} & \text{Isovector} & \text{Isoscalar} & \text{Isoscalar} \\
0^{++} & \pi & \eta' & \eta \\
0^{++} & a_0(980) & f_0(980) & f_0(600) \\
1^{++} & a_1(1260) & f_1(1285) & f_1(1420) \\
2^{++} & a_2(1320) & f_2(1270) & f_2'(1525) \\
\end{array}
\]
Two-photon fusion cross sections : modelisation

\[ \overline{M}(\nu) = \frac{4\nu^2}{\pi} \int_{\nu_0}^{\infty} d\nu' \frac{\sqrt{X' \sigma/\tau(\nu')}}{\nu'(\nu'^2 - \nu^2 - i\epsilon)} \]

- **Example :** contribution of the pseudoscalar to the amplitude \( \overline{M}_{TT}(\nu) \)

\[ \sigma_{TT} = 8\pi^2 \delta(s - m_P^2) \frac{2\sqrt{X}}{m_P^2} \times \frac{\Gamma_{\gamma\gamma}}{m_P} \times \left[ \frac{F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2)}{F_{P\gamma^*\gamma^*}(0, 0)} \right]^2 \]

\( \leftrightarrow \) Similar results for other mesons (we assume Breit-Wigner shape for resonances)

- We assume a constant mass shift in the spectrum (scalar, axial, tensor)

\[ m_X = m_X^{\text{phys}} + (m_\rho^{\text{lat}} - m_\rho^{\text{phys}}) \ , \ X = A, S, T \]

- The two-photons decay width \( \Gamma_{\gamma\gamma} = \frac{\pi\alpha^2}{4} m_S \left[ F_{S\gamma^*\gamma^*}^T(0, 0) \right]^2 \) is taken from experiment

All the non-perturbative information is encoded into the meson transition form factors
Assumptions on form factors

- **Pseudoscalar meson**
  
  $\xrightarrow{\text{We use our lattice results for}} F_{P\gamma^*\gamma^*}(Q_1^2, Q_2^2) \quad [\text{Gerardin, Nyffeler, Meyer '16}]$

  $\xrightarrow{\text{No free parameter}}$

- **Scalar mesons**
  
  $\xrightarrow{\text{can be produced by two transverse photons (T) or two longitudinal photons (L)}} F_{S\gamma^*\gamma^*}^T \text{ and } F_{S\gamma^*\gamma^*}^L$

  
  $$
  \frac{F_{S\gamma^*\gamma^*}^T(Q_1^2, Q_2^2)}{F_{S\gamma^*\gamma^*}^T(0, 0)} = \frac{1}{(1 + Q_1^2/M_S^2)(1 + Q_2^2/M_S^2)}
  $$

  $\xrightarrow{\text{Belle Collaboration (} Q^2 < 30 \text{ GeV}^2 \text{): }} M_S = 0.800(50) \text{ MeV for the } f_0(980) \text{ meson} \quad [\text{Masuda '15}]$

  $\xrightarrow{\text{In the following, } M_S \text{ is considered as a free parameter}}$
Assumptions on form factors

- Axial mesons

  - Two form factors $F_{A\gamma^*\gamma^*}^{(0)}$ and $F_{A\gamma^*\gamma^*}^{(1)}$ ($\Lambda = 0,1$ corresponds to the two helicity states of the axial meson)

  - Quark model inspired parametrisation [N. Cahn '87]

  \[
  F_{A\gamma^*\gamma^*}^{(0)}(Q_1^2, Q_2^2) = m_A^2 A(Q_1^2, Q_2^2),
  \]
  \[
  F_{A\gamma^*\gamma^*}^{(1)}(Q_1^2, Q_2^2) = -\frac{\nu}{X} \left( \nu + Q_2^2 \right) m_A^2 A(Q_1^2, Q_2^2),
  \]
  \[
  F_{A\gamma^*\gamma^*}^{(1)}(Q_2^2, Q_1^2) = -\frac{\nu}{X} \left( \nu + Q_1^2 \right) m_A^2 A(Q_1^2, Q_2^2)
  \]

  in which $2\nu = m_A^2 + Q_1^2 + Q_2^2$ with $m_A$ the meson mass,

  \[
  \frac{A(Q_1^2, 0)}{A(0, 0)} = \frac{1}{(1 + Q_1^2/M_A^2)^2},
  \]

  - Single virtual case: L3 Collaboration in the region $Q^2 < 5$ GeV$^2$ [Achard '01 '07]

  - $M_A = 1040(78)$ MeV for the isoscalar $f_1(1285)$ meson

  - One free fit parameter: $M_A$
Assumptions on form factors

- **Tensor mesons**
  - Amplitudes are described by four form factors with \( \Lambda = (0, T), (0, L), 1, 2 \)
  - Belle Collaboration: single-virtual form factors for helicities \( \Lambda = (0, T), 1, 2 \) (isoscalar) [Masuda ’15]
  - Data are compatible with a dipole form factor [Danilkin ’16]

\[
\frac{F_{T \gamma^* \gamma^*}^{(\Lambda)}(Q_1^2, Q_2^2)}{F_{T \gamma^* \gamma^*}^{(\Lambda)}(0, 0)} = \frac{1}{(1 + Q_1^2/M_{T, (\Lambda)}^2)^2(1 + Q_2^2/M_{T, (\Lambda)}^2)^2}
\]

- Four free fit parameters: \( M_{T, (\Lambda)} \)

- **Scalar QED**
  - \( \gamma^* \gamma^* \rightarrow \pi^+ \pi^- \) evaluated using scalar QED dressed with monopole form factors

- Monopole mass set to the (lattice) rho mass

- **Conclusion**
  - The model has 6 independent parameters (monopole and dipole masses of TFFs)
  - Global fit of the model to the eight lattice amplitudes (for all \( Q_1^2 \) and \( Q_2^2 \))
Introduction

The pion-pole contribution

Forward light-by-light scattering amplitudes

Direct lattice calculation

Conclusion

Results

The hadronic light-by-light scattering contribution to the muon $g - 2$ from lattice QCD
Results on F7 - dependence on $\nu$ and $Q_2^2$

- Each plot corresponds to a fixed $Q_1^2$
- Different colours correspond to different values of $\nu = Q_1^2 \cdot Q_2^2$

\[ Q_1^2 = 0.352 \text{ GeV}^2 \]

\[ \nu [\text{GeV}^2] \]

\[ Q_2^2 [\text{GeV}^2] \]

\[ M_{TT} \]

\[ M_{LT} \]

Antoine Gérardin
Preliminary results: F7 - contributions from different channels

$Q_1^2 = 0.352 \text{ GeV}^2 \quad Q_2^2 = 1.000 \text{ GeV}^2$

$Q_1^2 = 0.352 \text{ GeV}^2 \quad Q_2^2 = 1.000 \text{ GeV}^2$

$Q_1^2 = 0.352 \text{ GeV}^2 \quad Q_2^2 = 1.000 \text{ GeV}^2$

$Q_1^2 = 0.352 \text{ GeV}^2 \quad Q_2^2 = 1.000 \text{ GeV}^2$
Monopole and dipole masses: chiral extrapolations (preliminary, stat error only)

\[
M_s [\text{GeV}] = 1.00 \pm 0.05 \\
M_A [\text{GeV}] = 1.50 \pm 0.05 \\
M_{(2)}^T [\text{GeV}] = 2.00 \pm 0.05 \\
M_{(0,L)}^T [\text{GeV}] = 1.50 \pm 0.05 \\
M_{(1)}^T [\text{GeV}] = 1.50 \pm 0.05 \\
M_{(0,T)}^T [\text{GeV}] = 1.50 \pm 0.05
\]
Introduction

The pion-pole contribution

Forward light-by-light scattering amplitudes

Direct lattice calculation

Conclusion

Preliminary results: monopole and dipole masses

\begin{align*}
F_X(Q_1^2, Q_2^2) &= \frac{F_X(0,0)}{(1 + Q_1^2/\Lambda_X^2)(1 + Q_2^2/\Lambda_X^2)} \\
F_X(Q_1^2, Q_2^2) &= \frac{F_X(0,0)}{(1 + Q_1^2/\Lambda_X^2)^2(1 + Q_2^2/\Lambda_X^2)^2}
\end{align*}

- Global fit of the eight amplitudes

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<td>1.72(18)</td>
<td>0.51(8)</td>
<td>1.35</td>
</tr>
</tbody>
</table>

- $M_S = 1.04(14)$ GeV: slightly above the experimental result from the Belle Collaboration ($M_S = 796(54)$ MeV for the isoscalar scalar meson [Masuda '15])

- $M_A = 1.32(7)$ GeV to be compared with the experimental value by the L3 Collaboration $M_A = 1040(80)$ MeV for the isoscalar meson $f_1(1285)$ [Achard '01 '07].

- $M_T^{(2)} = 1.35(24)$ GeV, $M_{0,T}^{(1)} = 1.69(16)$ GeV and $M_{(0,T)}^{(1)} \approx 1.96(9)$ GeV, above the experimental values for the $f_2(1270)$ mesons obtained by fitting the single-virtual form factor [Masuda '15, Danilkin '16].
Direct lattice calculation of the hadronic light-by-light scattering contribution
Exact QED kernel in infinite volume

- For the HVP contribution: time momentum representation (TMR) [Bernecker, Meyer '12]

$$ a_{\mu}^{\text{HVP}} = \left( \frac{\alpha}{\pi} \right)^2 \int dx_0 \ K(x_0) \ G(x_0) \ , \ \ G(x_0) = -\frac{1}{3} \sum_{k=1}^{3} \sum_{i} \langle V_k(x)V_k(0) \rangle $$
For the HVP contribution: time momentum representation (TMR) [Bernecker, Meyer '12]

\[ a_{\mu}^{\text{HVP}} = \left( \frac{\alpha}{\pi} \right)^2 \int d\tau \ K(\tau) \ G(\tau), \quad G(\tau) = -\frac{1}{3} \sum_{k=1}^{3} \sum_{\vec{x}} \langle V_k(\vec{x})V_k(0) \rangle \]

Compute the QED part perturbatively in the continuum and in infinite volume (position space) [J. Green et al. '16] [N. Asmussen et al. '16 '17]

\[ a_{\mu}^{\text{HLbL}} = \frac{me^6}{3} \int \frac{d^4 y}{d^4 x} \int d^4 x \ L_{[\rho,\sigma];\mu\nu\lambda}(x, y) \ i\hat{\Pi}_{\rho,\mu\nu\lambda}(x, y) \]

\[ i\hat{\Pi}_{\rho,\mu\nu\lambda}(x, y) = -\int d^4 z \ z_{\rho} \langle J_\mu(x)J_\nu(y)J_\sigma(z)J_\lambda(0) \rangle \]

\[ \rightarrow \hat{\Pi}_{\rho,\mu\nu\lambda}(x, y) \] is the four-point correlation function computed on the lattice

\[ \rightarrow L_{[\rho,\sigma];\mu\nu\lambda}(x, y) \] is the QED kernel, computed semi-analytically (infra-red finite)

\[ \rightarrow \text{Avoid } 1/L^2 \text{ finite-volume effects from the massless photons} \]
For the HVP contribution: time momentum representation (TMR) \cite{Bernecker, Meyer '12}

\[ a_{HVP}^{\mu} = \left( \frac{\alpha}{\pi} \right)^2 \int dx_0 \ K(x_0) \ G(x_0) , \quad G(x_0) = -\frac{1}{3} \sum_{k=1}^{3} \sum_{x} \langle V_k(x)V_k(0) \rangle \]

Compute the QED part perturbatively in the continuum and in infinite volume (position space) \cite{J. Green et al. '16} [N. Asmussen et al. '16 '17]

\[ a_{HLbL}^{\mu} = \frac{m e^6}{3} \int d^4y \int d^4x \ L_{[\rho,\sigma];\mu\nu\lambda}(x, y) \ i\hat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x, y) \]

\[ i\hat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x, y) = -\int d^4z z_{\rho} \langle J_\mu(x)J_\nu(y)J_\sigma(z)J_\lambda(0) \rangle \]

\( \hat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x, y) \) is the four-point correlation function computed on the lattice

\( L_{[\rho,\sigma];\mu\nu\lambda}(x, y) \) is the QED kernel, computed semi-analytically (infra-red finite)

Avoid \( 1/L^2 \) finite-volume effects from the massless photons

On the lattice:

\( \rightarrow \) integration over \( x \) and \( z \) are performed explicitly on the lattice

\( \rightarrow \) the remaining part depends only on \( |y| \)

\( \rightarrow \) one-dimensional integral, can be sampled using different values of \( |y| \)
QED kernel : subtractions

\[ a_\mu^{\text{HLbL}} = \frac{m e^6}{3} \int \! d^4 y \int \! d^4 x \; \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x, y) \; i\hat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x, y) \]

- Conservation of the vector current : \( \partial_\mu J_\mu(x) = 0 \implies \text{The QED kernel is not unique} \quad \text{[RBC/UKQCD '17]}

\[ 0 = \sum_x \partial_\mu^{(x)} \left( x_\alpha \hat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x, y) \right) = \sum_x \hat{\Pi}_{\rho,\alpha\nu\lambda\sigma}(x, y) + \sum_x x_\alpha \partial_\mu^{(x)} \hat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x, y) \]

\( \rightarrow \) we can add any fonction \( f(y) \) to the standard QED kernel

\( \rightarrow \) differ by volume effects (and discretisation effects for the local vector current)

\( \rightarrow \) same argument valid for the other variable \( x \)
\begin{equation*}
a_{\mu}^{\text{HLbL}} = \frac{me^6}{3} \int d^4y \int d^4x \ L_{[\rho,\sigma];\mu\nu\lambda}(x, y) \ i\hat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x, y)
\end{equation*}

- Conservation of the vector current: $\partial_{\mu}J_\mu(x) = 0 \Rightarrow$ The QED kernel is not unique \cite{RBC/UKQCD '17}

\begin{equation*}
0 = \sum_x \partial_{\mu}^{(x)} \left(x_\alpha\hat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x, y)\right) = \sum_x \hat{\Pi}_{\rho,\alpha\nu\lambda\sigma}(x, y) + \sum_x x_\alpha \partial_{\mu}^{(x)}\hat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x, y)
\end{equation*}

$\Rightarrow$ we can add any function $f(y)$ to the standard QED kernel

$\Rightarrow$ differ by volume effects (and discretisation effects for the local vector current)

$\Rightarrow$ same argument valid for the other variable $x$

- Examples of possible subtractions (idea: subtract very short distance contributions)

\begin{align*}
L^{(0)}(x, y) &= L(x, y) \quad \Rightarrow L^{(1)}(0, 0) = 0 \\
L^{(1)}(x, y) &= L(x, y) - \frac{1}{2}L(x, x) - \frac{1}{2}L(y, y) \quad \Rightarrow L^{(1)}(x, x) = 0 \\
L^{(2)}(x, y) &= L(x, y) - L(0, y) - L(x, 0) \quad \Rightarrow L^{(2)}(x, 0) = L^{(2)}(0, y) = 0 \\
L^{(3)}(x, y) &= L(x, y) - L(0, y) - L(x, x) + L(0, x) \quad \Rightarrow L^{(3)}(0, y) = L^{(3)}(x, x) = 0
\end{align*}

- Different definitions may affect:

$\rightarrow$ Discretization effects / Finite-size effects / Statistical precision of the estimator
Lepton-loop on the lattice

\[ a_\mu^{\text{HLbL}} = \frac{m e^6}{3} \int d^4y \int d^4x \ \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x, y) \ i\hat{\Pi}_{\rho,\mu\nu\lambda}(x, y) \]

- We now compute \( \hat{\Pi}(x, y) \) on the lattice (unit gauge field, lattice propagators)

Lepton-loop contribution to \( a_\mu^{\text{LbL}} (m_l = 2m_\mu) \)

\[ a_\mu^{\text{LbL,ll}} = 0.1659 \]
\[ a_\mu^{\text{LbL,cc}} = 0.1596 \]

→ Use different lattice spacings / volumes
→ blue and black points correspond to two different discretizations of the vector current
→ standard kernel \( \mathcal{L}^{(0)}(x, y) \) : large discretization effects!
Lepton-loop on the lattice

\[ a_\mu^{\text{HLbL}} = \frac{m_e^6}{3} \int d^4y \int d^4x \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x, y) i\hat{\Pi}_{\rho,\mu\nu\lambda\sigma}(x, y) \]

- We now compute \( \hat{\Pi}(x, y) \) on the lattice (unit gauge field, lattice propagators)

\[ \mathcal{L}^{(2)}(x, y) = \mathcal{L}(x, y) - \mathcal{L}(x, 0) - \mathcal{L}(0, y) \]

\[ a_\mu^{\text{LbL},\text{ll}} = 0.1503 \]
\[ a_\mu^{\text{LbL},\text{cc}} = 0.1498 \]

→ \( \mathcal{L}^{(2)}(x, y) \) has much smaller discretization effects
→ we can reproduce the known result \( (a_\mu^{\text{LbL}} = 0.15031 \times 10^{-8}) \) for the lepton-loop with a very good precision
✓ check of the QCD code
The lattice QCD calculation

- Fully connected contribution

- Leading 2+2 disconnected contribution

- Sub-dominant disconnected contributions (3+1, 2+1+1, 1+1+1+1)

- 2+2 disconnected diagrams are not negligible!
  → Large-$N_c$ prediction: $2+2 \text{ disc} \approx -50\% \times \text{connected}$ [Bijnens '16], [A. G et al. '17]
  → Disconnected contributions: only $O(1\%)$ for the HVP!

- Other diagrams vanish in the SU(3) limit (at least one quark loop which couple to a single photon)
  → Smaller contributions, but might be relevant for $O(10\%)$ precision
There is a persistent $3 - 4 \sigma$ discrepancy between theory and experiment for the $(g - 2)_\mu$

Two new experiments (Fermilab and J-PARC) should reduce the experimental error by a factor 4

The error is dominated by hadronic uncertainties

Pion-pole contribution

→ First lattice QCD determination of $a_{\mu}^{\text{HLbL};\pi^0} = (65.0 \pm 8.3) \times 10^{-11}$

→ The (dominant) $\pi^0$-pole contribution can be computed precisely

→ In progress: new calculation with $N_f = 2 + 1$, more statistics, full $O(a)$-improvement ...

→ This calculation is important to correct FSE on the full lattice calculation

Beyond the pion-pole contribution

→ The forward LbL scattering amplitudes provide more information than $a_{\mu}^{\text{HLbL}}$ (single scalar)

→ The lattice data can be described by a simple phenomenological model

→ It would be interesting to consider the more realistic case with $N_f = 2 + 1$ dynamical quarks

Hadronic light-by-light scattering contribution

→ We are now starting the full QCD calculation. Goal: 20% accuracy in the near future
Conclusion

▶ There is a persistent $3 - 4 \sigma$ discrepancy between theory and experiment for the $(g - 2)_\mu$

▶ Two new experiments (Fermilab and J-PARC) should reduce the experimental error by a factor 4

▶ The error is dominated by hadronic uncertainties

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Thank you!
The hadronic light-by-light scattering contribution to the muon $g - 2$ from lattice QCD
Lattice calculation of the pion TFF

\[ \epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) = - \int d\tau e^{\omega_1 \tau} \int d^3z e^{-i\vec{q}_1 \cdot \vec{z}} \langle 0 | T \{ J_\mu(\vec{z}, \tau) J_\nu(\vec{0}, 0) \} | \pi^0(p) \rangle \]

- We consider the following 3-pt correlation function

\[ C_{\mu\nu}^{(3)}(\tau, t_\pi; \vec{q}_1) = \sum_{\vec{x}, \vec{z}} \langle T \{ J_\nu(\vec{0}, t_f) J_\mu(\vec{z}, t_i) P(\vec{x}, t_0) \} \rangle e^{-i\vec{q}_1 \cdot \vec{z}} \]

\[ \mathcal{F}_{\pi^0\gamma\gamma}(q_1^2, q_2^2) \propto \frac{2E_\pi}{Z_\pi} \int_{-\infty}^{\infty} d\tau \tilde{A}_{\mu\nu}(\tau) e^{\omega_1 \tau} \]

\[ A_{\mu\nu}(\tau) = \lim_{t_\pi \to \infty} C_{\mu\nu}(\tau, t_\pi) e^{E_\pi t_\pi} \]

\[ \tilde{A}_{\mu\nu}(\tau) = \begin{cases} A_{\mu\nu}(\tau) & \tau > 0 \\ A_{\mu\nu}(\tau) e^{-E_\pi \tau} & \tau < 0 \end{cases} \]

- Finite time-extent of the lattice:
  - Fit the 3-pt correlation function at large \( \tau \) (e.g. assuming a VMD)
  - Use the model to integrate up to \( \tau \to \infty \)

- There are also (quark) disconnected contributions
  - \( \mathcal{O}(0.5\%) \) on E5 with \( m_\pi = 340 \text{ MeV} \)

Antoine Gérardin

The hadronic light-by-light scattering contribution to the muon \( g - 2 \) from lattice QCD
The QED kernel $\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x, y)$ can be decomposed into several tensors

$$\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x, y) = \sum_{A=I,II,III} G_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda}^A T_{\alpha\beta\delta}^{(A)}(x, y)$$

- $G_{\delta[\rho\sigma]\mu\alpha\nu\beta\lambda}^A = \text{traces of gamma matrices} \rightarrow \text{sums of products of Kronecker deltas}$

- The tensors $T_{\alpha\beta\delta}^{(A)}$ are decomposed into a scalar $S$, vector $V$ and tensor $T$ part

\begin{align*}
T_{\alpha\beta\delta}^{(I)}(x, y) &= \partial^{(x)}_{\alpha}(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta})V_{\delta}(x, y) \\
T_{\alpha\beta\delta}^{(II)}(x, y) &= m\partial^{(x)}_{\alpha} \left( T_{\beta\delta}(x, y) + \frac{1}{4} \delta_{\beta\delta} S(x, y) \right) \\
T_{\alpha\beta\delta}^{(III)}(x, y) &= m(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta}) \left( T_{\alpha\delta}(x, y) + \frac{1}{4} \delta_{\alpha\delta} S(x, y) \right)
\end{align*}

They are parametrized by six weight functions

\begin{align*}
S(x, y) &= 0 \\
V_{\delta}(x, y) &= x_{\delta} \tilde{g}^{(1)} + y_{\delta} \tilde{g}^{(2)} \\
T_{\alpha\beta}(x, y) &= (x_{\alpha} x_{\beta} - \frac{x^2}{4} \delta_{\alpha\beta}) \tilde{\tau}^{(1)} + (y_{\alpha} y_{\beta} - \frac{y^2}{4} \delta_{\alpha\beta}) \tilde{\tau}^{(2)} + (x_{\alpha} y_{\beta} + y_{\alpha} x_{\beta} - \frac{x \cdot y}{2} \delta_{\alpha\beta}) \tilde{\tau}^{(3)}
\end{align*}

- the weight functions depend on the three variables $x^2$, $x \cdot y = |x||y| \cos \beta$ and $y^2$
- Semi-analytical expressions for the weight functions have been computed to about 5 digits precision