Functional Renormalization Group and some Applications

A. Wipf

Theoretisch-Physikalisches Institut, FSU Jena

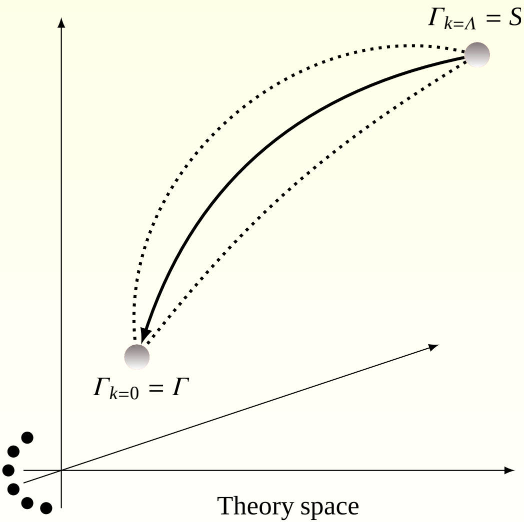
Humboldt Universität Berlin - NIC,DESY Zeuthen
Joint Lattice Seminar

Research Training Group
Quantum and Gravitational Fields

seit 1558

- particular implementation of the renormalization group
- for continuum field theory (symmetries ☺)
- functional methods + renormalization group idea
- conceptionally simple, technically demanding
- scale-dependent Schwinger functional or effective action
- large values of scale parameter k: high resolution $\to$ UV
- lowering k: including more and more fluctuations $\to$ IR
- known microscopic laws $\longrightarrow$ macroscopic phenomena

- flow of Schwinger functional $W_k[j]$: Polchinski equation
- flow of effective action $\Gamma_k[\varphi]$: Wetterich equation
- flow from classical action $S[\varphi]$ to effective action $\Gamma[\varphi]$
- applied to variety of physical systems
  - strong interaction
  - electroweak phase transition
  - asymptotic safety scenario
  - condensed matter systen
    e.g. Hubbard model, liquid He^4, frustrated magnets, superconductivity ...
  - effective models in nuclear physics
  - ultra-cold atoms


$\Gamma_{k=\Lambda} = S$
$\Gamma_{k=0} = \Gamma$
Theory space

1. K. Aoki, *Introduction to the nonperturbative renormalization group and its recent applications*,
Int. J. Mod. Phys. **B14** (2000) 1249.
2. J. Berges, N. Tetradis and C. Wetterich, *Nonperturbative renormalization flow in quantum field theory and statistical physics*,
Phys. Rept. **363** (2002) 223.
3. H. Gies, *Introduction to the functional RG and applications to gauge theories*,
in Lect. Notes Phys. Volume 852, Springer, Berlin (2012)
4. P. Kopietz, L. Bartosch and F. Schütz, *Introduction to the functional renormalization group*,
Lect. Notes in Phys., Volume 798, Springer, Berlin (2010).
5. A. Wipf, *Statistical Approach to Quantum Field Theory*,
Lect. Notes Phys., Volume 864, Springer, Berlin (2013).

- generating functional of (Euclidean) correlation functions

$$Z[j] = \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]+(j,\phi)}, \quad (j,\phi) = \int \mathrm{d}^d x \, j(x)\phi(x)$$

- Schwinger functional $W[j] = \log Z[j]$
- effective action = Legendre transform of $W[j]$

$$\Gamma[\varphi] = (j,\varphi) - W[j] \quad \text{with} \quad j(x) \quad \text{from} \quad \varphi(x) = \frac{\delta W[j]}{\delta j(x)}$$

- encodes properties of QFT in most economic way

- add scale-dependent IR-cutoff term ΔS_k to S

$$Z_k[j] = \int \mathcal{D}\phi \, \mathrm{e}^{-S[\phi]+(j,\phi)-\Delta S_k[\phi]}$$

- Scale-dependent Schwinger functional

$$W_k[j] = \log Z_k[j]$$

- regulator: quadratic functional

$$\Delta S_k[\phi] = \frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y \, \phi(x) R_k(x-y)\phi(y)$$

$\rightarrow$ one-loop structure of flow equation

- cutoff-function $\sim$ momentum-dependent mass

$$\Delta S_k[\phi] = \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\phi^*(p)R_k(p)\phi(p)$$

- recover effective action for $k \to 0$:

$$R_k(p) \stackrel{k\to 0}{\longrightarrow} 0 \quad \text{for fixed} \quad p$$

- recover classical action at UV-scale Λ:

$$R_k \stackrel{k\to\Lambda}{\longrightarrow} \infty$$

- regularization in the IR:

$$R_k(p) > 0 \quad \text{for} \quad p \to 0$$

exponential regulator: $R_k(p) = \frac{p^2}{e^{p^2/k^2} - 1}$,

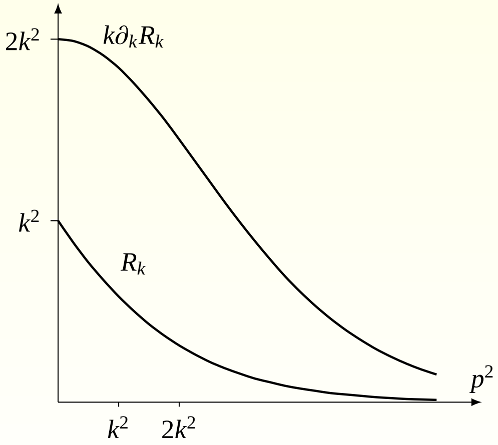
optimized regulator: $R_k(p) = (k^2 - p^2)\,\theta\left(k^2 - p^2\right)$,

quartic regulator: $R_k(p) = k^4/p^2$,

sharp regulator: $R_k(p) = \frac{p^2}{\theta\,(k^2 - p^2)} - p^2$,

Callan-Symanzik regulator: $R_k(p) = k^2$

exponential cutoff function and its derivative



- scale dependence of W_k from

$$\partial_k W_k[j] = -\frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y \; \langle \phi(x)\partial_k R_k(x,y)\phi(y)\rangle_k$$

- relates to connected two-point function

$$W_k^{(2)}(x,y) \equiv \frac{\delta^2 W_k[j]}{\delta j(x)\delta j(y)} = \langle \phi(x)\phi(y)\rangle_k - \varphi(x)\varphi(y)$$

Polchinski equation

$$\partial_k W_k[j] = -\frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y \, \partial_k R_k(x,y)\, W_k^{(2)}(x,y) - \int \mathrm{d}^d x \mathrm{d}^d y \, W_k^{(1)} \partial_k R_k(x,y) W_k^{(1)}(y)$$

- average field of cutoff theory with j

$$\varphi(x)=\frac{\delta W_k[j]}{\delta j(x)}\equiv W_k^{(1)}(x)$$

- fixed $j\to\varphi$ depends on k
- fixed $\varphi\to j$ depends on k
- modified Legendre transformation:

$$\Gamma_k[\varphi]=\big(\mathcal{L}W_k\big)[\varphi]-\Delta S_k[\varphi]$$

- Legendre transform $\big(\mathcal{L}W_k\big)(\varphi)=(j,\varphi)-W_k[j]$
- Γ_k not $\mathcal{L}$-transform of $W_k[j]$ for $k>0$!
- need not be convex, but $\Gamma_{k\to 0}=\Gamma$ convex

- vary effective average action (φ fixed)

$$\frac{\delta\Gamma_k[\varphi]}{\delta\varphi(x)} = \int \frac{\delta j(y)}{\delta\varphi(x)}\varphi(y) + j(x) - \int \frac{\delta W_k[j]}{\delta j(y)}\frac{\delta j(y)}{\delta\varphi(x)} - \frac{\delta\Delta S_k[\varphi]}{\delta\varphi(x)}$$

- red terms chancel $\to$ effective equation of motion

$$\frac{\delta\Gamma_k[\varphi]}{\delta\varphi(x)} = j(x) - \frac{\delta}{\delta\varphi(x)}\Delta S_k[\varphi] = j(x) - (R_k\varphi)(x)$$

- flow equation: φ fixed, j depends on scale, differentiate Γ_k

$$\partial_k\Gamma_k[\varphi] = \int \mathrm{d}^d x\,\partial_k j(x)\varphi(x) - \partial_k W_k[j] - \int \frac{\partial W_k[j]}{\partial j(x)}\partial_k j(x) - \partial_k\Delta S_k[\varphi]$$

$$\overset{PE}{=} \frac{1}{2}\int \mathrm{d}^d x\mathrm{d}^d y\,\partial_k R_k(x,y)\,W_k^{(2)}(x,y)$$

- $\partial_k W_k[j]$ only from k-dependence of the parameters

$$\begin{aligned}\partial_k \Gamma_k[\varphi] &= -\partial_k W_k[j] - \partial_k \Delta S_k[\varphi] \\ &= -\partial_k W_k[j] - \frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y\, \varphi(x) \partial_k R_k(x,y) \varphi(y)\end{aligned}$$

- use Polchinski equation $\rightarrow$

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y\, \partial_k R_k(x,y)\, W_k^{(2)}(x,y)$$

second derivative of W_k vs. second derivative of Γ_k:

$$\varphi(x) = W_k^{(1)}(x) \quad \text{and} \quad j(x) = \Gamma_k^{(1)}(x) + \int \mathrm{d}^d y\, R_k(x,y) \varphi(y)$$

- chain rule $\rightarrow$

$$\delta(x-y) = \int \mathrm{d}^d z \, \frac{\delta\varphi(x)}{\delta j(z)} \frac{\delta j(z)}{\delta\varphi(y)}$$
$$= \int \mathrm{d}^d z \, W_k^{(2)}(x,z) \left\{ \Gamma_k^{(2)} + R_k \right\} (z,y)$$

- relation between curvatures

$$W_k^{(2)} = \frac{1}{\Gamma_k^{(2)} + R_k}$$

- insert into $\partial_k \Gamma_k \;\Rightarrow$

Wetterich equation

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2} \mathrm{tr} \left(\frac{\partial_k R_k}{\Gamma_k^{(2)}[\varphi] + R_k} \right)$$

- non-linear functional integro-differential equation
- full propagator $\Gamma_k^{(2)}[\varphi]$ enters flow equation
- Polchinski and Wetterich equations = exact FRG equations
- Polchinski: simple polynomial structure (structural investigations)
- Wetterich: rational structure stabilizes flow (explicit calculations)
- UV and IR regulated
- in practice: truncation = projection onto finite-dim. space
- problem: reliable error estimate for flow
  $\rightarrow$ improve truncation, vary regulator, check stability

- anharmonic oscillator

$$S[q]=\int \mathrm{d}\tau\left(\frac{1}{2}\dot{q}^2+V(q)\right)\ ,$$

- here LPA (local potential approximation)

$$\Gamma_k[q]=\int \mathrm{d}\tau\left(\frac{1}{2}\dot{q}^2+u_k(q)\right)$$

- leading order in gradient expansion
- scale-dependent effective potential u_k
- neglected: higher derivative terms, mixed terms $q^n\dot{q}^m$

- in denominator of flow equation $\Gamma_k^{(2)} = -\partial_\tau^2 + u_k''(q)$
- LPA: may consider constant q $\to$ momentum space

$$(\partial_k \Gamma_k)[q] = \frac{\partial}{\partial k} \int \mathrm{d}\tau \, u_k(q) = \frac{1}{2} \int \mathrm{d}\tau \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \frac{\partial_k R_k(p)}{p^2 + u_k''(q) + R_k(p)}$$

- optimal regulator

$$R_k(p) = (k^2 - p^2)\, \theta\left(k^2 - p^2\right) \Longrightarrow \partial_k R_k(p) = 2k\theta(k^2 - p^2)$$

$\Rightarrow$ non-linear partial differential equation for u_k:

$$\partial_k u_k(q) = \frac{1}{\pi} \frac{k^2}{k^2 + u_k''(q)}$$

- free particle limit fixes subtraction in flow equation ($\Lambda \to \infty$)

$$\partial_k u_k(q) = \frac{1}{\pi}\left(\frac{k^2}{k^2 + u_k''(q)} - 1\right) = -\frac{1}{\pi}\frac{u_k''(q)}{k^2 + u_k''(q)}$$

- assume $u_\Lambda(q)$ even $\to$ $u_k(q)$ even
- polynomial ansatz

$$u_k(q) = E_k + \frac{\omega_k^2}{2} q^2 + \frac{\lambda_k}{4!} q^4 + \sum_{n=3,4,\dots} \frac{1}{(2n)!}\, a_{2n,k}\, q^{2n} \ ,$$

- insert and compare coefficients → infinite set of coupled ode's

$$\frac{\mathrm{d}E_k}{\mathrm{d}k} = -\frac{1}{\pi}\omega_k^2\Delta_0, \qquad \Delta_0 = \frac{1}{k^2+\omega_k^2}\,,$$

$$\frac{\mathrm{d}\omega_k^2}{\mathrm{d}k} = -\frac{k^2}{\pi}\lambda_k\Delta_0^2\,,$$

$$\frac{\mathrm{d}\lambda_k}{\mathrm{d}k} = -\frac{k^2\Delta_0^2}{\pi}\left(a_{6,k}-6\lambda_k^2\Delta_0\right)\,,$$

$$\frac{\mathrm{d}a_{6,k}}{\mathrm{d}k} = -\frac{k^2\Delta_0^2}{\pi}\left(a_{8,k}-30\lambda_k a_{6,k}\Delta_0+90\lambda_k^3\Delta_0^2\right)\,,$$

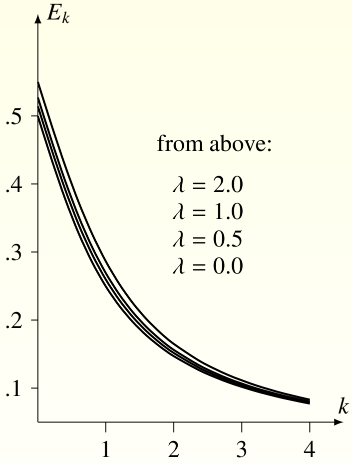
$$\vdots$$

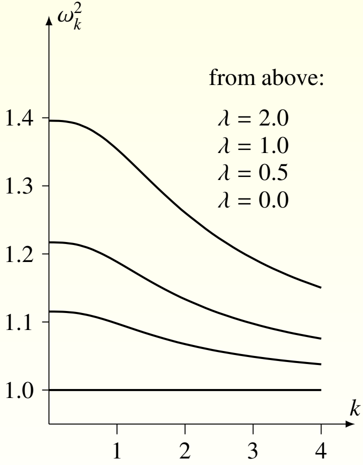
- initial condition: classical potential at cutoff
- projection onto space of polynomials up to given degree n

- e.g. crude truncation $a_6 = a_8 = \cdots = 0$
- truncated finite system of flow equations

$$\frac{\mathrm{d}E_k}{\mathrm{d}k} = -\frac{\omega_k^2}{\pi}\Delta_0, \quad \frac{\mathrm{d}\omega_k^2}{\mathrm{d}k} = -\frac{k^2\lambda_k}{\pi}\Delta_0^2, \quad \frac{\mathrm{d}\lambda_k}{\mathrm{d}k} = \frac{6k^2\lambda_k^2}{\pi}\Delta_0^3$$

- at cutoff $k = \Lambda$: $E_\Lambda = 0,\ \omega_\Lambda = 1$, varying λ_Λ at the cutoff scale
- $\to$ scale-dependent couplings
- dominant contribution from near typical scale $k \approx \omega$





The flow of the couplings E_k and ω_k^2 ($E_\Lambda = 0$, $\omega_\Lambda^2 = 1$).

- $\omega = \omega_{k=0} > 0 \;\Rightarrow$ minimum of u_0 at origin $\Rightarrow E_0 = u_0(0)$
- energy of *first excited state*

$$E_1 = E_0 + \sqrt{u_0''(0)} = E_0 + \omega_0$$

- reasonably good results with 3-parameter truncation

units of $\hbar\omega$: energies for different λ; different regulators

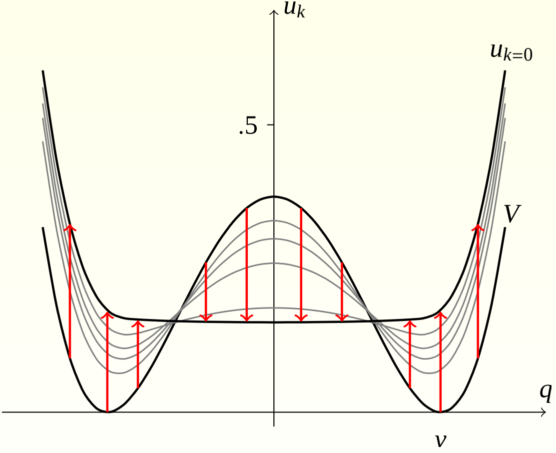
	ground state energy				energy of first excited state			
cutoff	optimal order 4	optimal order 12	Callan order 4	exact result	optimal order 4	optimal order 12	Callan order 4	exact result
$\lambda=0$	0.5000	0.5000	0.5000	0.5000	1.5000	1.5000	1.5000	1.5000
$\lambda=1$	0.5277	0.5277	0.5276	0.5277	1.6311	1.6315	1.6307	1.6313
$\lambda=2$	0.5506	0.5507	0.5504	0.5508	1.7324	1.7341	1.7314	1.7335
$\lambda=3$	0.5706	0.5708	0.5703	0.5710	1.8177	1.8207	1.8159	1.8197
$\lambda=4$	0.5885	0.5889	0.5882	0.5891	1.8923	1.8968	1.8898	1.8955
$\lambda=5$	0.6049	0.6054	0.6045	0.6056	1.9593	1.9652	1.9562	1.9637
$\lambda=6$	0.6201	0.6207	0.6196	0.6209	2.0205	2.0278	2.0168	2.0260
$\lambda=7$	0.6343	0.6350	0.6336	0.6352	2.0771	2.0857	2.0728	2.0836
$\lambda=8$	0.6476	0.6484	0.6469	0.6487	2.1299	2.1397	2.1250	2.1374
$\lambda=9$	0.6602	0.6611	0.6594	0.6614	2.1794	2.1905	2.1741	2.1879
$\lambda=10$	0.6721	0.6732	0.6713	0.6735	2.2263	2.2385	2.2205	2.2357
$\lambda=20$	0.7694	0.7714	0.7679	0.7719	2.5994	2.6209	2.5898	2.6166

- negative ω^2: mexican hat

$$\partial_k u_k(q) = -\frac{1}{\pi}\frac{u_k''(q)}{k^2 + u_k''(q)}$$

- denominator positive for large scales
  $\Rightarrow$ remains positive during the flow
- flow equation $\Rightarrow$
  $u_k(q)$ increases toward infrared if $u_k''(q)$ is positive
  $u_k(q)$ decreases toward infrared if $u_k''(q)$ is negative
  $k^2 + u_k''$ minimal at $q = 0$
  $\Rightarrow$ double-well potential flattens during flow, becomes convex
- expected on general grounds

solution of partial differential equation, $\omega^2 = -1$, $\lambda = 1$



- increasing barrier (weak coupling) $\rightarrow$ increasingly difficult
- numerical solution does better
- splitting induced by instanton effects: beyond leading order LPA

units of $\hbar\omega$: varying λ; optimized regulator,

	ground state energy				energy of first excited state			
	optimal order 4	optimal order 12	pde	exact	optimal order 4	optimal order 12	pde	exact
$\lambda = 1$			-0.8732	-0.8556			-0.7887	-0.8299
$\lambda = 2$		-0.2474	-0.2479	-0.2422		0.0049	0.0063	-0.0216
$\lambda = 3$	0.2473	-0.0681	-0.0679	-0.0652	-0.2241	0.3514	0.3500	0.3307
$\lambda = 4$	-0.0186	0.0286	0.0290	0.0308	0.3511	0.5753	0.5755	0.5598
$\lambda = 5$	0.0654	0.0949	0.0953	0.0967	0.5835	0.7455	0.7462	0.7324
$\lambda = 6$	0.1234	0.1457	0.1461	0.1472	0.7509	0.8842	0.8851	0.8723
$\lambda = 7$	0.1688	0.1871	0.1876	0.1885	0.8851	1.0021	1.0030	0.9909
$\lambda = 8$	0.2063	0.2223	0.2228	0.2236	0.9987	1.1052	1.1061	1.0944
$\lambda = 9$	0.2671	0.2530	0.2535	0.2543	1.1863	1.1972	1.1981	1.1866
$\lambda = 10$	0.2386	0.2803	0.2808	0.2816	1.0978	1.2805	1.2814	1.2701
$\lambda = 20$	0.4536	0.4632	0.4639	0.4643	1.7866	1.8638	1.8648	1.8538

- LP approximation

$$\Gamma_k[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2}(\partial_\mu \varphi)^2 + u_k(\varphi) \right)$$

- scale-dependent effective potential

$$\partial_k u_k(q) = \frac{1}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\partial_k R_k(p)}{p^2 + u_k''(q) + R_k(p)}$$

- optimized regulator: p-integration doable $\to$ flow equation

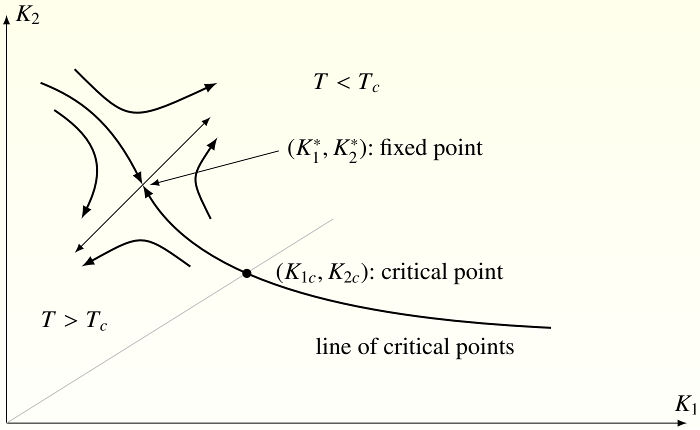
$$\partial_k u_k(\varphi) = \mu_d \frac{k^{d+1}}{k^2 + u_k''(\varphi)}\,, \quad \mu_d = \frac{(4\pi)^{-d/2}}{\Gamma(\frac{d}{2}+1)}$$

- space-time dimensions in k^{d+1} and μ_d
- nonlinear PDE

- series expansion (even) $\to$ flow equations for infinite set of couplings

$$
\begin{aligned}
k\frac{\mathrm{d}a_0}{\mathrm{d}k} &= +\mu_d k^{d+2}\Delta_0, \qquad \Delta_0 = \frac{1}{k^2+a_2}\ , \\
k\frac{\mathrm{d}a_2}{\mathrm{d}k} &= -\mu_d k^{d+2}\Delta_0^2 a_4\ , \\
k\frac{\mathrm{d}a_4}{\mathrm{d}k} &= -\mu_d k^{d+2}\Delta_0^2\left(a_6 - 6a_4^2\Delta_0\right)\ , \\
k\frac{\mathrm{d}a_6}{\mathrm{d}k} &= -\mu_d k^{d+2}\Delta_0^2\left(a_8 - 30a_4a_6\Delta_0 + 90a_4^3\Delta_0^2\right)\ , \\
\vdots \quad & \quad \vdots
\end{aligned}
$$

- point in space of dimensionless couplings where all β-functions vanish



- critical hyper-surface on which $\xi = \infty$
- RG trajectory moves away from critical surface
- If flow begins on critical surface $\rightarrow$ stays on surface
- generically: critical points are not fixed point
- $d \geq 3$: expect finite set of isolated fixed points $K^* = (K_1^*, K_2^*, \dots)$
- RG flow in the vicinity of fixed point $K = K^* + \delta K$
- linearize flow around fixed point

$$K_i' = K_i^* + \delta K_i' = R_i\big(K_j^* + \delta K_j\big) = K_i^* + \frac{\partial R_i}{\delta K_j}\Big|_{K^*} \delta K_j + O(\delta K^2)$$

- linearized RG transformation,

$$\delta K_i' = \sum_j M_i^{\,j} \delta K_j, \qquad M_i^{\,j} = \frac{\partial R_i}{\partial K_j}\Big|_{K^*}$$

- left-eigenvectors $\Phi_\alpha$and eigenvalues $\lambda_\alpha = \mathrm{e}^{y_\alpha}$ of M
- $y_\alpha > 0$: flow repelled by $K^* \to$ relevant perturbation
- $y_\alpha < 0$: flow attracted to $K^* \to$ irrelevant perturbation
- rel. couplings/exponents $\to$ IR-physics (scaling laws, TD critical exponents)

- introduce dimensionless field and potential,

$$\varphi = k^{(d-2)/2}\sqrt{\mu_d}\,\chi \quad \text{and} \quad u_k(\varphi) = k^d \mu_d v_k(\chi)$$

- flow equation in terms of dimensionless quantities (rescaled by $k^{\#}$)

$$k\partial_k v_k + dv_k - \frac{d-2}{2}\chi v_k' = \frac{1}{1+v_k''}, \quad v_k' = \frac{\partial v_k}{\partial \chi}\ldots$$

- fixed point equation for effective potential (ode)

$$dv_* - \frac{d-2}{2}\chi v_*' = \frac{1}{1+v_*''}$$

- constant solution $dv_* = 1 \to$ trivial Gaussian fixed point

scale invariance at fixed point $\Rightarrow$ fixed point $\sim$ CFT

- are there non-Gaussian fixed point solutions?
- answer depends on spacetime dimension d
- 2d theories:
  ∞ many fixed-point scalar models [min. models; Morris 1994]
  ∞ many fixed-point Yukawa models [Synatschke, AW, ...]
- 3d theories:
  small number of fixed-points
  from various truncations: series expansions and numerical solutions
- example: LPA and polynomial truncation for scalar field theory

$$w = \sum_{n=0}^{m} c_n \rho^n, \quad \rho = \frac{1}{2}\chi^2$$

- at fixed point: $\beta_n(c_0^*, c_1^*, \dots) = 0$ for $n = 0, 1, 2, \dots$
- always Gaussian fixed point: $c_0^* = \frac{1}{3},\ 0 = c_2^* = c_3^* = c_4^* = \dots$
- search for non-trivial fixed points

  only one for scalar field in $d = 3$ (even u):
  $m = 20 \Rightarrow c_1^* = -.186066$
  $m = 42 \Rightarrow c_1^* = -.186041$

  $\Rightarrow c_0^*, c_2^*, \dots, c_{m-1}^* \Rightarrow$ polynomial approximation to fixed point solution

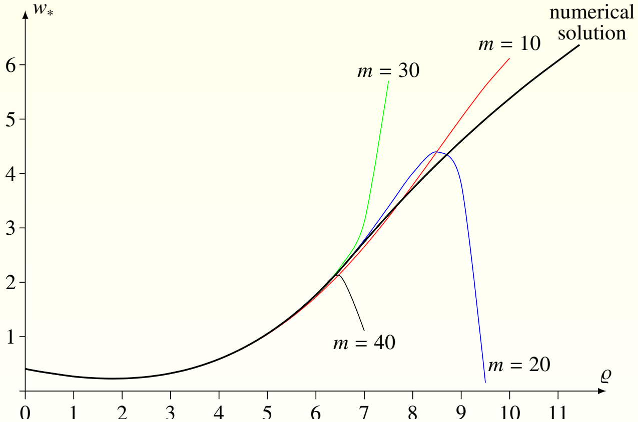
fixed-point coefficients $n! c_n^*$

	c_0^*	c_1^*	c_2^*	c_3^*	c_4^*	c_5^*	c_6^*
$m = 20$	0.409534	-0.186066	0.082178	0.018981	0.005253	0.001104	-0.000255
$m = 42$	0.409533	-0.186064	0.082177	0.018980	0.005252	0.001104	-0.000256

	c_7^*	c_8^*	c_9^*	c_{10}^*	c_{11}^*	c_{12}^*	c_{13}^*
$m = 20$	-0.000526	-0.000263	0.000237	0.000632	0.000438	-0.000779	-0.002583
$m = 42$	-0.000526	-0.000263	0.000236	0.000629	0.000431	-0.000799	-0.002643

	c_{14}^*	c_{15}^*	c_{16}^*	c_{17}^*	c_{18}^*	c_{19}^*	c_{20}^*
$m = 20$	-0.002029	0.007305	0.028778	0.034696	-0.077525	-0.381385	0.000000
$m = 42$	-0.002216	0.006677	0.026544	0.026320	-0.110498	-0.517445	-0.587152

numerics: shooting method with seventh-order Runge-Kutta



Critical exponents

m	$\nu = -1/\omega_1$	ω_2	ω_3	ω_4	ω_5
10	0.648617	0.658053	2.985880	7.502130	17.913494
14	0.649655	0.652391	3.232549	5.733445	9.324858
18	0.649572	0.656475	3.186784	5.853987	9.141093
22	0.649554	0.655804	3.170538	5.977066	8.522811
26	0.649564	0.655629	3.182910	5.897290	8.844632
30	0.649562	0.655791	3.180847	5.903039	8.907607
34	0.649561	0.655749	3.178636	5.922910	8.702583
38	0.649562	0.655731	3.180577	5.908885	8.814225
42	0.649562	0.655755	3.180216	5.909910	8.847386
46	0.649562	0.655746	3.179541	5.915754	8.738608

- convergence
- two relevant operators: $\omega_0 = -3$ and $\omega_1 = -1/\nu$
- LPA-prediction: $\nu = 0.649562$ (high-T expansion: $\nu = 0.630$)

- next-to-leading in derivative expansion $\rightarrow Z_k(p, \varphi)$
- simple LPA' truncation (no p and φ-dependence)

$$\Gamma_k[\varphi] = \int \mathrm{d}^d x \left(\frac{1}{2} Z_k \, (\partial_\mu \varphi)^2 + u_k(\varphi) \right) .$$

- flow of Z_k: project flow on operator $(\partial\phi)^2$
- must admit inhomogeneous φ in flow equation

  $\Rightarrow$ anomalous dimension

$$\eta = -k\partial_k \log Z_k$$

- scalar field $\phi \in \mathbb{R}^N$, O(N) invariant potential
- fixed-point analysis: dimensionless field χ and composite field

$$\varrho = \frac{1}{2}\chi \cdot \chi$$

- dimensionless potential $\nu_k(\varrho)$.
- $N-1$ Goldstone modes, one massive radial mode
- Goldstone modes drive flow for large N
- finds one Wilson-Fisher fixed point w_* for all N
- fixed point analysis: $w_k = w_* + \delta_k$
- fluctuation δ_k obeys the linear differential equation

- polynomial truncation to high order (40)

N	1	2	3	100	1000
$-w'_*(0)$	0.186064	0.230186	0.263517	0.384172	0.387935
$\nu = -1/\omega_1$	0.64956	0.70821	0.76113	0.99187	0.99923
ω_2	0.6556	0.6713	0.6990	0.97218	0.99844
ω_3	3.1798	3.0710	3.0039	2.98292	2.99554

- asymptotic formulas, e.g. $\nu \approx 0.9998 - \frac{0.9616}{N}$

Large-N limit

- flow equation can be solved (characteristics), exact relation $w_* \Leftrightarrow \delta$ in

$$w(t,\rho) \approx w_*(\rho) + \mathrm{e}^{\omega t}\delta(\rho), \quad \delta = {w'_*}^{(\omega+d)/2} \quad t = \log\frac{k}{\Lambda}$$

- perturbation non-singular $\to$ all critical exponents

$$\omega_n \in \{2n - d \,|\, n = 0, 1, 2, \dots\}$$

- on-shell supersymmetry: susy transformations non-linear
  $\Rightarrow$ no supersymmetric quadratic regulator exists
- off-shell supersymmetry: susy transformation linear
  supersymmetric quadratic regulator constructed Synatschke, Gies, AW
- cutoff function bosons $\Leftrightarrow$ cutoff function fermions

constructed manifest supersymmetric FRG

- $\mathcal{N} = 1$ theory in $d = 4$: non-renormalization of superpotential
- $\mathcal{N} = 1$ theories in $d < 4$: running superpotential, running couplings
- $\mathcal{N} = 2$ theories in $d < 4$: non-renormalization of superpotential
  instead: flow of Kähler-potential

- Lagrangian density: complex ϕ, F, Dirac ψ

$$\mathcal{L} = -\frac{1}{4}\int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \Phi\Phi^\dagger - \left\{\frac{1}{2\mathrm{i}}\int \mathrm{d}^2\theta W + \mathrm{h.c.}\right\}$$
$$= |\nabla\phi|^2 + \mathrm{i}\bar{\psi}\boldsymbol{\sigma}\nabla\psi - |F|^2 - \left\{\frac{\partial W}{\partial\phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi^T\sigma_2\psi + \mathrm{h.c.}\right\}$$

- (nonlocal) susy regulator with cutoff functions $r_1(k,p)$ and $r_2(k,p)$
- flow of Kähler metric

$$\Gamma_k = \int \mathrm{d}^3x\Big(Z_k^2(\phi)\big\{|\nabla\phi|^2 + \mathrm{i}\bar{\psi}\boldsymbol{\sigma}\nabla\psi - |F|^2\big\}$$
$$- \big\{\frac{\partial W_k}{\partial\phi}F - \frac{1}{2}\frac{\partial^2 W_k}{\partial\phi^2}\psi^T\sigma_2\psi + \mathrm{h.c.}\big\}\Big)$$

$\Rightarrow$ Kähler metric Z_*^2 at fixed point

P. Feldmann, L. Zambelli, AW, 2018

- FP-structure in $2 \leq d \leq 4$
- non-renormalization theorems — with F. Synatschke and M. Heilmann
- flow with higher derivative operators — with R. Flore, T. Hellwig, B. Knorr and O. Zanusso
- exact solution for W_k for susy $O(N \to \infty)$ model — with M. Heilmann and D. Litim
- fixed point of nonlinear susy $O(N)$ model — with R. Flore and O. Zanusso
- spike plots for susy theories — with T. Hellwig and O. Zanusso
- emergence of supersymmetry in Yukawa models — with T. Hellwig and L. Zambelli
- flow of Kähler metric in $\mathcal{N} = 2$ in $d = 3$ — with P. Feldmann and L. Zambelli

- spontaneouly breaking of susy — with H.Gies, F. Synatschke
- particle masses
- finite-temperature phase transition — with J. Braun, F. Synatschke

- many FRG studies of Gauge Theories and Gravity
  - problem with local gauge invariance
  - background-field method
  - bimetric approach, ...
- missing: convincing FRG studies for SYM
  - susy cutoffs exist (beyond WZ gauge)
  - mixing of susy transformations with gauge transformations
  - needed for comparison with lattice studies
  - extract effective theories
- supersymmetric lattice gauge theories

  many lattice-results in $d = 1, 2$ — many; D. August, B. Wellegehausen, AW

  view lattice simulations in $d = 4$ — Münster-DESY collaboration

  M. Steinhauser, A. Sternbeck, B. Wellegehausen, AW

Coworkers:

Jens Braun,Holger Gies, Ilya Shapiro; Boris Merzlikin, Luca Zambelli, Omar Zanusso; Polina Feldmann, Raphael Flore, Marianne Heilmann, Tobias Hellwig, Franziska Synatschke

JHEP **1712** (2017) 132 JHEP **1502** (2015) 109 JHEP **0903** (2009) 028
Phys. Rev. **D96** (2017) 125007; Phys. Rev. **D92** (2015) 085027; Phys. Rev. **D87** (2013) 06501
Phys. Rev. **D86** (2012) 105006; Phys. Rev. **D84** (2011) 125009; Phys. Rev. **D81** (2010) 125001

Phys. Rev. **D80** (2009) 085007; Phys. Rev. **D80** (2009) 101701