Total decay and transition rates from LQCD

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Let’s talk about particle decay

The survival probability

\[ P(t) = e^{-t/(\gamma\tau)} \]

motivates the definition of its lifetime (in the rest frame)

\[ \Gamma \equiv \frac{1}{\tau} \]

From QM we know that

\[ |\psi_j(t)\rangle = e^{-iE_j t} |\psi_j(0)\rangle \]

These type of states can be parametrised \textit{effectively} by

\[ E_j \longrightarrow E_j - i\Gamma/2 \]

\textit{where does this come from??}
Treatment of scattering in QM

\[ H = H_0 + V \]

\[ H |\psi_I(0)\rangle = E |\psi_I(0)\rangle \] (scattering states in the interaction picture)

\[ U_I(t, 0) = e^{iH_0 t} U(t, 0) e^{-iH_0 t} \]

\[ |\psi_I(t)\rangle = U_I(t, 0) |\psi_I(0)\rangle \]

Formally the S-Matrix is given by:

\[ \langle \psi_2 | S | \psi_1 \rangle = \langle \psi_2 | U_I(\infty, -\infty) | \psi_1 \rangle \]

Unitarity of the S-Matrix implies for the scattering amplitudes

\[ \mathcal{M}_\ell(E) = \frac{4\pi}{p} \frac{1}{\cot \delta_\ell - i} \]

\[ S = e^{2i\delta(p)} = \frac{E - E_R - i\Gamma_R/2}{E - E_R + i\Gamma_R/2} \]
Perturbation theory: quick reminder

2 real scalar fields $\phi, K$ with masses $m_K > 2m_\phi$

$$
\mathcal{L} = \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{2} (\partial K)^2 + \frac{1}{2} m_K^2 K^2 + \mathcal{L}_{\text{int}}
$$

$$
\mathcal{L}_{\text{int}} = \frac{1}{4!} \lambda \phi^4 + \frac{1}{2} g \phi^2 K + \frac{1}{4!} \lambda_k K^4 + \frac{1}{3!} g_k K^3 + \frac{1}{4} h K^2 \phi^2
$$

Let's calculate the width of $K \to \phi \phi$

$$
\Gamma_{K \to \phi \phi} = \frac{1}{2m_K} \left( \frac{1}{2} \right) \sqrt{\int d\Pi_2 |\langle \text{in;} K|\text{out;} \phi \phi \rangle|^2}
$$

$$
= \frac{g^2}{32\pi m_K} \sqrt{1 - \frac{4m_\phi^2}{m_K^2}}
$$

This is the easy and straightforward way but not too much insight…
Consider the following **Euclidean two-point function**:

\[
\int d^4 x \ e^{ipx} \langle K(x)K(0) \rangle = \frac{1}{p^2 + m^2_K - \Sigma_K(p^2)}
\]

calculate it to 1-loop

\[
\Sigma_K(p^2) = K
\]

\[
\theta(\lambda_k)
\]

\[
\phi
\]

\[
\theta(h)
\]

\[
\phi
\]

\[
\theta(g^2)
\]

\[
\theta(g^2_k)
\]

Obtain the spectral function:

\[
\rho(s) = \frac{1}{\pi} \text{Im} \left( \frac{1}{-s + m^2_K - \Sigma_K(-s - i\epsilon)} \right)
\]

\[
= \frac{1}{\pi} \left( \frac{\text{Im}\Sigma_K(-s - i\epsilon)}{(s - (m^2_K - \text{Re}\Sigma_K(s)))^2 + (\text{Im}\Sigma_K(-s - i\epsilon))^2} \right), \quad s = \omega^2 - p^2
\]

\[
\text{for} \quad s > 4m^2_\phi \quad \rightarrow \quad \text{Im}\Sigma_K \neq 0 \quad \rightarrow \quad \rho(s) \neq 0
\]
The spectral function has the form of a relativistic Breit-Wigner

\[ \propto \frac{1}{(E^2 - m_K^2)^2 + m_K^2 \Gamma^2} \]

\[ \text{Im}\Sigma_K(m_K^2) = m_K \Gamma \]

Agrees with my previous result and this relation is valid to all orders in PT.

\[ \rho(m_K^2) = \frac{1}{\pi m_K \Gamma} \]

since I chose the on-shell scheme

\[ \text{Re}\Sigma_K(m_K^2) = \text{Re}\Sigma_K'(m_K^2) = 0 \]

\textbf{The point is that widths can be calculated from spectral functions!}

(which is not very surprising since EVERYTHING is encoded in spectral functions).

The problem is that the extraction from the lattice is not an easy task.
To finish this story ...
To finish this story ...

\[
\frac{m_{\phi}^2}{\pi m_K \Gamma} = \frac{m_K}{m_{\phi}} = 3
\]

\[
g = 8
\]
To finish this story …

\[ \frac{m_\phi^2}{\pi m_K \Gamma} \]

\[ m_\phi^2 \rho(s) \]

\[ m_K \]

\[ g = 10 \]

\[ \frac{m_K}{m_\phi} = 3 \]
To finish this story …

\[ \frac{m_\phi^2}{\pi m_K \Gamma} \]

\[ \frac{m_K}{m_\phi} = 3 \]

\[ g = 12 \]
To finish this story …

\[ \frac{m_{\phi}^2}{\pi m_K \Gamma} \]

\[ m_K = 3 \]

\[ g = 15 \]
To finish this story …

\[
\frac{m_\phi^2}{\pi m_K \Gamma} \quad \text{versus} \quad \sqrt{s}
\]

\[
m_\phi^2 \rho(s) \quad \text{versus} \quad \sqrt{s}
\]

\[
m_K = 3
\]

\[
g = 20
\]
To finish this story …

\[
\frac{m_K}{m_\phi} = 3
\]

\[
g = 20
\]

\[
g_k = 100
\]
Suppose \( \mathcal{H} = \mathcal{H}_{\text{QCD}} + \mathcal{V} \) and \( \mathcal{V} \) is a small perturbation.

Consider a QCD-stable single-particle state \( |D, P\rangle \)

\[
\left[ \int d^3 x \mathcal{H}_{\text{QCD}}(x) \right] |D, P\rangle = E_D |D, P\rangle
\]

which can decay through \( \mathcal{V} \) into a multi particle state \( |E, p, \alpha; \text{out}\rangle \) (imagine for now that \( \mathcal{V} \) is some weak hamilton insertion)

\[
\Gamma_{D \to Q} \equiv \frac{1}{2M_D} \sum_{\alpha} \frac{1}{S_\alpha} \int d\Pi_\alpha(k_1, \cdots, k_{N_\alpha}) |\langle E_D, P, \alpha; \text{out}| \mathcal{V}_Q(0)|D, P\rangle|^2
\]

\[
= \frac{1}{2M_D} \int d^4 x \langle D, P| \mathcal{V}_Q(x) \mathcal{V}_Q(0)|D, P\rangle.
\]

This is the total width into ALL allowed multi particle states with quantum numbers \( Q \).
We can generalise the previous relation:

Let $\mathcal{V}_Q \to \mathcal{J}_Q$ be a local current that can inject or carry away energy and/or momentum.

(This is useful for describing scattering with leptons or photons).

We define the *transition spectral function*:

$$\rho_{Q,P}(E, p) = \int d^4x \ e^{i(E - E_N)t - i(p - P) \cdot x} \langle N, P | \mathcal{J}_Q^\dagger(x) \mathcal{J}_Q(0) | N, P \rangle.$$ 

and we obtain the less general result if we back-substitute: 

$\mathcal{J}_Q \to \mathcal{V}_Q$, $E = E_N$, $p = P$

$$\Gamma_{N \to Q} = \frac{1}{2M_N} \rho_{Q,P}(E_N, P)$$

again: widths can be obtained from spectral functions!

so far this was a continuum Minkowski discussion.
Possible issues...

- Finite volume does not allow the definition of in/out states.
- Finite volume energy-levels are discrete.
- Multi particle states in finite volume have power-law corrections instead of exponential.
- As the energy is increased, the density of finite volume level is very high. No possible resolving of those levels. No L-L approach.
- Minkowski real time is not Euclidean imaginary time. No real-time evolution study is possible.
- In many cases a lot of multi particle channels are open with more than two particles in the final state.
Write the Euclidean correlator (which is what we can calculate on the lattice) most closely related to the previous discussion.

\[ G_{Q,P}(\tau, x, L) \equiv 2E_N L^6 e^{-E_N \tau + iP \cdot x} \lim_{\tau_f \to \infty} \lim_{\tau_i \to -\infty} \frac{\langle \hat{N}(\tau_f, P) \mathcal{J}_Q^\dagger(\tau, x) \mathcal{J}_Q(0) \hat{N}^\dagger(\tau_i, P) \rangle_{\text{conn}}}{\langle \hat{N}(\tau_f, P) \hat{N}^\dagger(\tau_i, P) \rangle}, \]

\[ = 2E_N L^3 e^{-E_N \tau + iP \cdot x} \langle N, P | \mathcal{J}_Q^\dagger(\tau, x) \mathcal{J}_Q(0) | N, P \rangle_{L}. \]

Take the Fourier transform and use complete set of now finite volume states:

\[ \tilde{G}_{Q,P}(\tau, p, L) = 2E_N L^6 \sum_k e^{-E_k(L) \tau} |M_{k,N \to Q}(p, L)|^2 \tau > 0, \]

\[ M_{k,N \to Q}(p, L) \equiv \langle E_k(L), p, Q | \mathcal{J}_Q(0) | N, P \rangle_L. \]

We can rewrite it as:

\[ \tilde{G}_{Q,P}(\tau, p, L) = \int_0^\infty \frac{d\omega}{2\pi} e^{-\omega \tau} \rho_{Q,P}(\omega, p, L), \]

\[ \rho_{Q,P}(E, p, L) \equiv 2E_N L^6 \sum_k |M_{k,N \to Q}(p, L)|^2 2\pi \delta(E - E_k(L)), \]
Two big problems ...

\[ \rho_{Q,P}(E, p, L) \equiv 2E_N L^6 \sum_k |M_{k,N\to Q}(p, L)|^2 2\pi \delta(E - E_k(L)), \quad L \to \infty \]

\[ \rho_{Q,P}(E, p) \equiv \sum_\alpha \frac{1}{S_\alpha} \int d\Phi_\alpha(k_1, \cdots, k_{N_\alpha}) \langle E, p, \alpha; \text{out}|J_Q(0)|N, P \rangle^2, \]

a lot of progress has been done since Lellouch, Lüscher (2001)

\[ \Gamma_{N\to Q} = \frac{1}{2M_N} \rho_{Q,P}(E_N, P) \]

\[ \tilde{G}_{Q,P}(\tau, p, L) = \int_0^\infty \frac{d\omega}{2\pi} e^{-\omega \tau} \rho_{Q,P}(\omega, p, L), \]

This inverse Laplace transformation is an numerically ill-defined problem.

We aim at total decay rates! What about the infinite volume limit?

The Backus-Gilbert Method solves both problems at once!

\[ \hat{\rho}_{Q,P}(\bar{\omega}, p, L, \Delta) \equiv \int_0^\infty d\omega \ \hat{\delta}_\Delta(\bar{\omega}, \omega) \rho_{Q,P}(\omega, p, L). \]

\[ \int_0^\infty d\omega \ \hat{\delta}_\Delta(\bar{\omega}, \omega) = 1, \quad \lim_{\Delta \to 0} \int_0^\infty d\omega \ \hat{\delta}_\Delta(\bar{\omega}, \omega) \phi(\omega) = \phi(\bar{\omega}), \]

\[ \rho_{Q,P}(E, p) = \lim_{\Delta \to 0} \lim_{L \to \infty} \hat{\rho}_{Q,P}(E, p, L, \Delta), \]

order of limits is crucial!
Fermi Golden Rule motivation

The smearing idea goes back to Fermi's Golden rule:

\[(H_0 + V) |\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle, \quad |\psi(t)\rangle = \sum_n a_n(t) e^{-iE_n t/\hbar} |n\rangle\]

**Time-dependent PT:**
\[
\Gamma_{i\rightarrow k} = \frac{|a_k(t)|^2}{t} = \frac{4\langle k|V|i\rangle \sin^2(\omega t/2)}{\hbar^2 \omega^2 t}
\]

\[
\Gamma = 2\pi \lim_{t \to \infty} \lim_{L \to \infty} \sum_k |V_k(L)|^2 \delta_{1/t}(E_k(L) - M),
\]

\[
\delta_{1/t}(\omega) = \frac{2 \sin^2(\omega t/2)}{\pi \omega^2 t} \quad \text{(regularised delta function)}
\]

if interested in TOTAL decay rates, one can use other regularised delta functions. This is a key aspect that we exploit.
(open big parenthesis …)
Some remarks on the BG method

\[ G(\tau_i) = \int_0^\infty d\omega \, \phi(\omega) K(\omega, \tau_i) \]

- Developed in 1967 by geophysicists Bakus and Gilbert to study the propagation of earthquakes on the Earth.
- It is a linear method:
  \[ \hat{\delta}_\Delta(\bar{\omega}, \omega) = \sum_j C_j(\bar{\omega}, \Delta) \, e^{-\omega \tau_j}, \]
  \[ \hat{\phi}(\bar{\omega}, \Delta) = \int_0^\infty d\omega \, \hat{\delta}_\Delta(\bar{\omega}, \omega) \phi(\omega) = \sum_j C_j(\bar{\omega}, \Delta) \, \widetilde{G}(\tau_j). \]
- It converges to the exact solution:
  \[ \lim_{\Delta \to 0} \int_0^\infty d\omega \, \hat{\delta}_\Delta(\bar{\omega}, \omega) \phi(\omega) = \phi(\bar{\omega}) \]
- No a priori Ansatz. Model Independent estimator.
- No free lunch: The regularisation of the problem is translated into a trade off between resolving power and error estimation.
Some remarks on the BG method

- The optimal coefficients $C_j(\bar{\omega}, \Delta)$ are calculated by minimizing

$$\Delta = \int_0^\infty d\omega (\bar{\omega} - \omega)^2 \tilde{\delta}_\Delta(\bar{\omega}, \omega)^2 \quad \text{subject to} \quad \int_0^\infty d\omega \tilde{\delta}_\Delta(\bar{\omega}, \omega) = 1$$

In principle one can make the width very small if enough points are available...

... but the scaling is a Log.
The BG-method with other Kernels
(Finite-T applications)

\[ G(\tau_i, T, p) = \int_0^\infty d\omega \left( \frac{\rho_\lambda(\omega, p)}{\tanh(\omega/2)} \right) \left( \frac{\cosh(\omega(\beta/2 - \tau_i))}{\cosh(\omega\beta/2)} \right) \]

\[ \hat{\delta}_\Delta(\bar{\omega}, \omega) = \sum_j C_j(\bar{\omega}, \Delta) \left( \frac{\cosh(\omega(\beta/2 - \tau_j))}{\cosh(\omega\beta/2)} \right) \]

Brandt, Meyer, Francis, DR
1506.05732

a lot of BG in Granada ’17
see talks of Braguta, Harris, Steinberg, Hansen,…

its exact if the target is constant.
The BG-method with other Kernels

(Finite-T applications)

Residue estimation with the BG method:

\[
\hat{\rho}(\bar{\omega}) = \int_{0}^{\infty} d\omega \hat{\delta}_\Delta(\bar{\omega}, \omega) \left( \frac{\rho(\omega)}{\tanh(\omega \beta/2)} \right)
\]

assume the existence of a state at a given omega

\[
\rho(\omega) = \text{Res}(\omega) \delta(\omega^2 - \omega_1^2) + \ldots
\]

define an estimator for the residue

\[
\text{Res}(\bar{\omega}, \omega_1)_{BG} = \frac{2\omega_1 \hat{\rho}(\bar{\omega}) \tanh(\omega_1 \beta/2)}{\hat{\delta}(\bar{\omega}, \omega_1)}
\]

this definition is very stable against changes of the resolution function.
(... you better close it)
Back to our decay widths

So our strategy is:

1. Calculate the Euclidean 4-point function in question

\[
\frac{\langle \hat{N}(\tau_f, \mathbf{P}) \mathcal{J}^\dagger_Q(\tau, \mathbf{x}) \mathcal{J}_Q(0) \hat{N}^\dagger(\tau_i, \mathbf{P}) \rangle}{\langle \hat{N}(\tau_f, \mathbf{P}) \hat{N}^\dagger(\tau_i, \mathbf{P}) \rangle},
\]

This by itself can be quite challenging!

Boundaries, excited states contamination, signal to noise problems.

2. Run the Backus-Gilbert on it to obtain

\[
\rho_{Q, \mathbf{P}}(E, \mathbf{p}) = \lim_{\Delta \to 0} \lim_{L \to \infty} \hat{\rho}_{Q, \mathbf{P}}(E, \mathbf{p}, L, \Delta),
\]

\[
\Gamma_{N \to Q} = \frac{1}{2M_N} \lim_{\Delta \to 0} \lim_{L \to \infty} \hat{\rho}_{Q, \mathbf{P}}(E_N, \mathbf{P}, L, \Delta),
\]

3. Make sure you are in the right window of big volume and small Delta.
The order of the double limit

Take the scalar toy model again and consider only

$$L_{\text{int}} = \frac{1}{2} g K \phi^2$$

what does the smoothing when $L \to \infty$?

$$\rho_{Q,0}(E, 0, L) \equiv 2M_K L^6 \sum_k |M_{k,K \to \phi \phi}(0, L)|^2 2\pi \delta(E - E_k(L)),$$

$$|M_{k,K \to \phi \phi}(0, L)|^2 = g^2 M_K^2 \frac{\nu_k}{4M_K E_k(L)^2 L^9} \left| E_k(L) = 2\sqrt{M_\phi^2 + (2\pi/L)^2 q_k^2} \right.,$$

$$\hat{\rho}_{Q,0}(\bar{\omega}, 0, L, \Delta) = \int_0^\infty d\omega \hat{\delta}_\Delta(\bar{\omega}, \omega) \rho_{Q,0}(\omega, 0, L)$$

for now we consider a normalised gaussian resolution function to study the effect of smoothing and the double limit.

$$\hat{\delta}_\Delta(\bar{\omega}, \omega) = \frac{1}{\sqrt{2\pi \Delta^2}} e^{- (\omega - \bar{\omega})/(2\Delta^2)}$$
We can look at the same effect but doing a BG and starting with the finite volume Euclidean corrector itself.
\[ \tilde{G}_{K \to \phi \phi}(\tau, 0, L) = 2M_K L^6 \sum_k e^{-E_k(L)\tau} |M_{k,K \to \phi \phi}(0, L)|^2, \]

\[ \rho(\bar{\omega}) = \frac{\hat{\rho}(\bar{\omega})}{M_\pi^2} \quad \frac{\bar{\omega}}{M_\pi} \]

\( M_\pi L = 5 \)

\( N = 24 \)
\( N = 32 \)
\( N = 64 \)
\( L \to \infty \)
\[ \tilde{G}_{K \rightarrow \phi \phi}(\tau, 0, L) = 2M_K L^6 \sum_k e^{-E_k(L)\tau} |M_{k,K \rightarrow \phi \phi}(0, L)|^2, \]

\[ \tilde{G}_{K \rightarrow \phi \phi}(\tau, 0, L) = 2M_K L^6 \sum_k e^{-E_k(L)\tau} |M_{k,K \rightarrow \phi \phi}(0, L)|^2, \]

\[ \tilde{G}_{K \rightarrow \phi \phi}(\tau, 0, L) = 2M_K L^6 \sum_k e^{-E_k(L)\tau} |M_{k,K \rightarrow \phi \phi}(0, L)|^2, \]
\[ \tilde{G}_{K\to\phi\phi}(\tau, 0, L) = 2M_K L^6 \sum_k e^{-E_k(L)\tau} |M_{k,K\to\phi\phi}(0, L)|^2, \]

\[
\begin{align*}
\hat{\rho}(\bar{\omega})/M^2 = 0.035 & \quad N = 24 \\
\hat{\rho}(\bar{\omega})/M^2 = 0.03 & \quad N = 32 \\
\hat{\rho}(\bar{\omega})/M^2 = 0.025 & \quad N = 64 \\
L \to \infty & \quad M_{\pi} L = 10
\end{align*}
\]
A more general case

- Consider the total decay into two open channels: a 2-particle decay + 3-particle decay.
  (we can do it whereas L-L cannot)

\[ 3M_\pi < 2M_K < M_\phi \]

\[ \mathcal{L}_{\text{int}} = \frac{\lambda}{6} \phi(x) \pi(x)^3 + \frac{gM_\phi}{2} \phi(x) K(x)^2 \]

- Work out the infinite volume transition spectral function:

\[
\frac{1}{2M_\phi M_\pi} \rho_{Q,0}(\omega, 0) = \frac{\lambda^2}{3072\pi^3} \left( \frac{\omega}{M_\pi} \right) \frac{M_\pi}{M_\phi} \mathcal{F}(\omega/M_\pi) \theta(\omega - 3M_\pi) + \frac{g^2}{32\pi} \frac{M_\phi}{M_\pi} \sqrt{1 - \frac{4M_K^2}{\omega^2}} \theta(\omega - 2M_K),
\]

- Obtain the same from the BG output and compare to the exact result. In particular the width can be then extracted:

\[
\rho_{Q,0}(\bar{\omega}, 0) = \lim_{\Delta \to 0} \lim_{L \to \infty} \tilde{\rho}_{Q,0}(\bar{\omega}, 0, L, \Delta),
\]

\[
\frac{\Gamma_{\phi \to KK}}{M_\pi} + \frac{\Gamma_{\phi \to \pi\pi}}{M_\pi} = \frac{1}{2M_\phi M_\pi} \lim_{\Delta \to 0} \lim_{L \to \infty} \tilde{\rho}_{Q,0}(M_\phi, 0, L, \Delta).
\]
Be more realistic

- We assume uncertainty on the finite volume Euclidean correlator (as it would be the situation on the lattice).
- This implies regulating the problem in order to give an error estimate on \( \hat{\rho}_{Q,0}(\bar{\omega}, 0, L, \Delta) \)

\[
W_{ij}(\bar{\omega}) = \int_0^\infty d\omega e^{-\omega \tau_i} (\omega - \bar{\omega})^2 e^{-\omega \tau_j},
\]

\[
W_{ij}(\bar{\omega}) \to \lambda_{\text{reg}} W_{ij}(\bar{\omega}) + (1 - \lambda_{\text{reg}}) S_{ij}
\]

**Error estimation:**

\[
\Delta \hat{\rho}_{Q,0}(\bar{\omega}, 0, L, \Delta) = \sqrt{C_i(\bar{\omega}, \Delta) S_{ij} C_j(\bar{\omega}, \Delta)}
\]

Think of \( \lambda_{\text{reg}} \) by asking:
What final uncertainty I want?
Can I achieve a small enough resolution with the number of points that I have?
**DIS (e + p → e' + hadrons)**

"spin-averaged hadronic tensor:"

\[ W_{\mu\nu}(p, q) = \frac{1}{4\pi n_\lambda} \sum_\lambda \int d^4 x \, e^{iq \cdot x} \langle N, p, \lambda | j_\mu(x) j_\nu(0) | N, p, \lambda \rangle, \]

structure functions depend only on invariants and can be projected out:

\[ W_{\mu\nu} = F_1 \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) + \frac{F_2}{p \cdot q} \left( p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left( p_\nu - \frac{p \cdot q}{q^2} q_\nu \right), \]

and in terms of those …

\[ \nu = \frac{q \cdot p}{M}, \quad x = \frac{Q^2}{2M\nu}, \quad Q^2 = -q^2, \quad y = \frac{p \cdot q}{p \cdot k} \]

\[ \frac{d^2 \sigma}{dxdy} = \frac{e^4 M k^0}{2\pi Q^4} \left[ xy^2 F_1 + (1 - y) F_2 \right], \]

following Manohar ’92 hep-ph/9204208 (unpolarized cross-section)

\[ M\nu = E_p p_x^0 - E_p^2 - q \cdot p, \]

\[ Q^2 = q^2 - (p_x^0 - E_p)^2. \]

\[ \{\nu, Q^2\} \iff \{p^2, q^2, p \cdot q, p_x^0\} \]

this redundancy can be exploited in our advantage for building the optimal resolution function!
**DIS** \((e + p \rightarrow e' + \text{hadrons})\) on the lattice

1. Calculate the relevant four point function

\[
\tilde{G}_{\mu
u, p}(\tau, p_x, L) \equiv 2E_p L^6 e^{-E_p \tau} \int d^3 x e^{-i \mathbf{q} \cdot \mathbf{x}} \lim_{\tau_f \rightarrow \infty} \lim_{\tau_i \rightarrow -\infty} \frac{\sum_{\lambda} \langle \Psi_\lambda(\tau_f, \mathbf{p}) j_\mu(\tau, \mathbf{x}) j_\nu(0) \Psi_\lambda^\dagger(\tau_i, \mathbf{p}) \rangle_{\text{conn}}}{\sum_{\lambda} \langle \Psi_\lambda(\tau_f, \mathbf{p}) \Psi_\lambda^\dagger(\tau_i, \mathbf{p}) \rangle},
\]

2. It will contain the desired contribution.

\[
\tilde{G}_{\mu
u, p}(\tau, p_x, L) = e^{-E_p \tau} \langle N, \mathbf{p} | j_\mu(\tau, \mathbf{q}) j_\nu(0) | N, \mathbf{p} \rangle_L + \ldots
\]

\[
= \int_0^\infty dp_0^x W_{\mu\nu}(p_0^x, p_x; L) e^{-p_0^x \tau}
\]

all final states are taken into account!!

3. Apply the Backus-Gilbert method to obtain:

\[
\hat{W}_{\mu\nu}(p_0^x, p_x, L, \Delta) = \int_0^\infty d\omega \delta_\Delta(p_0^x, \omega) W_{\mu\nu}(\omega, p_x, L)
\]

\[
W_{\mu\nu}(p_0^x, p_x) = \lim_{\Delta \rightarrow 0} \lim_{L \rightarrow 0} \hat{W}_{\mu\nu}(p_0^x, p_x, L, \Delta)
\]
other possible decays one can study...

• Semi-leptonic weak decays:

\[ W_{\mu\nu}^{H_Q \to X}(v, q) = \frac{1}{2M_{H_Q}} \int d^4x e^{-iq \cdot x} \langle H_Q, p|J_\mu(x)J_\nu(0)|H_Q, p \rangle, \]

flavour changing current: \[ J_\mu = \bar{q}\gamma_\mu(1 - \gamma_5)Q \]

\[ q^\mu = p_\ell^\mu + p_\nu^\mu, \quad v^\mu = p_H^\mu / M_H \]

• Purely hadronic decays: \[ c \to su\bar{d} \]

\[ H_Q(x) = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} [\bar{s}(x)\gamma_\mu(1 - \gamma_5)c(x)] [\bar{u}(x)\gamma_\mu(1 - \gamma_5)d(x)] \]

D-meson decay

The approach is completely equivalent in both cases to DIS!

There does not seem to be any conceptual problem.
Conclusions

• We have found that it may be feasible to calculate **total widths** from the lattice for QCD-stable states.

• We have made progress on understanding finite volume effects when estimating spectral functions.

• The approach is complementary to other methods like LL. (density of states and subtraction of channels.) Best when slowly varying spectral function.

• As the energy is increased and more and more channels are open, it may be the way to go. Eventually the resolution will be bad.

• Do an exploratory study perhaps followed by a full LQCD calculation.