Condensation thresholds and scattering data - a study in the relativistic Bose gas at finite density

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Introduction

- Sign problem $\Rightarrow e^{-S[\psi]} \not\geq 0$

- Possible solution: Dual formulation $\Rightarrow e^{-S'[\psi']} \geq 0$

- Replacing old fields $\psi$ with new fields $\psi'$

- Different physical interpretation of the new fields, from gauge fields to fluxes with particles worldline variables

- Is it possible to use a dual formulation for something more? Dual formulation $\Rightarrow$ Charge condensation method $\Rightarrow$ Spectroscopy results

- Seminal paper for 2-D $O(3)$: F. Bruckmann, C. Gattringer, Christof, T. Kloiber and T. Sulejmanpasic **PhysRevLett.115.231601**
Dual formulation for $\phi^4$

- The Lagrangian of the system is:

$$S = \sum_x \left( \eta |\phi_x|^2 + \lambda |\phi_x|^4 - \sum_{\nu=1}^4 \left[ e^{\mu \delta_{\nu,4} } \phi_x^* \phi_{x+\hat{\nu}} + e^{-\mu \delta_{\nu,4} } \phi_x^* \phi_{x-\hat{\nu}} \right] \right)$$

Where $\eta = 2D + m^2 \Rightarrow 8 + m^2$

- The partition sum is:

$$Z = \int \mathcal{D}[\phi] e^{-S} \Rightarrow \mathcal{D}[\phi] = \prod_x \int_C \frac{d\phi_x}{2\pi}$$
Dual formulation for $\phi^4$

- The partition function can be rewritten with the following dual variables:

$$Z = \sum_{\{k,l\}} \left( \prod_{x,\nu} \frac{1}{(|k_x,\nu| + l_x,\nu)!l_x,\nu!} \right) \left( \prod_x \delta \left( \sum_\nu [k_x,\nu - k_{x-\hat{\nu}},\nu] \right) \right)$$

$$\times \left( \prod_x e^{\mu k_x,4} W \left( \sum_\nu [k_x,\nu + k_{x-\hat{\nu},\nu} + 2(l_x,\nu + l_{x-\hat{\nu},\nu})] \right) \right)$$

- The new variables are not gauge fields

- $l_{x,\nu} \in \mathbb{N}$ is a background variable

- $l_{x,\nu}$ can be updated with a standard Monte Carlo simulation
Dual formulation for $\phi^4$

- $k_{x,\nu} \in \mathbb{Z}$ is the flux variable subjected to constraints

- The constraints $\prod_x \delta \left( \sum_{\nu} \left[ k_{x,\nu} - k_{x-\hat{\nu},\nu} \right] \right)$ represents the flux conservation for every site $x$

- In two dimensions:

$$k_{x-\hat{0},0} + k_{x,0} + k_{x-\hat{1},1} + k_{x,1} = 0$$

- This means that the fluxes $k_{x,\nu}$ appear only in closed loops

- $k_{x,\nu}$ is usually updated using a Worm algorithm to satisfy the constraints
The chemical potential is acting through the term
\[ \prod_x e^{\mu k_{x,4}} \]

Its meaning is manifest looking at the particle number observable:
\[ N = T \frac{\partial \log Z}{\partial \mu} = \frac{1}{L_T} \left\langle \sum_x k_{x,4} \right\rangle \]

A particle is represented by line of fluxes winding around the temporal side.

The chemical potential helps at locale level:
- fluxes in the particle direction for chemical potential \( \mu > 0 \)
- fluxes in the antiparticle direction for chemical potential \( \mu < 0 \)

Causing the creation of particles or antiparticles
Dual formulation for $\phi^4$

- The weights
  \[ W\left(\sum_{\nu} \left[ |k_{x,\nu}| + |k_{x-\hat{\nu},\nu}| + 2(l_{x,\nu} + l_{x-\hat{\nu},\nu})\right]\right) \]
  are represented by the integral
  \[ W(n) = \int_{0}^{\infty} dr \ r^{n+1} \ e^{-\eta r^2 - \lambda r^4} \]

- These weights can be computed and stored in advance in a lookup table.

- Usually for a simulation $O(\#W) = 100$ is more than enough.

- The weights represent the parameters $\eta$ and $\lambda$ in the dual theory.
Simulation strategy:

1. Update the background $l_{x,\nu}$ with conventional local Monte Carlo steps
2. Update the variable $k_{x,\nu}$ with a worm step
3. Go back to 1

Strictly speaking a fixed order of the updates breaks Detailed Balance....

.... usually this is harmless because the balance condition is still satisfied

This approach can be harmful because the propagation of the worm in a fixed background can produce a large autocorrelation length

A more complex strategy with the update of the background $l_{x,\nu}$ during the worm propagation is advisable
Dual formulation for $\phi^4$

- The worm is a Random Walk on the lattice where the directions are weighted by the partition function.

- A closing worm automatically satisfies the constraints.
Dual formulation for $\phi^4$

- To describe the algorithm it is useful to separate the local part and the unlocal part of the partition function:

$$L_{x,\nu}(k_{x,\nu}) = \frac{e^{\mu \delta_{\nu,4} k_{x,\nu}}}{(|k_{x,\nu}| + l_{x,\nu})! l_{x,\nu}!},$$

$$S_x(\{k_x,\bullet\}) = W \left( \sum_{\nu} [|k_{x,\nu}| + |k_{x-\hat{\nu},\nu}| + 2(l_{x,\nu} + l_{x-\hat{\nu},\nu})] \right)$$

- So the rules are:

$$\rho^{S}_{x_0,\nu_0} = \frac{A}{S_{x_0}(\{k_{x_0,\bullet}\}) S_{x_1}(\{k_{x_1,\bullet}\})} \frac{L_{x_0,\nu_0}(k^{trial}_{x_0,\nu_0})}{L_{x_0,\nu_0}(k_{x_0,\nu_0})}$$

$$\rho^{R}_{x_j,\nu_j} = \frac{S_{x_j}(\{k^{new}_{x_j,\bullet}\})}{S_{x_{j+1}}(\{k_{x_{j+1},\bullet}\})} \frac{L_{x_j,\nu_j}(k^{trial}_{x_j,\nu_j})}{L_{x_j,\nu_j}(k_{x_j,\nu_j})}$$

$$\rho^{T}_{x_j,\nu_j} \bigg|_{j = n-1} = \frac{S_{x_{n-1}}(\{k^{new}_{x_{n-1},\bullet}\})}{A} \frac{S_{x_0}(\{k^{new}_{x_0,\bullet}\})}{L_{x_{n-1},\nu_{n-1}}(k^{trial}_{x_{n-1},\nu_{n-1}})} \frac{L_{x_{n-1},\nu_{n-1}}(k_{x_{n-1},\nu_{n-1}})}{L_{x_{n-1},\nu_{n-1}}(k_{x_{n-1},\nu_{n-1}})}$$
The worm has the following process:
Dual formulation for $\phi^4$

- The starting and terminating probabilities have a free parameter $A$
- It’s possible to use this parameter to tune the average length of a worm

\[ \eta = 7.44, \quad \lambda = 1.0, \quad \mu = 0.275 \]
Dual formulation for $\phi^4$

\[ \eta = 6.0, \quad \lambda = 0.4, \quad \mu = 0.275 \]

- It’s important to have worm long enough if we want to change the winding number
Dual formulation for $\phi^4$

- We check the correctness of the algorithm
- At $\mu = 0$ is possible to compute $|\phi|^2$ with a standard MC
- In the dual formulation the observable $|\phi|^2$ is defined as:

$$\langle |\phi|^2 \rangle = -\frac{T \partial \ln Z}{V} = \frac{1}{L_s^3 L_T} \left\langle \sum_x \frac{W(f_{x+2})}{W(f_x)} \right\rangle$$
Dual formulation for $\phi^4$

For the coupling $\lambda = 0$ there is an analytical expression for the particle number density:

$$n = \frac{T}{V} \frac{\partial \log Z}{\partial \mu} = \frac{1}{L_s^3 L_T} \left\langle \sum_{x} k_{x,4} \right\rangle$$
Charge condensation method

- At finite volume and $T = 0$; 1,2,... particles sectors are separated by finite energy steps.

- The energy steps are represented by threshold for the chemical potential $\mu_1, \mu_2, \ldots$

- Using Lüscher approach it is possible to address the finite volume effects.

- From $\mu_1$ we extract the mass of the lightest particle of the theory.

- From $\mu_1 + \mu_2$ the interaction of two particles: scattering length.
Charge condensation method

- The partition function of the system is described by the Grand canonical partition sum:
  \[ Z(\mu) = \text{Tr}\left( e^{-\frac{(\hat{H} - \mu \hat{Q})}{T}} \right) = e^{-\frac{\Omega(\mu)}{T}} \]

- For \( T \to 0 \), every sector is dominated by the minimum of the Hamiltonian in that sector:
  \[
  \Omega(\mu) \xrightarrow{T \to 0} \left\{ \begin{array}{l}
  E_{\text{min}}^{Q=0} = 0 & \mu \in [0, \mu_1) \\
  E_{\text{min}}^{Q=1} - \mu = m - \mu & \mu \in (\mu_1, \mu_2) \\
  E_{\text{min}}^{Q=2} - 2\mu = W - 2\mu & \mu \in (\mu_2, \mu_3) 
  \end{array} \right.
  \]

- The first transition occurs at:
  \[ 0 = m - \mu_1 \Rightarrow m = \mu_1 \]

- The second transition occurs at:
  \[ m - \mu_2 = W - 2\mu_2 \Rightarrow W = \mu_1 + \mu_2 \]
Charge condensation method

- The limit to a step function can be showed going to larger $L_T$:

$$L_S = 4$$
**Charge condensation method**

- In our case we simulate with parameters $\eta = 7.44$ and $\lambda = 1.0$

- In general it is sufficient to clearly separate the 1 and 2 particles sectors

- To find the threshold chemical potentials: $\mu_1, \mu_2, \ldots$, we fit the particle number with a sigmoid function

- In particular, the logistic function:

$$f(x) = \frac{1}{1 + e^{-k(\mu - \mu_0)}}$$
We can show some examples for the particle number

1-particle, $L_s = 3$, $L_T = 320$

![Graph showing particle number $N$ as a function of chemical potential $\mu$. The graph plots $N$ against $\mu$ with data points and a fitted curve.]
Charge condensation method

2-particles, $L_s = 3$, $L_T = 640$
Charge condensation method

1-particle, $L_s = 10$, $L_T = 640$
Charge condensation method

2-particles, $L_s = 10, L_T = 640$
Charge condensation method

For one particle it's simple to compare our results with conventional spectroscopy results:

![Graph showing spectroscopy and dual formulation results](image-url)
Charge condensation method

- We can use Lüscher’s formulas to describe the finite volume scaling

- For 1 particle in a box we use:

\[ m(L) = m_0 + c L^{-\frac{3}{2}} e^{-m_0 L} \]

- For 2 particles in a box we use:

\[ W(L) = 2m_0 - \frac{4\pi a_0}{mL^3} \left\{ 1 + c_1 \frac{a_0}{L} + c_2 \frac{a_0^2}{L^2} \right\} + O\left( \frac{1}{L^6} \right) \]

\[ c_1 = -2.837297 \quad , \quad c_2 = 6.375183 \]
The results of the fits are:
Charge condensation method

- Comparing with the point at $L_s = 3$ and $L_s = 4$:

- We suppose that order $\mathcal{O}\left(\frac{1}{L^6}\right)$ is not negligible in this case
Giving a more detailed overview of the 1 particle fit we have:
Charge condensation method

For the two particle fit we have:

\[ \mu \]

\[ L \]

\[ \mu_1 + \mu_2 \]
Charge condensation method

The results of the fits are:

- For $m(L)$ we have:
  
  \begin{align*}
  m_0 &= 0.168 \pm 0.001 \\
  c &= 1.508 \pm 0.025
  \end{align*}

- For $W(L)$ we find:
  
  \begin{align*}
  m_0 &= 0.175 \pm 0.002 \\
  a_0 &= -0.232 \pm 0.011
  \end{align*}
Conclusions

- Dual formulation is an approach not limited to the sign problem solution

- Spectroscopy results are available using dual formulation

- A large $L_T$ is necessary to separate 1 and 2 particles sectors

- Up to today there is no dual formulation of a non-abelian theory