Towards P to $\gamma\gamma$ form factors from twisted mass Lattice QCD

ETMC: Bartosz Kostrzewa, Konstantin Ottnad,
Marcus Petschlies, Carsten Urbach

HISKP (Theory), University of Bonn

DESY Zeuthen, February 13th, 2017

Motivation: Chiral Anomaly in QCD

- neutral pion: dominant contribution to $\pi^0 \rightarrow \gamma\gamma$ from ADLER-BELL-JACKIW (ABJ) anomaly

- axial current conservation violated by quantum effects

⇒ so–called chiral anomaly

- ABJ precisely predicts decay rate

$$\mathcal{F}_{\pi^0\gamma\gamma}(0,0) = \frac{1}{4\pi^2 f_0}$$

in the chiral limit and at vanishing photon momenta

- first calculation available

[Feng et al., PRL, (2012)]

Motivation: muon $g-2$ theory vs. experiment

- experiment:

$$a_\mu^{\text{exp}} = (11\,659\,209.1 \pm 6.3) \times 10^{-10}$$

[BNL, (2006), PRD 73 072003, Jegerlehner (2015)]

- theory:

$$a_\mu^{\text{SM}} = (11\,659\,176.37 \pm 5.2) \times 10^{-10}$$

[Jegerlehner (2015)]

- discrepancy in units of 10^{-10}

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = 32.73 \pm 5.2_{\text{th}} \pm 6.3_{\text{exp}} = 32.73 \pm 8.15$$

$\Rightarrow$ 3–5 σ deviation

- planned/upcoming FNAL and J-PARC experiments

$$6.3_{\text{exp}} \quad \to \quad 1.6_{\text{exp}}$$

- theory error?!

SM contributions to $g-2$

SM contributions to a_μ

[Jegelehner, (2015)]

- QED at 5–loops:

$$(11\ 658\ 471.8851 \pm 0.036) \times 10^{-10}$$

- EW at 2–loops

$$(15.40 \pm 0.10) \times 10^{-10}$$

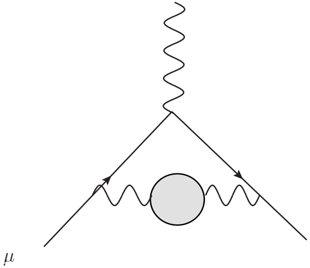
- QCD total

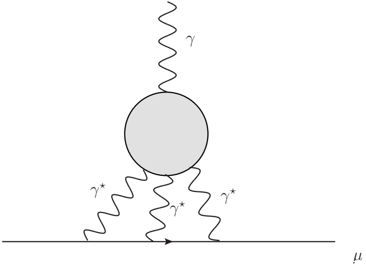
$$(689.082 \pm 5.18) \times 10^{-10}$$

QCD contributions to $g-2$

- QCD total

$$(689.082 \pm 5.18) \times 10^{-10}$$



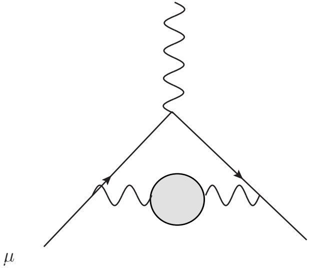


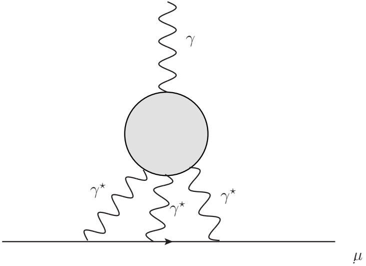
- LO HVP: $(687.19 \pm 3.48) \times 10^{-10}$
- NLO: $(-9.934 \pm 0.019) \times 10^{-10}$
- NNLO: $(1.226 \pm 0.012) \times 10^{-10}$
- HLbL: $(10.6 \pm 3.9) \times 10^{-10}$

QCD contributions to $g-2$

- QCD total

$$(689.082 \pm 5.18) \times 10^{-10}$$





- LO HVP: $(687.19 \pm 3.48) \times 10^{-10}$
- NLO: $(-9.934 \pm 0.019) \times 10^{-10}$
- NNLO: $(1.226 \pm 0.012) \times 10^{-10}$

- HLbL: $(10.6 \pm 3.9) \times 10^{-10}$

HLbL in Lattice QCD

two approaches

- inclusive approach

  [Blum, Chowdbury, Hayakawa, Izubuchi, PRL, (2014)]

  attempt to compute the complete HLbL contribution directly

- dispersive approach

  [Jegerlehner, Nyffeler, Phys. Rept. (2009), Colangelo, Hoferichter, Procura, Stoffer, JHEP, (2014, 2015), Pauk, Vanderhaegen, PRD (2014)]

$$\Pi^{\mathrm{HLbL}}_{\mu\nu\lambda\sigma} = \Pi^{\pi^0}_{\mu\nu\lambda\sigma} + \Pi^{\mathrm{FsQED}}_{\mu\nu\lambda\sigma} + \Pi^{\eta}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \Pi^{\eta'}_{\mu\nu\lambda\sigma} + \ldots$$

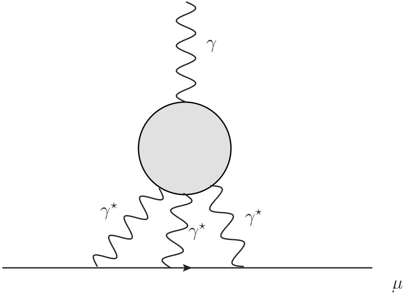
Inclusive Approach

- direct computation in QCD only prohibitively costly

- way out: QCD + QED computation

[Hayakawa, Blum, Izubuchi, Yamada, (2005)]

$$\langle \psi\, j_\mu\, \bar{\psi} \rangle_{\mathrm{HLbL}} = \langle A \cdot B \rangle_{\mathrm{QCD+QED}} \\ - \langle A \rangle_{\mathrm{QCD+QED}} \cdot \langle B \rangle_{\mathrm{QED}}$$



- $\psi, \bar{\psi}$: state with muon QN

- j_μ: EM current

- relies on delicate cancellation

  - first benchmark calculation shows feasibility

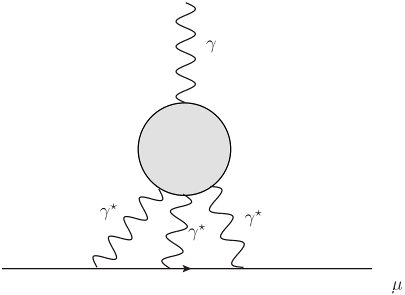
[Blum, Chowdhury, Hayakawa, Izubuchi, PRL, 2014)]

Dispersive approach

- HLbL tensor decomposed

$$\Pi^{\mathrm{HLbL}}_{\mu\nu\lambda\sigma} = \Pi^{\pi^0}_{\mu\nu\lambda\sigma} + \Pi^{\mathrm{FsQED}}_{\mu\nu\lambda\sigma} + \Pi^{\eta}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \Pi^{\eta'}_{\mu\nu\lambda\sigma} + \ldots$$

- uniquely defined by analytic structure
- unambiguously related to certain physical intermediate states
- FsQED: scalar QED dressed by pion vector form factor (completely determined by F_π^V)

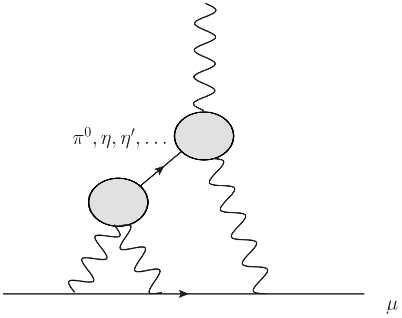


Dispersive approach

- HLbL tensor decomposed

$$\Pi^{\mathrm{HLbL}}_{\mu\nu\lambda\sigma} = \Pi^{\pi^0}_{\mu\nu\lambda\sigma} + \Pi^{\mathrm{FsQED}}_{\mu\nu\lambda\sigma} + \Pi^{\eta}_{\mu\nu\lambda\sigma} + \Pi^{\pi\pi}_{\mu\nu\lambda\sigma} + \Pi^{\eta'}_{\mu\nu\lambda\sigma} + \ldots$$

- uniquely defined by analytic structure
- unambiguously related to certain physical intermediate states
- FsQED: scalar QED dressed by pion vector form factor (completely determined by F_π^V)



Dispersive approach

- e.g. the pion pole contribution

$$a_\mu^{\pi^0} \propto \int \frac{\mathrm{d}^4 q_1}{q_1^2}\ \frac{\mathrm{d}^4 q_2}{q_2^2} \times \\ \left[\frac{\mathcal{F}(q_1^2, q_2^2)\mathcal{F}(s,0)}{s - M_{\pi^0}^2} T_1(q_1, q_2, p) + \frac{\mathcal{F}(s, q_2^2)\mathcal{F}(q_1^2, 0)}{q_1^2 - M_{\pi^0}^2} T_2(q_1, q_2, p)\right]$$

- $\mathcal{F}$: transition form factor $\pi^0 \to \gamma^\star\gamma^\star$
- p momentum of the muon, $s = (q_1 + q_2)^2$
- $T_{1,2}$ known kinematic functions (weight functions)
- Wick rotation $\rightarrow$ only space–like momenta

Dispersive approach

- inclusive dispersive approach appears not feasible
- expansion in mass can allow for a data driven way of computing HLbL
- and provides an alternative way for a Lattice contribution
- con: no obvious power counting, apart from the particle mass
- π^0, η, η' are expected to contribute around 80% to total HLbL
- η, η' still responsible for around 25%

[Nyffeler, PRD94 (2016)]

Model Predictions

approximate contributions from vector meson dominance (VMD) model

- neutral pion

$$a_{\mu;\mathrm{VMD}}^{\pi^0} \; = \; 57.0 \times 10^{-11}$$

- η meson

$$a_{\mu;\mathrm{VMD}}^{\eta} \; = \; 14.5 \times 10^{-11}$$

- η' meson

$$a_{\mu;\mathrm{VMD}}^{\eta'} \; = \; 12.5 \times 10^{-11}$$

- $\sum_P = 84 \times 10^{-11}$ compared to 106×10^{-11} HLbL total ($\approx 80\%$)

[e.g. Nyffeler, PRD94 (2016)]

Weight Functions and Momentum Range

- after WICK rotation, we are left with integrals like

$$\int_0^\infty \mathrm{d}q_1 \int_0^\infty \mathrm{d}q_2 \int_{-1}^1 \mathrm{d}t\; T_1(q_1, q_2, t)\, \mathcal{F}(-q_1^2, -(q_1+q_2)^2)\, \mathcal{F}(-q_2^2, 0)$$

- weight functions $T_{1,2}$ are universal and known

[Jegerlehner, Nyffeler, Phys.Rept.477(2009), Nyffeler, PRD94 (2016)]

- dependence on the state P only via the mass M_P

$\Rightarrow$ pion: weight functions peaked at small momenta below 1 GeV

$\Rightarrow$ η, η': need momenta maybe up to 2.5 GeV

[e.g. Nyffeler, PRD94 (2016)]

$\mathcal{F}_{P\to\gamma\gamma}$ from LQCD

- in MINKOWSKI space-time defined

$$\varepsilon_{\mu\nu\alpha\beta}\; q_1^\alpha\; q_2^\beta\; \mathcal{F}_{\pi^0\gamma^\star\gamma^\star}(q_1^2, q_2^2) \;=\; \mathcal{M}_{\mu\nu}(p, q_1) \;=\; i^n \mathcal{M}_{\mu\nu}^E$$

- EUCLIDEAN matrix element M^E

$$\mathcal{M}_{\mu\nu}^E \;=\; -\int \mathrm{d}t e^{\omega_1 t} \int \mathrm{d}^3 z e^{-i\vec{q}_1 \vec{z}}\; \langle 0|T\{J_\mu(\vec{z}, t) J_\nu(0)\}|\pi(p)\rangle$$

with four momentum $q_1 = (\omega_1, \vec{q}_1)$

[Ji, Jung, (2001,2001), Dudeck, Edwards, (2006), Feng et al., (2012)]

$\rightarrow$ consider EUCLIDEAN three-point function

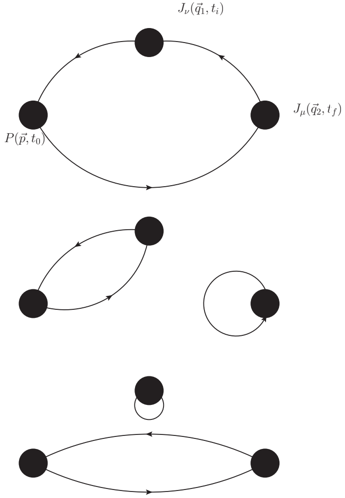
$$C(t', t'') \;=\; \sum_{\vec{x},\vec{y}} \langle T\{J_\mu(\vec{y}, t_f)\; J_\nu(\vec{0}, t)\; P^\dagger(\vec{x}, t_i)\}\rangle e^{i\vec{p}\vec{x}} e^{-i\vec{q}_1\vec{y}}$$

with $t' = t_f - t$
and $t'' = \min\{t - t_i, t_f - t_i\}$ [see e.g., Gérardin et al., (2016)]

$\mathcal{F}_{P\to\gamma\gamma}$ from LQCD: contractions

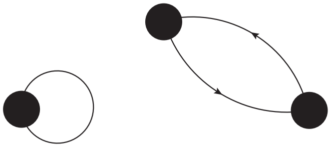
- dominant connected contribution
- two disconnected contractions with vector loops
- disconnected contributions found to be small (actually negligible within errors)

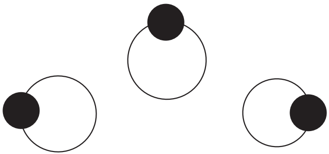
  [Gérardin et al., (2016)]
- this might depend on the pion mass!
- physically: this is all for the π^0



$\mathcal{F}_{P\to\gamma\gamma}$ from LQCD: contractions

- η, η' require contractions including the strange quark
- and additional contraction types needed
- one singly disconnected contribution
- one triply disconnected contribution
- in twisted mass: those also contribute for the π^0 case





$\mathcal{F}_{P\to\gamma\gamma}$ from LQCD

- asymptotically, for large t'' one finds (for $t' > 0$)

$$\lim_{t''\to\infty} C(t',t'') \;\propto\; \sum_{\vec{z}} \langle 0|J_\mu(\vec{z},t')J_\nu(0)|P(p)\rangle e^{-iq_1\vec{z}} e^{-E_\pi t''}$$

  and similarly for $t' < 0$

$\to$ we first need to extrapolate in t''
(minimal distance between pion and the two vector currents)

- EUCLIDEAN matrix element M^E then obtained from an integral in t'

- normalisation obtained from pion two-point function

Choice of Lattice Actions

ETMC provides a variety of ensembles

- $N_f = 2 + 1 + 1$ twisted mass
  - three values of the lattice spacing
  - large range of pion mass values, $M_\pi \geq 230\ \mathrm{MeV}$
- $N_f = 2$ twisted mass plus clover
  - one value of the lattice spacing $a \approx 0.1\ \mathrm{fm}$
  - pion mass from $130\ \mathrm{MeV}$ up to $350\ \mathrm{MeV}$
  - two volumes at the physical point: $L \approx 4\ \mathrm{fm}$ and $L \approx 6\ \mathrm{fm}$
- $N_f = 2 + 1 + 1$ twisted mass plus clover
  - two values of the lattice spacing
  - pion mass from $130\ \mathrm{MeV}$ up to $250\ \mathrm{MeV}$
  - in production...

Twisted Mass Opportunity: η and η'

- in Wilson twisted mass LQCD at maximal twist

$\Rightarrow$ $\mathcal{O}(a)$ improvement paired with isospin breaking

- in tmLQCD π^0:
  - obtains significant disconnected contributions!
  - $M_{\pi^\pm}$ in general not equal to M_{π^0}

- but, η, η' can be computed quite efficiently!

  $\Rightarrow$ efficient noise reduction available

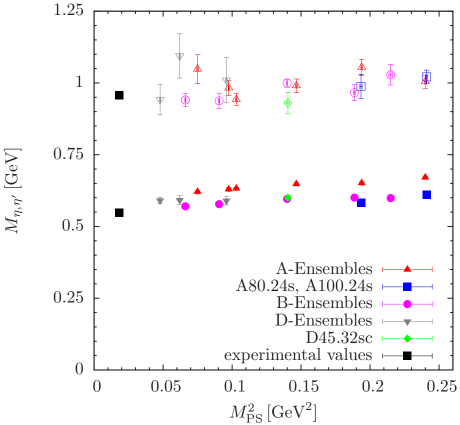
  [Michael, CU, PoS LAT2007 (2007)]

  $\Rightarrow$ lot of prerequisites available

  [Ottnad, Michael, CU, (2011-2014)]

M_η and $M_{\eta'}$ from tmLQCD

- η and η' mesons computed for $N_f = 2+1+1$ ensembles
- three lattice spacing values
- able to resolve strange quark mass dependence in M_η
- lattice artefacts small, but visible
- optimal interpolating operator for η and η' from variational approach



$N_f = 2$ Twisted plus Clover

- fermion action

$$S_\ell^{\mathrm{tm}} = \sum_x \bar{\chi}_\ell \left[D_W(U) + m_0 + i\mu_\ell\gamma_5\tau^3 + \frac{i}{4}c_{\mathrm{sw}}\sigma^{\mu\nu}\mathcal{F}^{\mu\nu}(U) \right] \chi_\ell(x)\,,$$

- $c_{\mathrm{sw}} = 1.57551$ set to its non-perturbative value

  [Aoki et al., (2006)]

- Iwasaki gauge action with $\beta = 2.10$ $\quad a \approx 0.09\ \mathrm{fm}$

  [Iwasaki, (1985)]

- clover term not added for $\mathcal{O}(a)$ improvement

- but to reduce the effects of isospin splitting

- pion masses range from 130 to 350 MeV

Neutral Pion

- neutral pion acquires sizable disconnected contribution in tmLQCD
- interpolating operator projected to zero-momentum

$$O_{\pi^0}(t) \ = \ \sum_{\vec{x}} \frac{1}{\sqrt{2}}(\bar{u}i\gamma_5 u - \bar{d}i\gamma_5 d)(\vec{x},t)$$

- and correlation function

$$C_{\pi^0}(t-t') \ = \ \langle\, O_{\pi^0}(t)\; O^{\dagger}_{\pi^0}(t')\,\rangle$$

- in tmLQCD, the corresponding loop has a VEV

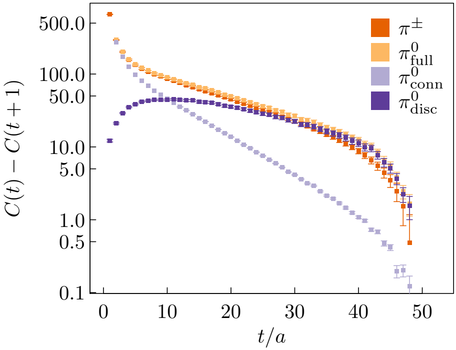
$\Rightarrow$ can be subtracted via

$$C^{\mathrm{sub}}_{\delta t}(t) \ = \ C_{\pi^0}(t) - C_{\pi^0}(t+\delta t)\,, \qquad \delta t > 0$$

- also reduces correlation between different t-values significantly

Neutral Pion

- example: physical point ensemble with $L/a = 48$
- disconnected contribution dominant for large t
- charged an neutral pion show very similar decay rate

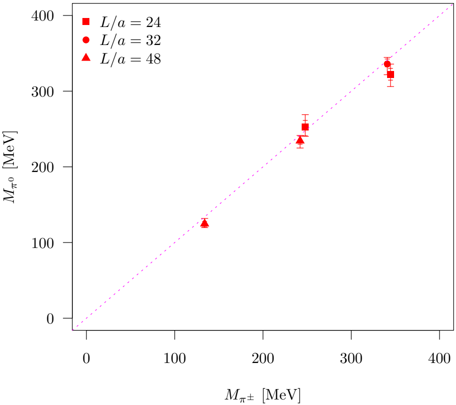


- extracted masses agree within errors
- compare to 40% mass splitting observed without clover term

Neutral Pion

- neutral versus charged pion mass
- all five ensembles show similar picture

$\Rightarrow$ pion mass splitting compatible with zero



- important implications
  - reduced finite size effects
  - reduces isospin splitting effects

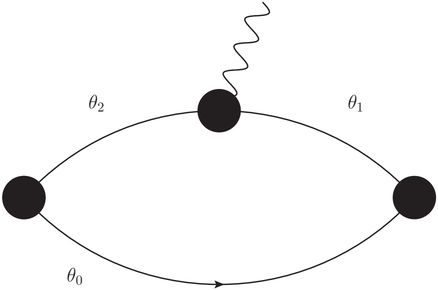
Test case: Pion Vector Form Factor

Electromagnetic pion form factor

$$F_\pi(-(p-p')^2)(p+p')_\mu = \langle \pi^+(p')|V_\mu(0)|\pi^+(p)\rangle$$

- vector current V_μ
- small momenta due to twisted boundary conditions

  [Divitis, Petronzio, Tantalo, (2004)]
- BREIT-frame $\theta_1 = -\theta_2$, $\theta_0 = 0$
- sufficient to study V_0



Pion Vector Form Factor

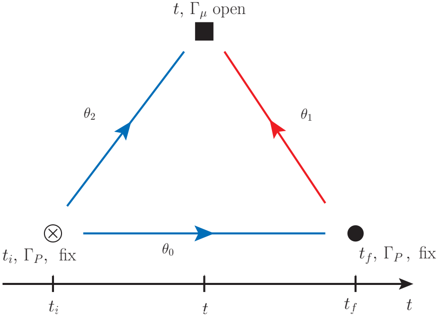
requires to estimate the 3pt function

$$C_0(t'', t', \vec{p}, \vec{p}') = \sum_{x,y,z} \left\langle O_\pi(y) V_0(x) O_\pi^\dagger(z) \right\rangle \delta_{t'',t_x-t_z} \delta_{t',t_y-t_z} e^{-i\vec{p}(\vec{x}-\vec{z})+i\vec{p}'(\vec{x}-\vec{y})}$$

- then, up to normalisation

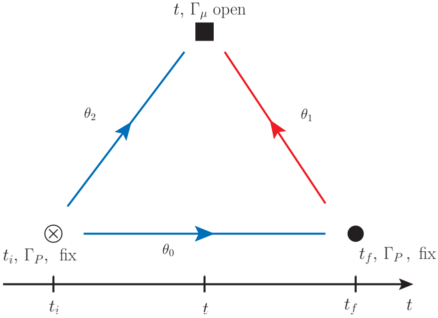
$$\lim_{t'',t'-t''\to\infty} C_0(t'', t', \vec{p}, -\vec{p}) \propto F_\pi(q^2)$$

- typically, t' kept fixed
- here $t' = T/2$
- can be estimated using sequential propagators



Stochastic Estimator

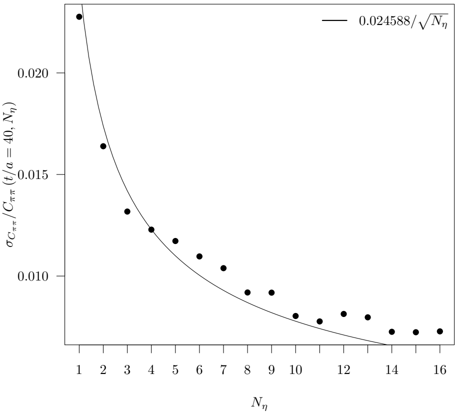
- time slice Z_2 noise sources in spin, colour and space at t_i
- $t_f - t_i = T/2$ fix
- insertion open at t
- sequential propagator (red) at t_f



- perform inversions on N_η sources per gauge
- sample t_i uniformly in $\{0, 1, \ldots, T-1\}$

Error Scaling

- keep the number of gauges fix to 60
- N_η: number of sources per gauge
- relative error at $t/a = 40$
- physical point ensemble with $T/a = 96$



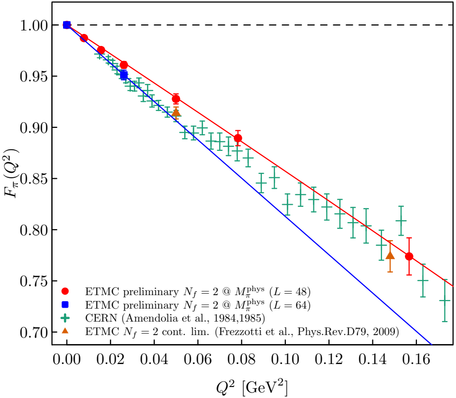
- $N_\eta^{-1/2}$ scaling observed until $N_\eta \approx T/a/8$
- similar results for other volumes

Pion Vector Form Factor

- two volumes ($L/a = 48, 64$) at the physical point
- compared to experimental data

  [CERN, Amendolia et al., (1984, 1985)]
- also compared to previous $N_f = 2$ computation by ETMC

  [Frezzotti et al., (2009)]



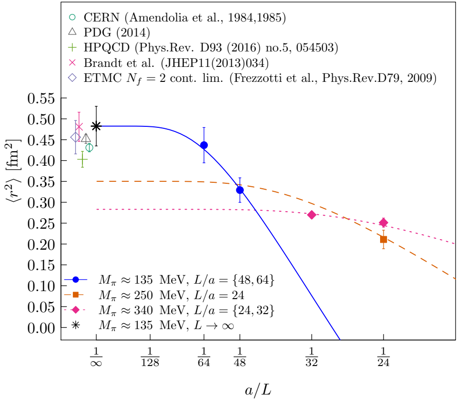
- further momenta in production
- remarkable precision, sizable finite volume effects

Pion Charge Radius: Volume Effects

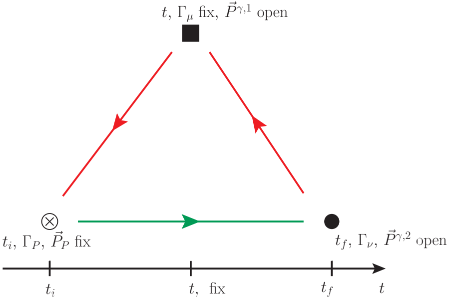
$$F_\pi(Q^2) = 1 - \frac{\langle r^2 \rangle}{6} Q^2 + cQ^4 + \ldots$$

- radius determined for every ensemble separately
- effective fit ansatz for chiral and infinite volume extrapolations

  [Koponen et al., (2016)]
- good agreement with phenomenology and lattice
- more data in production
- no continuum extrapolation



Strategy for $P \to \gamma^\star\gamma^\star$: Connected Contribution



- in principle:
  all-to-all method needed

$\Rightarrow$ we need all t_i and t_f

- assume we fix t and invert from point sources

$\Rightarrow$ need to close with sequential inversions for all t_i and $\vec{x}$

- or close with all-to-all propagators

- one possibility: stochastic volume sources ξ with

$$\lim_{R\to\infty}[\xi_i^\star \cdot \xi_j]_R \;=\; \delta_{ij}\,, \qquad \lim_{R\to\infty}[\xi_i \cdot \xi_j]_R \;=\; 0$$

Strategy for $P \to \gamma^\star\gamma^\star$: Connected Contribution

- one possibility: stochastic volume sources ξ with

$$\lim_{R\to\infty}[\xi_i^\star \cdot \xi_j]_R \;=\; \delta_{ij}\,, \qquad \lim_{R\to\infty}[\xi_i \cdot \xi_j]_R \;=\; 0$$

- reconstruct propagator stochastically

$$\phi^r = D^{-1}\xi^r\,, \qquad [\xi_i^{r\star}\phi_j^r]_R = D_{ij}^{-1} + \mathrm{noise}$$

- known to not work well for case at hand $\quad\to\quad$ too expensive

- the signal to noise ratio is quite low, in particular for large volume

- can be improved by dilution

Strategy for $P \to \gamma^\star\gamma^\star$: Connected Contribution

- problem can be overcome using low and high mode separation

  [DeGrand, Schäfer, (2004), Giusti et al., (2004)]

- compute the N_{EV} lowest eigenvalues and eigenvectors of the hermitian DIRAC operator exactly

- those can be used to build the low-mode part of the propagator without noise

$$S \;=\; S^{\mathrm{low}} + S^{\mathrm{high}}$$

- works well when low modes are dominant

- the part corresponding to the high modes can be estimated stochastically

- eigenvectors can also be used to deflate the CG solver

Even/Odd Preconditioning

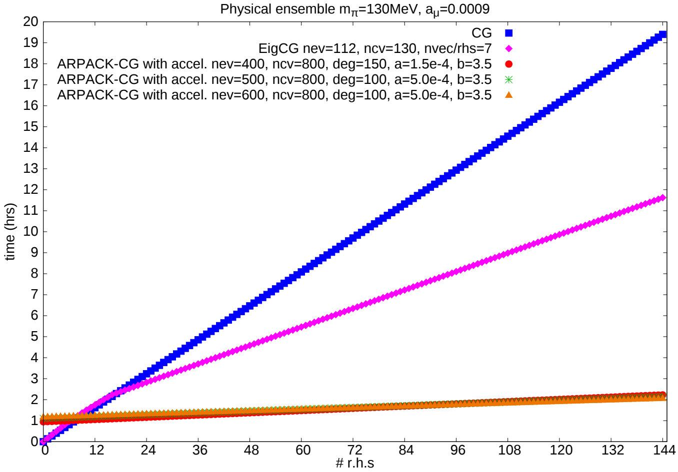
- even/odd preconditioning speeds up CG inversion by a factor > 2
- reduces memory food print
- use deflated even/odd preconditioned CG
- eigenvalues computed using ARNOLDI with TSCHEBYSHEFF acceleration

$\Rightarrow$ we have generalised low-mode-averaging to the case of twisted mass even/odd preconditioning

[M. Petschlies]

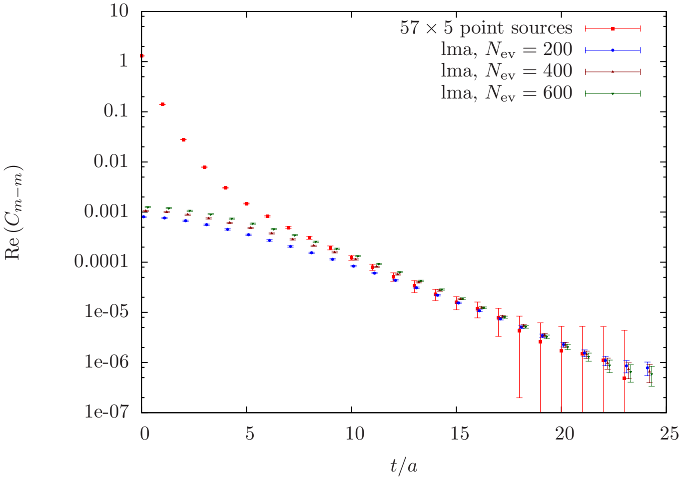
- derivation is lengthy, but does not give much insights

Solver Speedup @M_π^{phys}



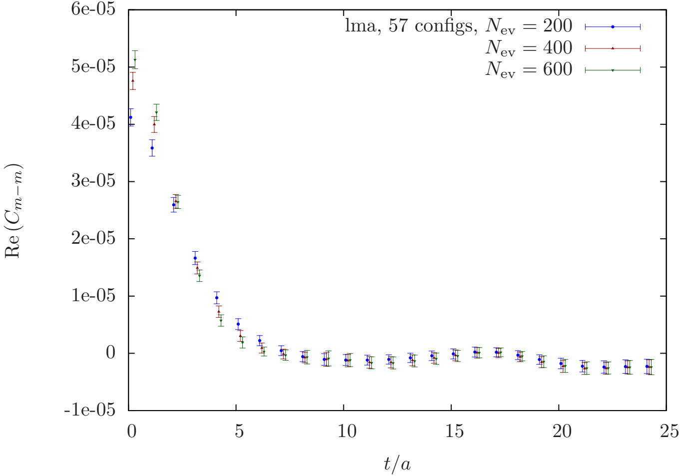
$\Rightarrow$ exactly deflated CG improves by large factors over CG

Neutral Connected 2pt Vector Correlator

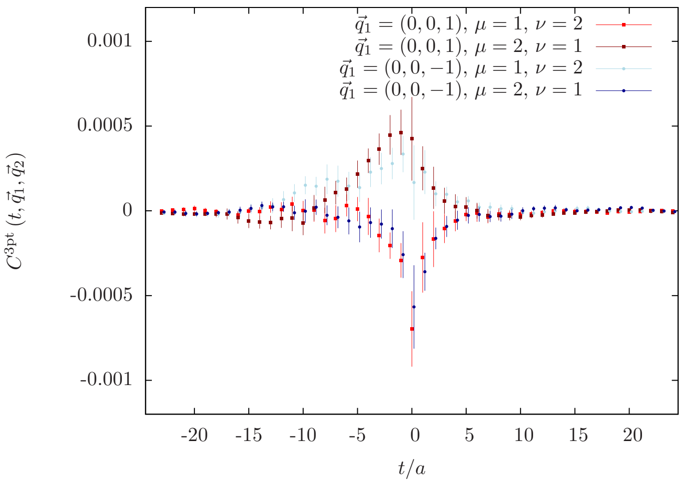


- example for $N_f = 2 + 1 + 1$ ensemble with $M_\pi \approx 250$ MeV, $T/a = 48$

Neutral Disconnected 2pt Vector Correlator



Example Three Point Function



- here $q_1 = -q_2$

Summary and Outlook

- discussed ways to estimate $P \to \gamma^\star\gamma^\star$ form factors from lattice QCD
- discussed our strategy to estimate the corresponding 3pt function
- showed results for the pion vector form factor as a test case
- showed preliminary results for the 3pt function
- calculation appears feasible and has started
- low mode averaging can be extended to further propagators

Thanks to ...

- the lattice QCD group in Bonn:
  C. Helmes, C. Jost, B. Knippschild, B. Kostrzewa, L. Liu, M. Oehm, M. Petschlies, F. Pittler, M. Werner

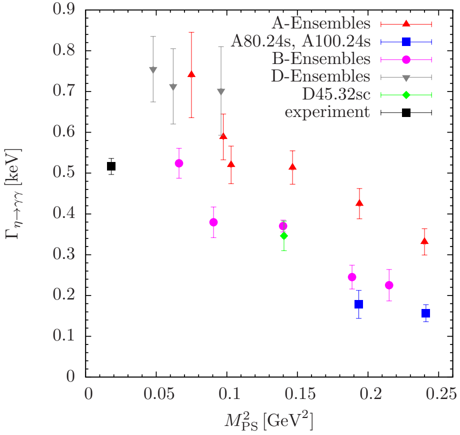
- the DFG funding the Sino-German CRC 110

- R. Frezzotti, K. Jansen, C. Michael, U.-G. Meißner, A. Rusetsky

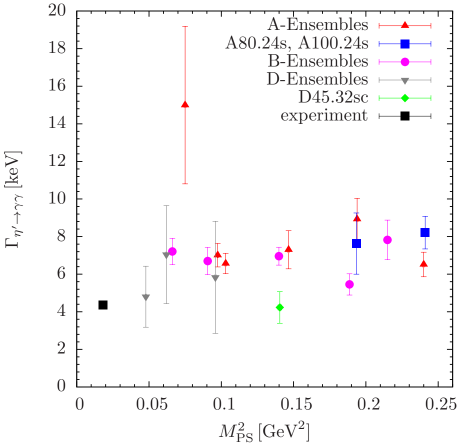
- the ETM collaboration

- ... and for your attention!

$\eta, \eta' \rightarrow \gamma\gamma$ decay-width



$\eta, \eta' \quad \to \quad \gamma\gamma$ decay-width



Pion Charge Radius

- effective fit ansatz

$$\left\langle r_L^2(M_\pi)\right\rangle = \left(1 - c_L \cdot \frac{\exp(-M_\pi \cdot L)}{(M_\pi \cdot L)^\alpha}\right)$$
$$\left[\left\langle r_\infty^2(M_\pi^{\mathrm{phys}})\right\rangle - \frac{1}{\Lambda^2} \cdot \ln\left(\left[\frac{M_\pi}{M_\pi^{\mathrm{phys}}}\right]^2\right)\right] ,$$