Pseudoscalar mesons in a finite cubic volume with twisted boundary conditions

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Outline

- Introduction
  - Finite volume effects in ChPT
  - Twisted Boundary Conditions (TBC)
- FVE for TBC in ChPT to one loop
- Asymptotic formulae
  - Lüscher formula
  - Resummation of higher exponentials
  - Extension to decay constants
  - Generalization of AF to TBC
  - Chiral Ward Identities for TBC
- Summary

Work done in collaboration with Alessio Vaghi, JHEP 2016

Introduction

ChPT: expansion in m_{q_l}/Λ and p/Λ

In finite volume the momentum is quantized: $p = \frac{2\pi}{L} n$

Condition of applicability of ChPT:

$$m_{q_l} \ll \Lambda \quad \text{and} \quad \frac{2\pi}{L} \ll \Lambda$$

$$\Lambda \sim 4\pi F_\pi \quad \Rightarrow \quad 2LF_\pi \gg 1$$

Once this condition is respected we still have two different physical situations

$$LM_\pi \lesssim 1 \quad \Rightarrow \quad \epsilon\text{–regime} \quad M_\pi \sim \frac{1}{L^2} \sim O(\epsilon^2)$$

$$LM_\pi \gg 1 \quad \Rightarrow \quad p\text{–regime} \quad M_\pi \sim \frac{1}{L} \sim O(p)$$

p– or ϵ–regime?

Two alternatives:

- Chiral limit on the lattice $\Rightarrow$ ϵ–regime

  (unless one can simulate enormous volumes)

  $\Rightarrow$ Rely on ChPT to relate unphysical observables to physical quantities

- $M_\pi > M_\pi^{\text{phys}}$: choose $L \gg 1/M_\pi$, $\Rightarrow$ p–regime

  (e.g. $M_\pi = 300$ MeV, $L = 2$ fm, $M_\pi L \sim 3$)

  $\Rightarrow$ Rely on ChPT to make the chiral and the large volume extrapolation

p-regime

Calculational rule in ChPT for isotropic finite box with periodic boundary conditions (PBC):

- the Lagrangian is the same as in infinite volume
- the propagators must be made periodic:

$$G_L(\vec{x}, t) = \sum_{\vec{n}} G_\infty(\vec{x} + \vec{n}L, t)$$

p-regime

Calculational rule in ChPT for isotropic finite box with periodic boundary conditions (PBC):
Examples: Gasser and Leutwyler (88)

$$\begin{aligned} M_\pi(L) &= M_\pi \left[1 + \frac{1}{2N_f}\xi\, g_1(\lambda) + O(\xi^2)\right] \\ F_\pi(L) &= F_\pi \left[1 - \frac{N_f}{2}\xi\, g_1(\lambda) + O(\xi^2)\right] \end{aligned}$$

with

$$\lambda = M_\pi L, \;\; \xi = (M_\pi / 4\pi F_\pi)^2$$

$$g_1(\lambda) = {\sum_{\vec{n}}}' \int_0^\infty dz\; e^{-\frac{1}{z} - \frac{z}{4}\vec{n}^2\lambda^2} = \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L, t)|_{t=\vec{x}=0}$$

Twisted Boundary Conditions (TBC)

- PBC $\Leftrightarrow$ $\vec{p} = \frac{2\pi}{L}\vec{n}$ pose a serious limitation:
  testing small variations of $\vec{p}$ only possible for $L \to \infty$
- TBC offers a solution: $\vec{p}_{\rm min} \neq \vec{0}$ can be chosen freely

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- Define fields as periodic up to an $SU(3)$ transformation – a diagonal phase factor de Divitiis, Petronzio, Tantalo (2004)

$$q_f(x + L\hat{e}_j) = e^{-iL\hat{e}_j\vartheta_f} q_f(x) \qquad \text{for} \quad j = 1, 2, 3$$

$$\vartheta_f^\mu = \begin{pmatrix} 0 \\ \vec{\vartheta}_f \end{pmatrix} \qquad \text{twisting angles}$$

- Momenta are shifted by an external parameter $\vec{\vartheta}_f$

$$\vec{p}_{\rm TBC} = \vec{p}_{\rm PBC} + \vec{\vartheta}_f = \frac{2\pi}{L}\vec{m} + \frac{\vec{\theta}_f}{L} \qquad \text{with} \quad \theta_f^j \in [0, 2\pi)$$

TBC - a different formulation

Relate twisted to periodic quark fields: Sachrajda, Villadoro 2005

$$q_T(x) = \mathcal{V}(x)q(x) \qquad \mathcal{V}(x) = e^{-iv_\vartheta x}$$

where $v_\vartheta^\mu = \mathrm{diag}(\vartheta_u^\mu, \vartheta_d^\mu, \vartheta_s^\mu)$: $\langle v_\vartheta^\mu \rangle = 0$, $[v_\vartheta^\mu, \mathcal{M}] = 0$.

Change in the Lagrangian

$$\bar{q}_T(x)\left[i\not{D} - \mathcal{M}\right]q_T(x) \longrightarrow \bar{q}(x)\left[i\left(\not{D} - i\not{v}_\vartheta\right) - \mathcal{M}\right]q(x)$$

Generators not commuting with v_ϑ^μ are broken symmetries:
e.g. cubic symmetry
For three different twisting angles:
$\mathrm{SU}(3)_V$ and isospin symmetry
Still conserved: I_3, S, Q_e

ChPT with TBC

The latter formulation is very practical for ChPT with TBC:

Sachrajda, Villadoro 2005

$$\mathcal{L}_{\mathrm{QCD}}(q, v, a, s, p) \longrightarrow \mathcal{L}_{\mathrm{QCD}}(q, v + v_\vartheta, a, s, p)$$
$$\mathcal{L}_{\mathrm{ChPT}}(U, v, a, s, p) \longrightarrow \mathcal{L}_{\mathrm{ChPT}}(U, v + v_\vartheta, a, s, p)$$

Alternatively, we can start with U_T satisfying:

$$\text{TBC:} \quad U_T(x + L\hat{e}_j) = e^{iL\hat{e}_j v_\vartheta} U_T(x) e^{-iL\hat{e}_j v_\vartheta}$$

transform it to U, $U_T(x) = e^{iv_\vartheta x} U(x)\, e^{-iv_\vartheta x}$, which satisfies

$$\text{PBC:} \quad U(x + L\hat{e}_j) = U(x)$$

The change in the Lagrangian is:

$$\mathcal{L}_{\mathrm{ChPT}}(U_T, v, a, s, p) = \mathcal{L}_{\mathrm{ChPT}}(U, v + v_\vartheta, a, s, p)$$

Since $D^\mu U_T = D^\mu U - i[v_\vartheta^\mu, U] =: \hat{D}^\mu U$

ChPT with TBC

We work with:

$$\mathcal{L}_{\mathrm{ChPT}}(U, v + v_\vartheta, a, s, p)$$

The shifts in the momenta of the mesonic fields follow from their quark content and the shifts for the quarks:

E.g.

$$p^\mu + \vartheta^\mu_{\pi^+} = p^\mu + (\vartheta^\mu_u - \vartheta^\mu_d) \qquad \vartheta^\mu_{\pi^-} = -\vartheta^\mu_{\pi^+}$$

and similarly for the other mesons

Antiparticles have twisting angles of opposite sign

π^0, η have no twisting angles

ChPT with TBC

- Twisting angles enter expressions of external states and propagators, e.g.

$$G_{\pi^+,L}(x) = \frac{1}{L^3} \sum_{\substack{\vec{k}=\frac{2\pi}{L}\vec{m} \\ \vec{m}\in\mathbb{Z}^3}} \int_{\mathbb{R}} \frac{dk_0}{(2\pi)} \frac{e^{-ikx}}{M_\pi^2 - (k+\vartheta_{\pi^+})^2}$$

- Calculation of physical observables is similar as for PBC:
  - tree graphs generate same contribution with momenta shifted by twisting angles
  - in loop diagrams, one must keep track of virtual particles and their twisting angles
  - breaking of cubic invariance $\Longrightarrow$ extra terms

Outline

Mass Corrections

- Mass corrections are defined by poles of full propagators, e.g.

$$M_K^2 - (p + \vartheta_{K^+})^2 - \Delta\Sigma_{K^+} = 0$$

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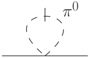
$$\Delta\Sigma_{K^+} = \frac{1}{2}\Bigg[$$



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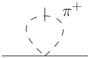

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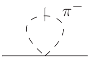

$$+ \dots \Bigg]$$

$$= \Delta A_{K^+} + \Delta B_{K^+} (p + \vartheta_{K^+})^2 + 2\,(p + \vartheta_{K^+})_\mu \Delta\vartheta^\mu_{\Sigma_{K^+}} + \dots$$

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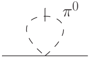
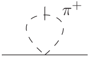
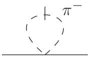
$\Delta\Sigma_{K^+} = \frac{1}{2}\Big[$ 

$+$ 

$+$ 

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$$= \Delta A_{K^+} + \Delta B_{K^+} (p + \vartheta_{K^+})^2 + 2\,(p + \vartheta_{K^+})_\mu \Delta\vartheta^\mu_{\Sigma_{K^+}} + \dots$$

- Terms also present for PBC

$$\Delta A_{K^+},\ \Delta B_{K^+} \propto \frac{1}{L^3} \sum_{\substack{\vec{k}=\frac{2\pi}{L}\vec{m} \\ \vec{m}\in\mathbb{Z}^3\setminus\{\vec{0}\}}} \int_{\mathbb{R}} \frac{dk_0}{2\pi} \frac{1}{i\left[M_P^2 - (k + \vartheta_P)^2\right]}$$

can be evaluated with Poisson resummation formula

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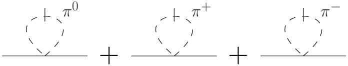

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$$= \Delta A_{K^+} + \Delta B_{K^+} (p + \vartheta_{K^+})^2 + 2\,(p + \vartheta_{K^+})_\mu \Delta\vartheta^\mu_{\Sigma_{K^+}} + \dots$$

- Extra terms originating from breaking of cubic invariance

$$\Delta\vartheta^\mu_{\Sigma_{K^+}} \propto \frac{1}{L^3} \sum_{\substack{\vec{k} = \frac{2\pi}{L}\vec{m} \\ \vec{m} \in \mathbb{Z}^3 \setminus \{\vec{0}\}}} \int_{\mathbb{R}} \frac{dk_0}{2\pi} \frac{k^\mu}{i\left[M_P^2 - (k + \vartheta_P)^2\right]}$$

can be evaluated with Poisson resummation formula

Mass Corrections

- At NLO extra terms can be reabsorbed in on-shell conditions

$$(p + \vartheta_{K^+} + \Delta\vartheta_{\Sigma_{K^+}})^2 = M_{K^+}^2(L)$$

Sachrajda, Villadoro 2005, Jiang, Tiburzi 2007, see also Cherman, Sen, Wagman, Yaffe 2016

Bijnens, Relefors 2014 adopt a different definition

Mass Corrections

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$$(p + \vartheta_{K^+} + \Delta\vartheta_{\Sigma_{K^+}})^2 = M_{K^+}^2(L)$$

- Corrections are momentum-independent and given by

$$\begin{aligned}\delta M_{K^+}^2 &= \frac{M_{K^+}^2(L) - M_K^2}{M_K^2} \\ &= -\frac{\Delta A_{K^+}}{M_K^2} - \Delta B_{K^+} + \mathcal{O}(p^4/F_\pi^4)\end{aligned}$$

Sachrajda, Villadoro 2005, Jiang, Tiburzi 2007, see also Cherman, Sen, Wagman, Yaffe 2016

Bijnens, Relefors 2014 adopt a different definition

Remarks

- Derivations of mass corrections for $\pi^{\pm}, K^{-}, \bar{K}^{0}$ are similar:
  - extra terms ($\Delta\vartheta^{\mu}_{\Sigma_{\pi^{\pm}}}, \ldots$) originate from breaking of cubic invariance
  - $\Delta\vartheta^{\mu}_{\Sigma_{\pi^{\pm}}}, \ldots$ can be reabsorbed in on-shell conditions
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  - $\Delta\vartheta^{\mu}_{\Sigma_{\pi^{\pm}}}, \ldots$ can be reabsorbed in on-shell conditions
  - corrections are momentum-independent
- π^{0}, η do not have external twisting angles:
  - no extra terms
  - corrections are momentum-independent

Decay constant corrections

FVC calculation for decay constants is analogous Sachrajda, Villadoro 2005

Bijnens, Relefors 2014, see also Cherman, Sen, Wagman, Yaffe 2016

- other extra terms appear e.g.

$$\langle 0| A_P^\mu(0) |P(p+\vartheta_P)\rangle_L = iF_P(L)\big(p+\vartheta_P+\Delta\bar{\vartheta}_{\mathscr{A}_P}\big)^\mu$$

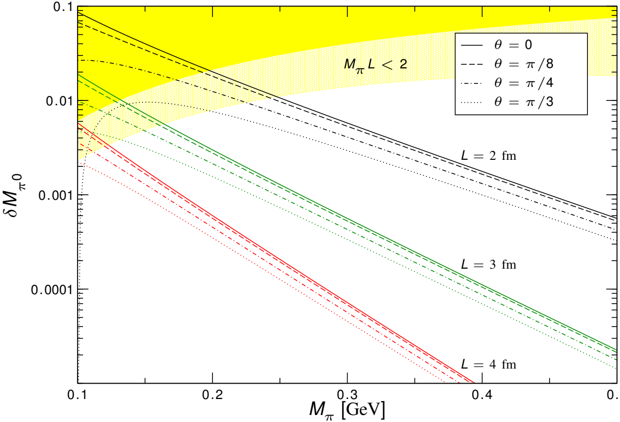
- $\Delta\vartheta^\mu_{\mathscr{A}_{\pi^\pm}}, \ldots$ can also be treated as twist renormalization
- in fact the explicit calculation shows that

$$\Delta\vartheta^\mu_{\mathscr{A}_{\pi^\pm}} = \Delta\vartheta^\mu_{\Sigma_{\pi^\pm}}$$

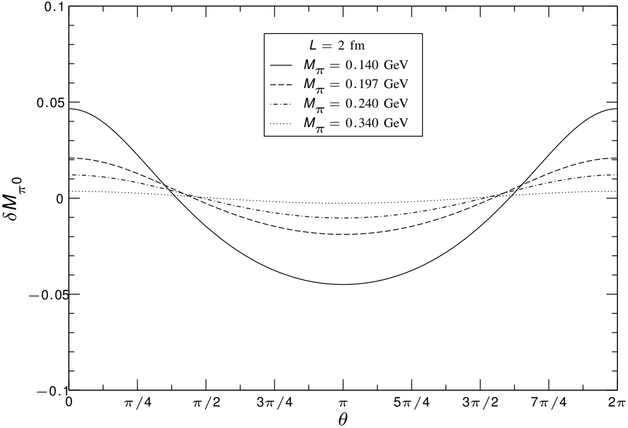
  confirming the interpretation
- the Ward identity

$$\hat{m}\,\langle 0| P_P(0) |P(p+\vartheta_P)\rangle_L = (\partial - i\vartheta_P)_\mu \langle 0| A_P^\mu(0) |P(p+\vartheta_P)\rangle_L$$

  implies that no such terms appear in G_P at this order ✓



$$\delta M_{\pi^0} = \frac{M_{\pi^0}(L) - M_\pi}{M_\pi}$$



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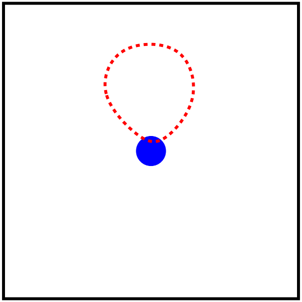
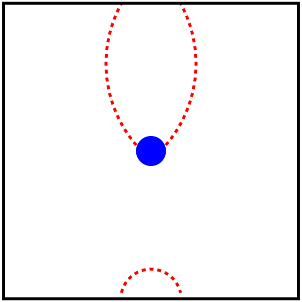
Outline

Masses in finite volume



Loop-diagram

Masses in finite volume



Loop diagram with PBC

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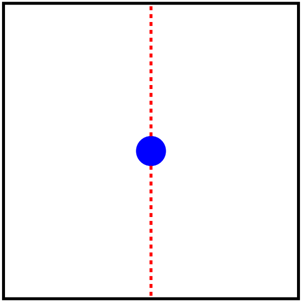

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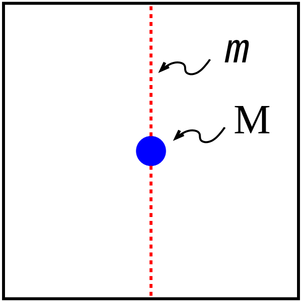
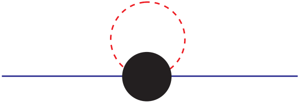


Loop diagram with PBC

This diagram exists only for $L \neq \infty$

Its effect is of the order $\exp\left[-mL\right]$

Lüscher's Formula

 $= \int d\ell\, \Gamma(p,\ell,-\ell,-p) G_L(\ell)$

$$G_L(\ell) = \sum_{\vec{n}} G_\infty(\ell) e^{i\vec{\ell}\cdot\vec{n}L} \qquad G_\infty(\ell) \sim \frac{1}{\ell^2 + m^2}$$

$$\begin{aligned} M_L - M &= \int d\ell\, \Gamma(p,\ell,-\ell,p)\left[G_L(\ell) - G_\infty(\ell)\right] \\ &= \sum_{\vec{n}\neq\vec{0}} \int d\ell\, \Gamma(p,\ell,-\ell,p) G_\infty(\ell) e^{i\vec{\ell}\cdot\vec{n}L} \end{aligned}$$

Lüscher's Formula

Leading correction for $mL \gg 1$: (Lüscher 86)

$$M_L - M = C \int_{-\infty}^{\infty} dy\ e^{-\sqrt{m^2+y^2}L} \mathcal{F}(iy) + \dots$$

$\mathcal{F}(\nu)$ is the scattering amplitude between the red (m) and blue (M) particle, and C a constant that depends on L, m and M

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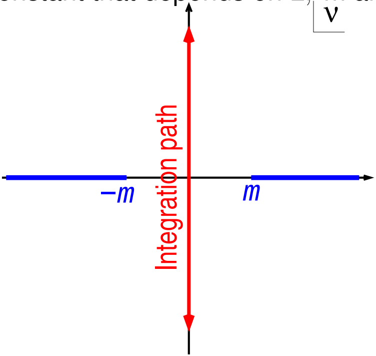

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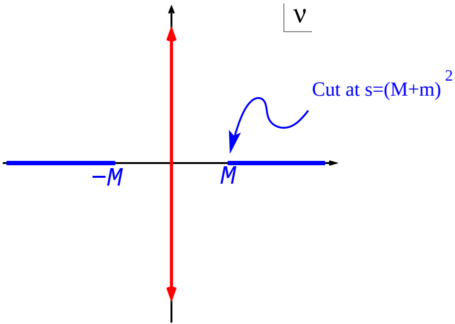
- The formula expresses the corrections as an integral over a physical amplitude (analytically continued)
- What matters for the behaviour of the corrections is not the mass of the particle itself, but rather
  the mass of the lightest particle to which it is coupled
- e.g. both the corrections for the pion as well as those for the proton mass depend exponentially on $M_\pi L$

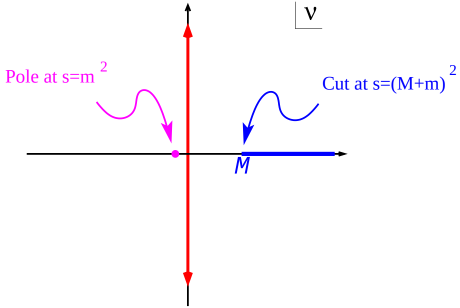
Cuts and poles in the scattering amplitude

Any scattering amplitude must have a cut at
$s, u = (M+m)^2 \Rightarrow \nu = \frac{s-u}{4M} = \pm m$



Cuts and poles in the scattering amplitude

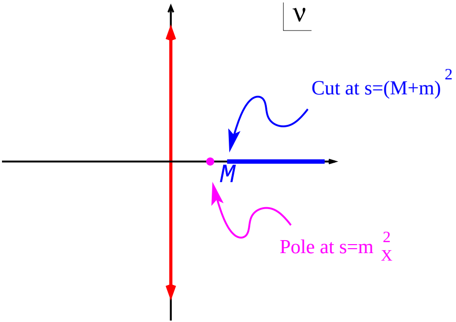
In addition it may have poles,
e.g. at $s, u = M^2 \Rightarrow \nu = \frac{s-u}{4M} = \mp\frac{m^2}{2M}$



Poles on the lhs of the imaginary axis generate an extra term in the Lüscher's formula

Cuts and poles in the scattering amplitude

In addition it may have poles,
or at $s, u = M_X^2 \Rightarrow \nu = \frac{s-u}{4M} = \mp \frac{m^2}{2M} + \Delta M \left(1 + \frac{\Delta M}{2M}\right)$ $\Delta M = M_X - M$

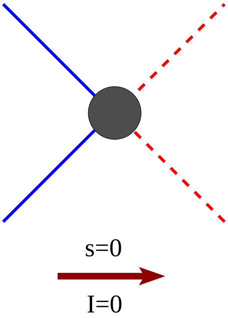


Poles on the rhs of the imaginary axis do not generate an extra term in the Lüscher's formula, cf. Arndt and Lin (04)

Corrections for M_π

$\pi\pi$ scattering amplitude with forward kinematics

$$F_{\pi\pi}(\nu) = T^{I=0}\left[0, 2M_\pi(M_\pi+\nu), 2M_\pi(M_\pi-\nu)\right] = -\frac{M_\pi^2}{F_\pi^2} + O(p^4)$$



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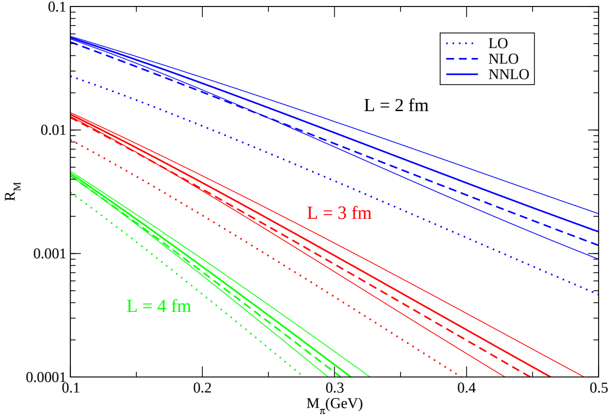

$$R_M = M_\pi(L)/M_\pi - 1$$

GC and S. Dürr 03

Lüscher's Formula or ChPT?

$$\Delta M_{\pi\ \text{Lüscher}} = \frac{-3}{16\pi^2\lambda}\int_{-\infty}^{\infty} dy\ \mathcal{F}(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

$$\Delta M_{\pi\ \mathrm{ChPT}} = \frac{1}{4}\xi\, g_1(\lambda) + O(\xi^2)$$

$$g_1(\lambda) = \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L,t)\big|_{t=\vec{x}=0} = \sum_{|\vec{n}|=1}^{\infty}\frac{4m(|\vec{n}|)}{|\vec{n}|\,\lambda}\,K_1\left(|\vec{n}|\,\lambda\right)$$

where $m(|\vec{n}|)$ is the multiplicity of a vector of length $|\vec{n}|$ in 3-dimensional discretized space

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These represent the leading term in two different expansions

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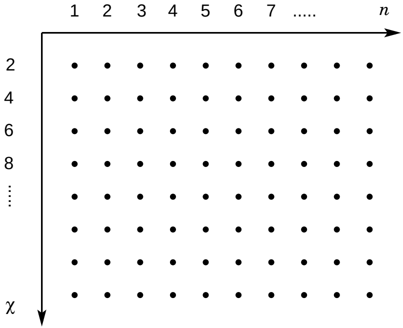

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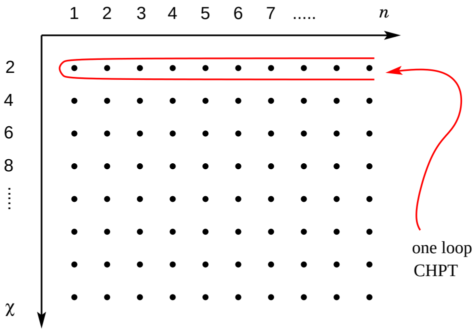
$$\Delta M_{\pi\ \text{ChPT}} = \frac{1}{4}\xi\, g_1(\lambda) + O(\xi^2)$$



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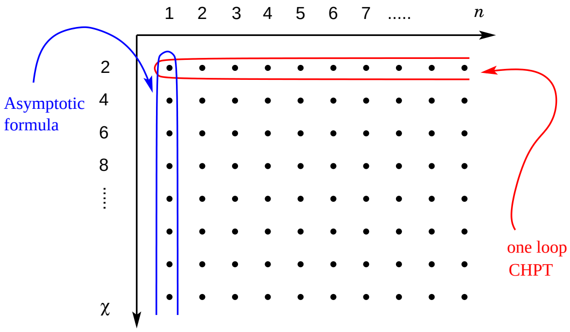

Extension of Lüscher's Formula

One-loop ChPT corrections

$$g_1(\lambda) = \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L, t)\big|_{t=\vec{x}=0} = \sum_{|\vec{n}|=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\,\lambda}\, K_1\left(|\vec{n}|\,\lambda\right)$$

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Analogously one can extend the Lüscher's Formula so that it contains contributions from all $|\vec{n}|$ of a single propagator:

$$\Delta M_\pi = -\frac{1}{32\pi^2\lambda} \sum_{|\vec{n}|=1}^{\infty} \frac{m(|\vec{n}|)}{|\vec{n}|} \int_{-\infty}^{\infty} dy \mathcal{F}(iy) e^{-\sqrt{\vec{n}^2(M_\pi^2+y^2)}L}$$

The extension does not give all exponentially subleading terms!

Extension of Lüscher's Formula

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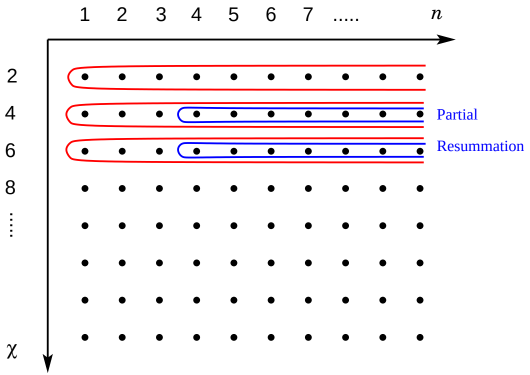
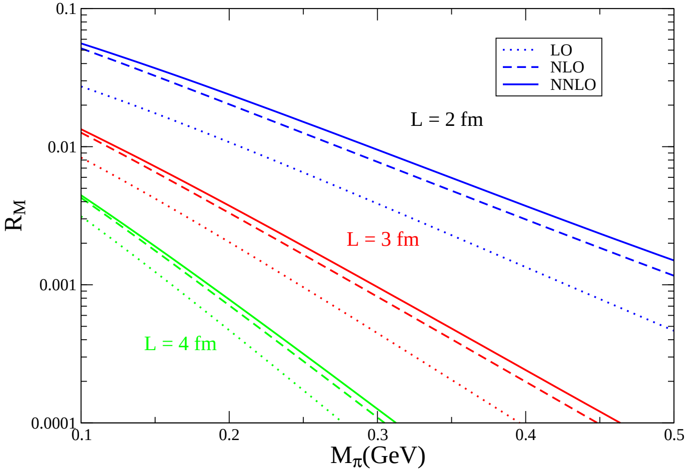


Nonleading exp. terms in $M_\pi(L)$



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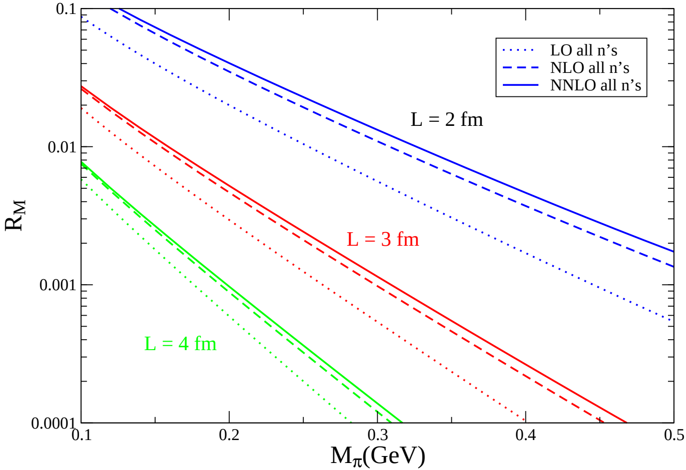
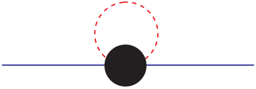
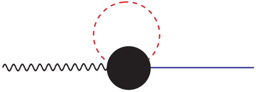


Extension to decay constants



$$\Rightarrow \quad \Delta M \propto \int_{-\infty}^{\infty} dy\; e^{-\sqrt{M_\pi^2+y^2}L}\mathcal{F}(iy) + \ldots$$

$$\mathcal{F}(\nu) \Leftrightarrow \langle \pi\pi | T | \pi\pi \rangle$$



$$\Rightarrow \quad \Delta F \propto \int_{-\infty}^{\infty} dy\; e^{-\sqrt{M_\pi^2+y^2}L}\mathcal{N}(iy) + \ldots$$

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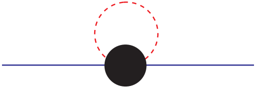
Extension to decay constants



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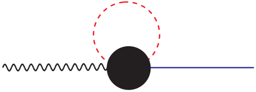

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The $\langle 0 | A_\mu | \pi\pi\pi \rangle$ amplitude must be subtracted:

$$\mathcal{N}(\nu) = \langle (2\pi)_{I=0}\pi | A_\mu(0) | 0 \rangle - iQ_\mu \frac{F_\pi \mathcal{F}(\nu)}{M_\pi^2 - Q^2}$$

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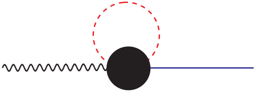
Extension to decay constants



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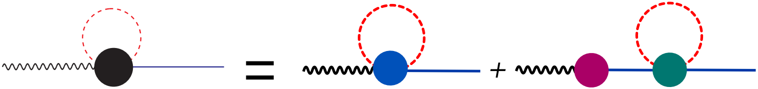

Ward identity in finite volume

$$\langle 0|A^i_\mu(0)|\pi^k(p)\rangle = i\delta^{ik} F_\pi p_\mu \qquad \langle 0|P^i(0)|\pi^k(p)\rangle = i\delta^{ik} G_\pi$$

Ward identity

$$F_\pi M_\pi^2 = \hat{m} G_\pi$$

The identity is valid also in finite volume

$$R_G = R_F + 2R_M \qquad (R_X := \Delta X_\pi / X_\pi)$$

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$$C\int_{-\infty}^{\infty} dy e^{-\sqrt{M_\pi^2+y^2}L}\left[\frac{1}{G_\pi}\mathcal{C}(iy) - \frac{1}{F_\pi}\mathcal{N}_F(iy) + \frac{1}{M_\pi}\mathcal{F}(iy)\right] = 0$$

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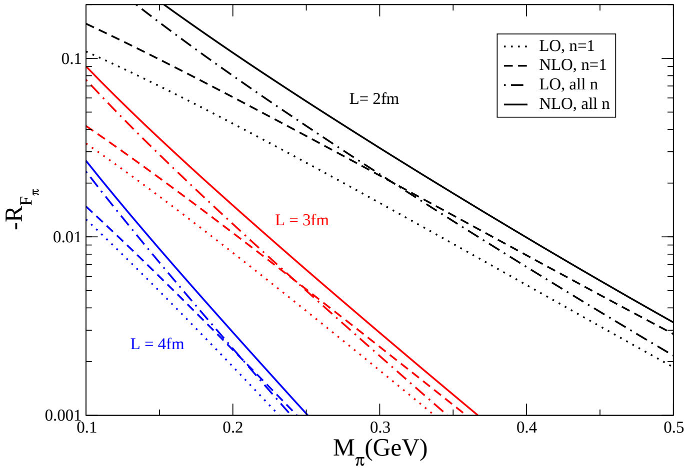
– in particular for the integrands:

$$\frac{1}{G_\pi}\mathcal{C}(\nu) - \frac{1}{F_\pi}\mathcal{N}_F(\nu) + \frac{1}{M_\pi}\mathcal{F}(\nu) = 0$$

This is a Ward identity for 4-point functions

Corrections for F_π



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Other applications

Quantity	Amplitude	Theory status
M_K	$A(\pi K \to \pi K)$	$O(p^6)$ (Bijnens et al.)
F_K	$A(K_{l4})$	$O(p^6)$ (Bijnens et al.)
M_η	$A(\pi\eta \to \pi\eta)$	$O(p^4)$ (Bernard et al.)
F_η	$A(\eta_{l4})$	?
M_N	$A(\pi N \to \pi N)$	$O(p^4)$ various Authors
M_B	$A(\pi B \to \pi B)$	?
F_B	$A(B_{l4})$	?

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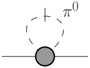
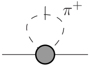
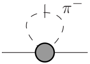
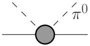
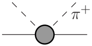
$\Delta\Sigma_{K^+} = \frac{1}{2}\Big[$ 

$+$ 

$+$ 

$\Big] + \mathcal{O}(e^{-\bar{M}L})$

$= \frac{1}{2}\sum_{\substack{\vec{n}\in\mathbb{Z}^3\\ |\vec{n}|\neq 0}} \int \frac{d^4k}{(2\pi)^4}\frac{e^{iL\vec{n}\vec{k}}}{M_\pi^2+k^2}\Big[$ 

$+$ 

$e^{-iL\vec{n}\vec{\vartheta}_{\pi^+}} + \ldots\Big]$

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Generalization of AF to TBC

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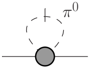
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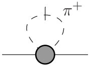
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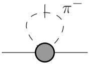
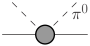

$$\Big] + \mathcal{O}(e^{-\bar{M}L})$$

$$= \frac{1}{2}\sum_{\substack{\vec{n}\in\mathbb{Z}^3 \\ |\vec{n}|\neq 0}} \int \frac{d^4k}{(2\pi)^4} \frac{e^{iL\vec{n}\vec{k}}}{M_\pi^2 + k^2}\Big[$$

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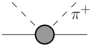
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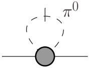

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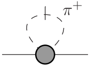

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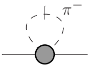

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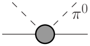

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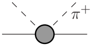

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- Integral substitution $\vec{k} \mapsto \vec{k} - \vec{\vartheta}_{\pi^\pm}$
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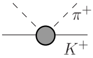
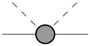
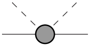


$$= \Bigg[$$

$$\Bigg]_{\vartheta^\mu_{K^+}=0} + \frac{k_\mu \vartheta^\mu_{K^+}}{M_K}\Bigg[$$

$$\Bigg]'_{\vartheta^\mu_{K^+}=0} + \mathcal{O}[(\vartheta^\mu_{K^+})^2]$$

- Integration over $\vec{k}$ can be performed within $\mathcal{O}(e^{-\bar{M}L})$ and remains integral on k_0

Generalization of AF to TBC

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$$\Delta\Sigma_{K^+} = -2M_K^2\,\delta M_{K^+} - 2\left(\vec{p} + \vec{\vartheta}_{K^+}\right)\Delta\vec{\vartheta}_{\Sigma_{K^+}} + \mathcal{O}(e^{-\bar{M}L})$$

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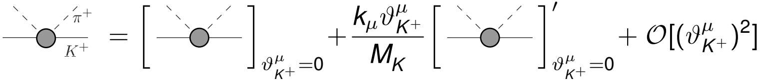

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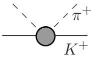
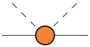
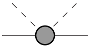


$= \Big[$  $\Big]_{\vartheta^\mu_{K^+}=0} + \frac{k_\mu \vartheta^\mu_{K^+}}{M_K} \Big[$  $\Big]'_{\vartheta^\mu_{K^+}=0} + \mathcal{O}[(\vartheta^\mu_{K^+})^2]$

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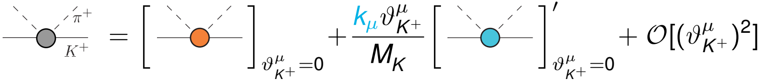


Explicit expressions

Expressions for masses of particles with or without twist $\lambda_\pi \equiv LM_\pi$

$$\delta M_{\pi^0} = \frac{-1}{2(4\pi)^2\lambda_\pi} \sum_{\substack{\vec{n}\in\mathbb{Z}^3 \\ |\vec{n}|\neq 0}} \int_{\mathbb{R}} \frac{dy}{|\vec{n}|}\, \mathrm{e}^{-\lambda_\pi |\vec{n}|\sqrt{1+y^2}} \mathcal{F}_{\pi^0}(iy, \vartheta_{\pi^+}) + \mathcal{O}(e^{-\bar{\lambda}})$$

$$\delta M_{\pi^\pm} = \frac{-1}{2(4\pi)^2\lambda_\pi} \sum_{\substack{\vec{n}\in\mathbb{Z}^3 \\ |\vec{n}|\neq 0}} \int_{\mathbb{R}} \frac{dy}{|\vec{n}|}\, \mathrm{e}^{-\lambda_\pi |\vec{n}|\sqrt{1+y^2}} \left(1 + y\,\frac{D_{\pi^\pm}}{M_\pi}\,\frac{\partial}{\partial y}\right) \mathcal{F}_{\pi^\pm}(iy, \vartheta_{\pi^+}) + \mathcal{O}(e^{-\bar{\lambda}})$$

where

$$D_{\pi^\pm} = \sqrt{M_\pi^2 + |\vec{\vartheta}_{\pi^\pm}|^2} - M_\pi$$

and

$$\mathcal{F}_{\pi^\pm}(\tilde{\nu}, \vartheta_{\pi^+}) = T_{\pi^\pm\pi^0}(0, -4M_\pi\nu) + [T_{\pi^\pm\pi^+}(0, -4M_\pi\nu) + T_{\pi^\pm\pi^-}(0, -4M_\pi\nu)]\mathrm{e}^{iL\vec{n}\vec{\theta}_{\pi^+}}$$

Explicit expressions

$$\Delta\vec{\vartheta}_{\Sigma_{\pi^\pm}} = \frac{-M_\pi}{2(4\pi)^2} \sum_{\substack{\vec{n}\in\mathbb{Z}^3 \\ |\vec{n}|\neq 0}} \frac{\vec{n}}{|\vec{n}|} \int_{\mathbb{R}} dy e^{-\lambda_\pi |\vec{n}|\sqrt{1+y^2}} y\, \mathcal{G}_{\pi^\pm}(iy, \vartheta_{\pi^+}) + \mathcal{O}(\mathrm{e}^{-\bar{\lambda}})$$

$$\Delta\vec{\vartheta}_{\Sigma_{K^\pm(K^0)}} = \frac{-1}{2(4\pi)^2} \frac{M_\pi^2}{M_K} \sum_{\substack{\vec{n}\in\mathbb{Z}^3 \\ |\vec{n}|\neq 0}} \frac{\vec{n}}{|\vec{n}|} \int_{\mathbb{R}} dy e^{-\lambda_\pi |\vec{n}|\sqrt{1+y^2}} y\, \mathcal{G}_{K^\pm(K^0)}(iy, \vartheta_{\pi^+}) + \mathcal{O}(\mathrm{e}^{-\bar{\lambda}})$$

The amplitudes $\mathcal{G}_P$ are given by differences of isospin components:

$$\mathcal{G}_P(\tilde{\nu}, \vartheta_{\pi^+}) = [T_{P\pi^+}(0, -4M_P\nu) - T_{P\pi^-}(0, -4M_P\nu)]\, \mathrm{e}^{iL\vec{n}\vec{\theta}_{\pi^+}}$$

Remarks on Asymptotic Formulae with TBC

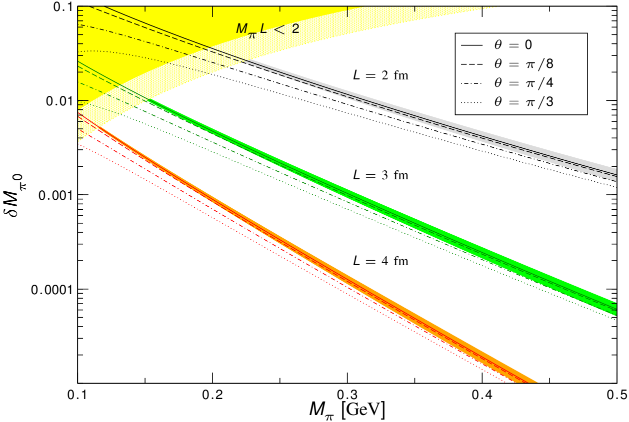
- AF for decay (pseudoscalar) constants are derived analogously
  like for PBC, amplitudes in integrands are defined after a pole subtraction
- Also derived:
  AF for matrix elements of pion scalar form factors ($t = 0$)

Remarks on Asymptotic Formulae with TBC

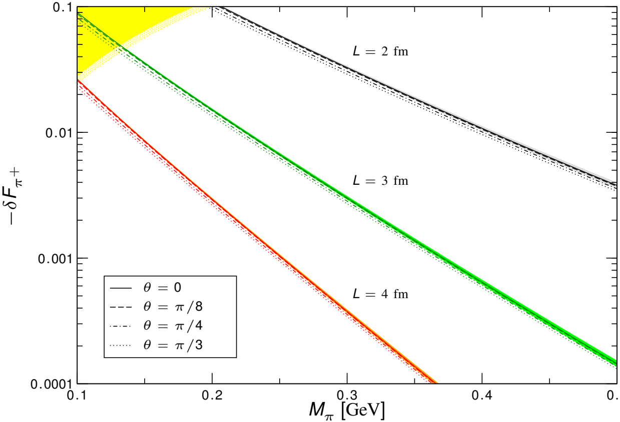
- AF for decay (pseudoscalar) constants are derived analogously
  like for PBC, amplitudes in integrands are defined after a pole subtraction
- Also derived:
  AF for matrix elements of pion scalar form factors ($t = 0$)

In general:

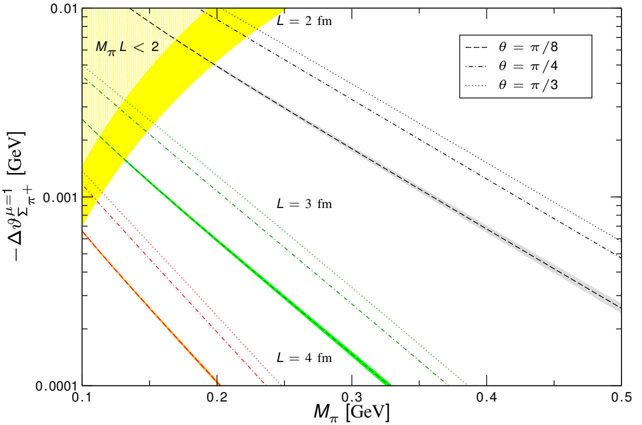
- AF for $\pi^\pm, K^\pm, \bar{K}^0$ are valid for small external twisting angles
- AF for π^0, η are valid for arbitrary twisting angles
- Inserting amplitudes at LO $\Longrightarrow$ FVC at NLO in ChPT ✓
- If all twisting angles are zero $\Longrightarrow$ AF for PBC ✓



$$\delta M_{\pi^0} = \frac{M_{\pi^0}(L) - M_\pi}{M_\pi}$$



$$\delta F_{\pi^+} = \frac{F_{\pi^+}(L) - F_\pi}{F_\pi}$$


0.01
0.001
0.0001
0.1
0.2
0.3
0.4
0.5
$M_\pi L < 2$
$L = 2$ fm
$L = 3$ fm
$L = 4$ fm
$\theta = \pi/8$
$\theta = \pi/4$
$\theta = \pi/3$
$-\Delta\vartheta^{\mu=1}_{\Sigma_{\pi^+}}$ [GeV]
M_π [GeV]

Chiral Ward Identities in Finite Volume

$$\langle 0 | P_a(0) | \pi_b(p) \rangle = i\delta_{ab} G_\pi \qquad \langle 0 | A_a^\mu(0) | \pi_b(p) \rangle = ip^\mu \delta_{ab} F_\pi$$

- Matrix elements are related by chiral Ward identities

$$\hat{m} G_\pi = M_\pi^2 F_\pi$$

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$$\delta G_{\pi^+} - 2\,\vec{\vartheta}_{\pi^+} \Delta\vec{\vartheta}_{\mathscr{G}_{\pi^+}} = 2\,\delta M_{\pi^+} + \delta F_{\pi^+} - \frac{2}{M_\pi^2}\,\vec{\vartheta}_{\pi^+} \left(\Delta\vec{\vartheta}_{\mathscr{A}_{\pi^+}} - \Delta\vec{\vartheta}_{\Sigma_{\pi^+}}\right) + \mathcal{O}(e^{-\bar{M}L})$$

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$$\delta G_{\pi^\pm} = 2\,\delta M_{\pi^\pm} + \delta F_{\pi^\pm} + \mathcal{O}(\mathrm{e}^{-\bar\lambda}) ,$$
$$\Delta\vec{\vartheta}_{\mathscr{G}_{\pi^\pm}} = \frac{1}{M_\pi^2}\left(\Delta\vec{\vartheta}_{\mathscr{A}_{\pi^\pm}} - \Delta\vec{\vartheta}_{\Sigma_{\pi^\pm}}\right) + \mathcal{O}(\mathrm{e}^{-\bar\lambda}) ,$$

These relations hold if the amplitudes entering the asymptotic formulae satisfy

$$\frac{\hat{m}}{M_\pi}\,\mathcal{C}_{\pi^\pm}(\tilde\nu,\vartheta_{\pi^+}) = \mathcal{N}_{\pi^\pm}(\tilde\nu,\vartheta_{\pi^+}) - \frac{F_\pi}{M_\pi}\,\mathcal{F}_{\pi^\pm}(\tilde\nu,\vartheta_{\pi^+}) ,$$
$$\frac{\hat{m}}{M_\pi}\,\mathcal{K}_{\pi^\pm}(\tilde\nu,\vartheta_{\pi^+}) = \mathcal{H}_{\pi^\pm}(\tilde\nu,\vartheta_{\pi^+}) - \frac{F_\pi}{M_\pi}\,\mathcal{G}_{\pi^\pm}(\tilde\nu,\vartheta_{\pi^+}) ,$$

which they do.

Outline

Summary

- For large volumes ($2LF_\pi \gg 1$), finite–volume effects can be calculated analytically within ChPT
- several one-loop ChPT calculations in the p–regime ($M_\pi L \gg 1$) are available in the literature
- the combined use of ChPT and asymptotic formulae *à la* Lüscher offers the most efficient way to get to higher orders in ChPT
- one-loop ChPT calculations have been extended to TBC by several groups
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- one-loop ChPT calculations have been extended to TBC by several groups
- I have reviewed these and shown how asymptotic formulae can be extended to the case of TBC
- even with TBC the extrapolation $L \to \infty$ can be controlled analytically