A New Way Around Topological Barriers in Lattice QCD

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Outline

Conventional: Axion Mass from Lattice QCD
  Topological Susceptibility from the Lattice
  Quenched Results

New Idea: Non-orientability in Lattice QCD
  Bosons
  Fermions

Conclusion
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Why Axions?

Simple extension of SM:
- complex scalar field $\phi$ dynamics at high scale $f_a$

But powerful:
- Peccei-Qinn solution of strong $CP$ problem
- Dark Matter candidate

Axion cosmology depends on QCD:
- $m_a(T)^2 f_a^2 = \chi_t(T)$ important at $\sim 3$ GeV
- Lattice QCD?
Topological Charge

Integral

\[ Q = \int_{\mathcal{M}} d^4 x q(x) \]

over the topological charge density

\[ q(x) = \frac{1}{4\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr} (F_{\mu\nu}(x) F_{\rho\sigma}(x)) \]

- discretized in finite volume on \( \mathcal{M} = \mathbb{T}^4 \)
- sectors with different \( Q \) separated by infinite action barrier in continuum
- problem for ergodicity of MC algorithms with small "step" size in field space
Topological Susceptibility

Integral of $qq$ correlator

$$\chi = \int_{\mathcal{M}} d^4 x \langle q(0) q(x) \rangle$$

With global translation symmetry on $\mathcal{M} = T^4$

$$\chi = \frac{1}{V_4} \langle Q^2 \rangle$$

- measurement must sample sectors with $Q \neq 0$
- difficult close to continuum
- difficult when $\chi V_4 = \langle Q^2 \rangle \ll 1$
Quenched Study

How far can we go with conventional brute force?

→ test in "cheap" quenched case

- roughly same scaling with temperature expected
- test bed for deviations from brute force strategy
- result with control over all errors valuable as crosscheck for models which can also do fermions
Previous lattice studies

- [Alles:1996nm,Gattringer:2002mr] etc. 1st gen results
- [Berkowitz:2015aua] large volume/statistics up to $2.5T_c$
- [Kitano:2015fla] reweighting to get $Q \neq 0$
Renormalization of $\chi$

$\chi$ has additive and multiplicative renormalization

[Alles:1997nu]

$$\chi^R = Z\chi + \chi_0$$

- cooling makes $Z \to 1$ and $\chi_0 \to 0$

$\chi(t)$ at finite flow time is already renormalized [Luscher:2010iy]

- sufficient to perform a continuum limit at flow time fixed in physical units, e.g. $t = w_0^2$
- choice of $t$ impacts size of lattice artefacts
Flow dependence of $\chi(t)$

- $\chi(t)$ has weak dependence on choice of $t$
- we choose $t = w_0^2 \approx (0.176 \text{ fm})^2$

\[ \chi \propto t^2 / w_0^2 \]

\[ \chi w_0^4 \]

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Lattice Setup

Pure SU(3)

- Symanzik improved gauge action: $O(a^2)$ errors
- gluonic $q(x)$ from clover field strength tensor $F_{\mu\nu}$: $O(a^2)$ errors
- update sweep: 1 heatbath + 4 overrelaxation $\Rightarrow z = 1$

Parameters

- $0.1 \, T_c \leq T \leq 4.0 \, T_c$
- $n_t = 5, 6, 8$
- spatial volume fixed in physical units $L_{x,y} = 2/T_c$
- $L_z = 2L_{x,y}$ to enable subvolume analysis

Simulations on the QPACE machines in Wuppertal and Jülich
Subvolume Trick [Brower:2014bqa]

Possible solution

- discretization of $Q$ is finite volume effect
- continuous $Q_{sub}$ on finite subvolumes of $\mathbb{R}^4$ and $T^4$
- calculate $\chi_{sub} = \langle Q_{sub}^2 \rangle / V_{sub}$
- make infinite $V_{sub}$ limit instead of infinite $V_4$ limit

Quenched and $T = 0$: large $\chi$

- plausible, works

Dynamic or $T \neq 0$: small $\chi$

- finite volume errors are $T$ independent
- errors larger than $\chi$ for reasonable volumes
Subvolume Trick - Finite Volume Errors

\[ T = 2T_c, \; N_t = 5, \; L_{sub} = L_z/2 \]

error scales like \[ 1/L \]

\[ \chi^4 w_0 \]

\[ Q^2 \quad Q^2_{sub} \]

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Slide 13
Subvolume Trick 2

- step from $\chi = \int_{\mathcal{M}} d^4 x \langle q(0) q(x) \rangle$ to $\chi = \langle Q^2 \rangle / V_4$ required translation invariance of $\mathcal{M} = \mathbb{T}^4$
- not valid for subvolume with boundary $\Rightarrow$ finite volume error
- large cancellations in integral of correlator $\Rightarrow$ large finite volume error

Alternative:
- evaluate $\chi = \int_0^{L_{\text{sub}}} dz \int d^3 x \langle q(L_{\text{sub}}/2, 0) q(z, x) \rangle$ directly
- correlator is only evaluated at distances in $z$ smaller than $L_{\text{sub}} / 2$
  $\Rightarrow$ reduced finite volume errors
Subvolume Trick 2 - Finite Volume Errors

\[ T = 2T_c, \ N_t = 5, \ L_{sub} = L_z/2, \ \text{identical configs} \quad \text{no } 1/L \]

\[
\chi_w^4, \quad 1/L_x T_c
\]

\[
Q^2, \quad Q_{sub}^2, \quad q_{q_{sub}}
\]
Results - Full Volume VS Sub Volume

\[ T / T_C \]

\[ \chi w_0^4 \]

\[ Q^2, N_t = 5 \]
\[ Q_{sub}^2, N_t = 5 \]
\[ qq_{sub}, N_t = 5 \]
Result - Continuum Extrapolation: $b = 7.1(4)(2)$
Quenched Lattice $\leftrightarrow$ DIGA

$T$ dependence OK

normalization off by $\mathcal{O}(10)$

fixed by comparison to lattice
Calibrated Dynamic DIGA

dynamic DIGA
quenched calibrated cosmology
⇒ axion mass
About statistics

Cost of the conventional algorithm at relative error $\delta \chi_t$

$$\propto V_4 N_c \approx \frac{1}{(\delta \chi_t)^2 \chi_t(T)}$$

Relative cost $(4 T_c)/(1 T_c)$ ($4 T_c \sim 1200 \text{ MeV}$)

- From measured $\chi_t(T)$: $4^{7.1} \approx 2 \times 10^4$
- From measured $\delta \chi_t$: $10^5 - 10^6$

- Quenched $\chi_t(T = 0)$ calculated $\sim 20$ years ago
- Moores law gives factor of $\sim 10^5$

$\Rightarrow$ Just possible
About statistics

Dynamic relative cost \(\frac{(7T_c)}{(1T_c)} \) \( (7T_c \sim 1200 \text{ MeV}) \)

from estimated \( \chi_t(T) \) \[7^7-8 \approx 10^6 - 10^7\]

increasing \( \tau_{int} \) with \( T \) \[10^7 - 10^9\]

- dynamic \( \chi_t(T = 0) \) in 2010, Moore factor of \( \sim 10^1 \)
- using \( \chi_{PT} \) for extrapolation to physical point
- \( \not\exists \chi_{PT} : T > T_c \)

\( \Rightarrow \) conventional dynamical study absolutely not possible
at physical quark masses and high enough temperature
Literature

Interesting result: [Bonati:2015vqz]

- conventional
- fully dynamic
- up to $\approx 4 T_c \sim 600$ MeV
- continuum extrapolation

Result: $b \sim 3$

- completely unexpected (DIGA etc. $b \sim 8$)
- crosses quenched result at $\sim 600$ MeV

$\Rightarrow$ Investigation necessary
What did we learn?

Only quenched result, but
- up to $T = 4 T_c$ with full systematic errors
- can be used to calibrate models

Finite volume errors in subvolume method
- space-time is important for this business
- loosing (translational) symmetry is to be avoided

Brute force method
- this far, not further $\rightarrow$ need new ideas
- in particular by orders of magnitude not for fermions (unless "a miracle" occurs)
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Motivation

Serious Problem: Getting stuck in topological sector

- auto-correlation time of certain slow modes diverges
- slow modes $\leftrightarrow$ topological structure of field space $\rightarrow$ topological charge $Q$ $\leftrightarrow$ pseudoscalar $\rightarrow$ pseudoscalar $\rightarrow$ orientability?

Several Ansätze in Literature

- manually increase tunnelling + reweighting [Kitano:2015fla]
- metadynamics [Laio:2015era]
- open boundary conditions [Luscher:2011kk]

Not much literature on QFT on non-orientable manifolds...
Open Boundaries: Solution

- slow modes $\leftrightarrow$ topological structure of field space
- Topological structure depends on gauge group and space-time
- gauge group and local structure of space-time define the local QFT $\rightarrow$ cannot change that

Open Boundaries:
- replace the periodic boundary conditions in one direction by open boundaries
- Topology of space-time changes $\rightarrow$ topology of field space changes
- charge is not discretized, not topologically conserved, no problem for simulations
Open Boundaries: Problems

- break translational invariance
- local structure of space-time is changed very strongly at the boundaries
  → changes the local QFT
- need to handle boundaries explicitly
- effectively reduces usable volume

Alternative:
- different change in the topology of space-time without any local changes of space-time
- still possible to keep charge non-discretized?
Orientability

orientable: The One Ring

non-orientable: The Möbius Strip
Topological Charge and Orientability

$q(x)$ is a pseudoscalar density

orientable roundtrip: no effect on charge

non-orientable roundtrip: changes sign of charge
Integration on Non-Orientable Manifolds

Differential geometry differs on non-orientable manifolds

- **No global volume form** on non-orientable manifolds → integration?
- **Volume element** on non-orientable manifolds → integrate scalar densities
- Cannot integrate **pseudoscalar density** $q(x)$
Integration on Non-Orientable Manifolds

Current workaround:

Define a total charge $Q_m$ on a maximal oriented submanifold

$$Q_m = \int_0^T dt \int d^3x q(x)$$

It is the same for open boundaries

We drop index "$m$" from now on and call this charge the charge.
P-Boundaries: Solution

- Replace the periodic boundary conditions in one direction by P-periodic boundaries
- i.e. implement an additional parity transformation P on all fields in the boundary condition

Result:
- Topology of **space-time** changes $\rightarrow$ topology of **field space** changes
- Charge is P-odd, continuous translation in P-direction $\rightarrow$ charge cannot be discretized
- No hard local breaking of **translation invariance**
P-Boundaries: Actual Implementation (Pure Gauge)

- Easier to implement in parallel than $P$ transformation: Reflection $R_x$ of single coordinate $x$
- $R_x \equiv P \times \text{rotation by } \pi$

$$U_x(x, y, z, t + T) = U_x^\dagger(L - x - 1, y, z, t),$$
$$U_i(x, y, z, t + T) = U_i(L - x, y, z, t)$$

for $i = y, z, t$. In the other three directions we keep the usual periodic boundary condition.
P-Boundaries: Construction on the Universal Cover
Diffusion Model

Describe autocorrelations in simulation time
[McGlynn, Mawhinney, PRD 90 (2014) 7]

\[ C(t, t_0, \tau) \equiv \langle Q(t + t_0, \tau_0 + \tau)Q(t_0, \tau_0) \rangle \]

\[ \frac{\partial}{\partial \tau} C(t, t_0, \tau) = D \frac{\partial^2}{\partial t^2} C(t, t_0, \tau) - \frac{1}{\tau_{\text{tunn}}} C(t, t_0, \tau), \]

- topological charge \( Q(t, \tau) \) on a time slice \( t \) at simulation time \( \tau \)
- diffusion constant \( D \)
- timescale for topological charge tunneling \( \tau_{\text{tunn}} \)
Diffusion Model

- Solutions determine integrated autocorrelation time
- Solutions determined by symmetry of boundaries
- All cases here give even solutions in $t$

<table>
<thead>
<tr>
<th>Boundary</th>
<th>Periodicity</th>
<th>$\tau_{\text{int}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Torus</td>
<td>$C(t + T, t_0, \tau) = C(t, t_0, \tau)$</td>
<td>$\propto \tau_{\text{tunn}}$</td>
</tr>
<tr>
<td>$P$</td>
<td>$C(t + T, t_0, \tau) = -C(t, t_0, \tau)$</td>
<td>$\propto \frac{T^2}{\pi^2 D}$</td>
</tr>
<tr>
<td>Open $t_0 = \frac{T}{2}$</td>
<td>&quot;$C(t + T, \frac{T}{2}, \tau) = -C(t, \frac{T}{2}, \tau)$&quot;</td>
<td>$\propto \frac{T^2}{\pi^2 D}$</td>
</tr>
</tbody>
</table>

- Same $\tau_{\text{int}}$ for $P$ and (middle of) open
- Only torus is dependent on $\tau_{\text{tunn}}$ for small $\tau_{\text{tunn}}
P-Boundaries: Claims

1. $Q$ is not quantized
   - better than periodic boundaries
   - like open boundaries

2. scaling of $\tau_{\text{int}}(Q)$ with lattice spacing is OK
   - better than periodic boundaries
   - like open boundaries

3. breaking of translational symmetry is suppressed
   - similar to periodic boundaries
   - better than open boundaries
P-Boundaries: History of the topological charge $Q$

\[ \beta = 5.1, \text{ lattice spacing } a = 0.040 \text{ fm, and box size } 1.6 \text{ fm.} \]
P-Boundaries: Claims

1. $Q$ is not quantized ✓
   better than periodic boundaries
   like open boundaries

2. Scaling of $\tau_{\text{int}}(Q)$ with lattice spacing is OK
   better than periodic boundaries
   like open boundaries

3. Breaking of translational symmetry is suppressed
   similar to periodic boundaries
   better than open boundaries
P-Boundaries: Integrated autocorrelation time

\[ \tau_{\text{int}}(Q) \]

- periodic
- open
- P-periodic

box size 1.6 fm.
P-Boundaries: Claims

1. $Q$ is not quantized ✓
   better than periodic boundaries like open boundaries

2. Scaling of $\tau_{\text{int}}(Q)$ with lattice spacing is OK ✓
   better than periodic boundaries like open boundaries

3. Breaking of translational symmetry is suppressed similar to periodic boundaries
   better than open boundaries
P-Boundaries: Time slice averaged action density

$\beta = 5.1$, lattice spacing $a = 0.040$ fm, and box size 1.6 fm.
P-Boundaries: Claims

1. \( Q \) is not quantized ✓
   better than periodic boundaries
   like open boundaries

2. Scaling of \( \tau_{\text{int}}(Q) \) with lattice spacing is OK ✓
   better than periodic boundaries
   like open boundaries

3. Breaking of translational symmetry is suppressed ✓
   similar to periodic boundaries
   better than open boundaries
Remarks on Observables

Topological Susceptibility on P-boundaries:

\[ \chi = \int_{\mathcal{M}} d^4 x \langle q(0) q(x) \rangle \neq \frac{1}{V_4} \langle Q^2 \rangle \]

just like for the subvolume method, due to missing translational symmetry of \( q(x) \)

Alternative again:

- evaluate \( \chi = \int_0^T dt \int d^3 x \langle q(0, T/2) q(x, t) \rangle \) directly
- correlator is only evaluated at distances in \( t \) smaller than \( T/2 \)
  \( \Rightarrow \) reduced finite volume errors
Observables: FV dependence

\[ \beta = 4.42466, \text{ lattice spacing } a = 0.093 \text{ fm}, \]
\[ \text{fixed spatial size } L, \text{ only temporal size } T \text{ changes} \]
Extension to Fermions

Construction straightforward:

- P-boundaries for gauge field
- reflect fermion fields
- include the $\gamma$ structure corresponding to reflection

$$T = i\gamma_5\gamma^x$$

$$\psi(x, y, z, t + T) = T\psi(L - x, y, z, t)$$

Remark:
Weyl-Fermions are not possible, as reflection mixes handedness
Problems with Fermions

Simulation problematic:
- \( T\gamma_5 = -\gamma_5 T \rightarrow [\gamma_5, T] \neq 0 \)
- Dirac operator \( D \) not \( \gamma_5 \) hermitian
- \( \det(D) \) is complex
  \( \Rightarrow \) complex action!

Possible Solution:
- find \( T, \Gamma_5 \) with \([\Gamma_5, T] = 0\)
- such that new Dirac operator \( D \) is \( \Gamma_5 \) hermitian
Solution for 2 Flavors

Two degenerate flavors $u, d$ (like $G$-parity [Wiese:1991ku])

- $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$, $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- choose $T = i \gamma_5 \gamma_x \tau_1$
- choose $\Gamma_5 = \gamma_5 \tau_3$

→ real action:

$$[\Gamma_5, T] = 0 \quad \Gamma_5 D \Gamma_5 = D^\dagger$$

- flavor violation suppressed [Lucini:2015hfa]
- works only for pairs of degenerate flavor
Solution for 1 Flavor

Include $C$ at the boundary

- $C = i\gamma_y\gamma_t = C^\dagger = C^{-1} = -C^T$
- $\psi \rightarrow C\overline{\psi}^T$, $\overline{\psi} \rightarrow -\psi^T C$
- $U_\mu \rightarrow U_\mu^*$

charge $Q$ is $P$ odd and $CP$ odd $\rightarrow$ same improvements

couples $\psi$ and $\overline{\psi}$

- $S = \overline{\psi}D\psi$ not applicable
- rewrite using vectors containing both $\psi$ and $\overline{\psi}$
Rewrite action

Change to eigenbasis of charge conjugation: [Lucini:2015hfa]

\[ \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi + C\psi^T \\ -i\psi + iC\psi^T \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & C \\ -i & iC \end{pmatrix} \begin{pmatrix} \psi \\ \psi^T \end{pmatrix} \]

Charge conjugation: \( \eta \rightarrow \begin{pmatrix} \eta_1 \\ -\eta_2 \end{pmatrix} = \rho_3 \eta, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

Action: \( S = -\frac{1}{2} \eta^T C\hat{D}\eta \rightarrow \text{pfaffian instead of determinant} \)

\( \hat{D} \) is a usual Dirac operator, but with \( 6 \times 6 \) real component links:

\[ \hat{D} = D \left[ \text{Re}(U) \cdot 1_{2\times2} - i \text{Im}(U) \cdot \rho_2 \right] \]
Hermiticity using $CP$ boundaries

Transformation $T \sim CP$:
- in $C$ eigenbase: $\rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- choose $T = -\gamma_5 \gamma_x \rho_2 \rho_3$
- choose $\Gamma_5 = \gamma_5 \rho_2$

$\rightarrow$ real action:

$[\Gamma_5, T] = 0 \quad \Gamma_5 \hat{D} \Gamma_5 = \hat{D}^\dagger$

- flavor violation suppressed [Lucini:2015hfa]
- works also for single flavors
Numerical test

- $16^3 \times 32$ quenched CP-bc, $\beta = 4.35$, $w_0 = 1.57$
- Wilson-Dirac, 4 stout with $\varrho = 0.125$, bare $m_0 = -0.16$

Backward propagation suppressed

Translational invariance
Outlook

New method in uncharted territory

⇒ Needs further exploration, e.g.

- Implementation in dynamic HMC
- Integration of pseudo-scalars
- Two classically disconnected sectors (?)
What did we learn?

P/CP-Boundaries are useful:
- improve topological autocorrelations
  ⇒ like open boundaries
- suppressed breaking of translational symmetry
  ⇒ better than open boundaries

P/CP-Boundaries are interesting:
- parity in-sensitive fields are straightforward (e.g. scalar)
- parity sensitive fields need additional thoughts (e.g. pseudoscalar, spinor)
- fermions are possible
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Topological structure of QCD still interesting

- Quenched result for $\chi_t(T)$
- Subvolume sensitive to finite volume errors
- Conventional method insufficient for dynamical case

- New idea: Non-orientable background for lattice QCD
- Same local QFT everywhere
- Promising and interesting properties
What are Axions?

Simple candidate for extension of SM:

- add a complex scalar field $\phi$
- with symmetry breaking Mexican hat at high scale $f_a$
- couple the Goldstone mode $\arg(\phi)$ as dynamic $\theta$ angle to QCD
- Goldstone mode is called axion

$$L_a = \partial^\mu \phi \partial_\mu \phi - \frac{\lambda}{8} \left( \phi^* \phi - f_a^2 \right)^2 + \chi t \frac{|\phi|}{f_a} \cos(\arg(\phi))$$
Strong CP Problem

Full QCD can include an effective CP breaking $\theta$ term:

$$\mathcal{L}_{QCD} = \sum_f \bar{\psi}_f (D_{\mu} \gamma^\mu + m_f) \psi_f + \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} - i\theta \frac{\alpha}{16\pi} \epsilon_{\mu\nu\rho\sigma} \tilde{F}^a_{\mu\nu} F^a_{\rho\sigma}$$

with $-\pi < \theta \leq \pi$, so natural $\theta \sim \mathcal{O}(1)$

From experiment we know $|\theta| < 10^{-10}$, so nature is unnatural $\rightarrow$ fine-tuning?
Peccei-Quinn solution

Promote $\theta$ to a dynamical field

- as phase of a global $U(1)$ symmetric scalar field $\phi$
- with spontaneous symmetry breaking potential

Absorb all phases from chiral rotations in redefinition of $\arg(\phi) := \theta_{\text{eff}}$

Effective potential for $\phi$ has minimum for $0 = \theta_{\text{eff}} = \arg(\phi) = 0$

$$\mathcal{L}_a = \partial_\mu \phi^* \partial^\mu \phi - \frac{\lambda}{8} \left( \phi^* \phi - f_a^2 \right)^2 + \chi t \frac{|\phi|}{f_a} \cos(\theta_{\text{eff}})$$
Axion Mass – Suszeptibility

Effective potential for $\theta_{\text{eff}}$:

$$V(\theta_{\text{eff}}) \equiv - \lim_{V_4 \to \infty} \frac{1}{V_4} \ln \left( \frac{Z(\theta_{\text{eff}})}{Z(0)} \right)$$

Effective axion field Lagrangian should be standard real scalar field:

$$\partial_\mu A \partial^\mu A - V(A) = \partial_\mu A \partial^\mu A - m_a^2 A^2 + \mathcal{O}(A^4)$$

Identify: $A = f_a \theta_{\text{eff}}$

Curvature of $V$ gives mass:

$$m_a^2 = \left. \frac{\partial^2 V(\theta_{\text{eff}})}{\partial A^2} \right|_{A=0} = \frac{1}{f_a^2} \lim_{V_4 \to \infty} \frac{1}{V_4} \langle Q^2 \rangle = \frac{\chi t}{f_a^2}$$
Massive Modes

Two massive oscillations of $\phi$

- heavy "string" mode in magnitude; with mass $m_s \approx \sqrt{\lambda f_a}$
- light "axion" mode in phase; with mass $m_a \approx \sqrt{\chi t} / f_a$

Given $\chi t$, cosmology gives expected abundance of axions

Axions can give substantial/total amount of dark matter

Two axion production mechanisms:
- misalignment
- dynamics and decay of string/wall networks
Topological Structures

Spontaneous symmetry breaking + causality:

different $\theta_{\text{eff}}$ in causally disconnected patches

$\Rightarrow$ Strings

with QCD potential

$\Rightarrow$ Walls
String/Wall Networks

- string-like defects arise and form networks
  \[ \Rightarrow \text{axion radiation} \]
- when \( \chi_t \) becomes relevant, formation of walls between strings
  \[ \Rightarrow \text{axion radiation} \]
- walls accelerate annihilation of topological defects
  \[ \Rightarrow \text{axion radiation} \]

\( \chi_t \) influences string dynamics, needed as input for total axion production
Misalignment

- alignment of misaligned neighbouring patches → axion radiation
- when $\chi_t$ becomes relevant, $\theta_{\text{eff}}$ "rolls" down to $\theta = 0$ → axion radiation

$\chi_t$ influences field dynamics, needed as input for total axion production
Both production mechanisms
- depend on $\chi_t$
- depend dynamics over cosmological time
⇒ need $\chi_t(t)$ over cosmological time

- $\chi_t$ is temperature dependent (not explicitly on time)
- cosmology also gives $T(t)$
⇒ need $\chi_t(T)$ over cosmological temperatures
What We Know About $\chi_t(T)$

Low $T \ll T_c$: $\chi_{PT}$
- $\chi_t(T) \approx \chi_0$
- $\chi_t \propto m_f$
  $\rightarrow$ very small $\chi_t$

High $T \gg T_c$: perturbation theory
- $\chi_t(T) \sim (T/T_c)^{-b}$, $b \sim 7 - 8$
- $\chi_t \propto m_f^{N_f}$ (from DIGA)
  $\rightarrow$ even smaller $\chi_t$

Interesting region in between: lattice might help
Wilson Flow

flow $B_\mu(t, x)$ defined by

$$\frac{d}{dt} B_\mu = D_\nu G_{\nu \mu},$$

$B_\mu|_{t=0} = A_\mu,$

with field strength tensor $G$ at finite $t$

Generated by stout smearing steps

$$\frac{d}{dt} V_\mu = i Q_\mu V_\mu,$$

$V_\mu|_{t=0} = U_\mu,$

→ approximation by small stout smearing steps with

$r_{smear}^2 = 8t = 8N_{stout}\rho_{stout}$
Stout Smearing

damp UV fluctuations i.e. high energy part of the spectrum generate ”fat” links by

\[ U'_\mu = e^{iQ_\mu} U_\mu, \]

\[ Q_\mu = \frac{i}{2} \left( \Omega^\dagger_\mu - \Omega_\mu - \frac{1}{3} \text{tr}[\Omega^\dagger_\mu - \Omega_\mu] \right), \]

\[ \Omega_\mu = \left( \rho_{\text{Stout}} \sum_{\nu \neq \mu} C_{\mu\nu} \right) U^\dagger_\mu, \]

with staples \( C_{\mu\nu} \)

effective smearing radius

\[ r_{\text{smear}} = a \sqrt{8 \rho_{\text{Stout}} N_{\text{Stout}}}. \]
Lattice Setup

Pure SU(3)

- Symanzik improved gauge action: $O(a^2)$ errors
- gluonic $q(x)$ from clover field strength tensor $F_{\mu\nu}$: $O(a^2)$ errors
- update sweep: 1 heatbath + 4 overrelaxation $\Rightarrow z = 1$

Parameters

- $N_t = N_x = N_y = N_z \in \{16, 20, 24, 32, 40\}$
- spatial volume fixed in physical units $L = 1.6\text{fm}$
Grassmann integral

- \[ S = \overline{\psi} D \psi = -\frac{1}{2} \xi^T \tilde{D} \xi \]
- \[ \int d\xi \exp \left( -\frac{1}{2} \xi^T \tilde{D} \xi \right) = \text{pf}(\tilde{D}) \]
- \( \text{pf}(M) \): Pfaffian of \( 2n \times 2n \) matrix \( M \)
  - \[ \text{pf}(M) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^{n} M_{\sigma(2i-1), \sigma(2i)} \]
  - \[ \text{pf}(M) = \text{pf} \left( \frac{M - M^T}{2} \right) \]
  - \[ \text{pf}(A^T MA) = \det(A) \text{pf}(M) \]
  - if \( M = -M^T \) then \( \text{pf}(M)^2 = \det(M) \)
- In our case
  - \[ \text{pf}(\tilde{D}) = \text{pf} \left( \begin{pmatrix} -D & D^T \end{pmatrix} \right) = \det(D) = \int d\psi \ d\overline{\psi} \ \exp(\overline{\psi} D \psi) \]
Suppressed backward propagation

\[
\langle O_{\pi^-}(t) \overline{O}_{\pi^-}({\bar{t}}) \rangle = = \text{Tr} \left[ CP \, e^{- (T - t) H} \, O_{\pi^-} \, e^{-(t - \bar{t}) H} \, \overline{O}_{\pi^-} \, e^{-\bar{t} H} \right] = \\
= \sum_{n,k} \langle n | \, CP \, e^{- (T - t) H} \, O_{\pi^-} \, | k \rangle \langle k | \, e^{-(t - \bar{t}) H} \, \overline{O}_{\pi^-} \, e^{-\bar{t} H} \, | n \rangle = \\
= \sum_{n,k} \langle CP(n) | \, O_{\pi^-} \, | k \rangle \langle k | \, \overline{O}_{\pi^-} \, | n \rangle \, \exp \left( - T E_n - (t - \bar{t}) (E_k - E_n) \right)
\]

- Lowest term: \( n = \text{vacuum}, k = \pi^- \longrightarrow \exp \left( - (t - \bar{t}) M_\pi \right) \)
- Missing: \( n = \pi^+, k = \text{vacuum} \)
- 2nd lowest: \( n = \pi^- + \pi^+, k = \pi^- \)
  \[ \longrightarrow \exp \left( - T E_{\pi^- + \pi^+} + (t - \bar{t}) (E_{\pi^- + \pi^+} - M_\pi) \right) \]