Three topics with twisted boundary conditions

Alberto Ramos
<alberto.ramos@cern.ch>

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Overview

Topology in infinite volume

$SU(N)$ Yang-Mils theory: Field configurations with finite action

$$S = -\frac{1}{2g^2}\int d^4x \,\mathrm{tr}\left(F_{\mu\nu}F_{\mu\nu}\right)$$
$$Q = \frac{1}{16\pi^2}\int d^4x \,\mathrm{tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \text{ and } \tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$$

Since

$$\mathrm{tr}\left(F_{\mu\nu}\tilde{F}_{\mu\nu}\right) = \partial_\mu K_\mu = 2\partial_\mu \varepsilon_{\mu\nu\rho\sigma}\mathrm{tr}\left(A_\nu\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma\right)$$

Topological charge is the integral of a total divergence

$$Q = \int_{\mathbb{S}^3_\infty} d\sigma_\mu K_\mu = 0(??)$$

Topology in infinite volume

Chern-Simons does not vanish at infinity!

In spherical coordinates $x = (r, \theta, \phi, \varphi)$, at $r = \infty$ the action density must be zero

$$F_{\mu\nu}(x)|_{r=\infty} = 0 \Longrightarrow A_\mu(x)|_{r=\infty} = \Lambda \partial_\mu \Lambda^\dagger$$

But Chern-Simons is not gauge invariant $A_\mu \to \Lambda A_\mu \Lambda^\dagger + \Lambda \partial_\mu \Lambda^\dagger$

$$K_\mu[A_\mu] \longrightarrow K_\mu[A_\mu] + \frac{2}{3}\varepsilon_{\mu\nu\rho\sigma}\bar{A}_\nu \bar{A}_\rho \bar{A}_\sigma + 2\varepsilon_{\mu\nu\rho\sigma}\partial_\rho \mathrm{tr}(\bar{A}_\nu A_\sigma)\,,$$

with $\bar{A}_\mu = \Lambda \partial_\mu \Lambda^\dagger$.

$$Q = \frac{1}{16\pi^2}\int_{\mathbb{S}^3_\infty} d\sigma_\mu \frac{2}{3}\varepsilon_{\mu\nu\rho\sigma}\bar{A}_\nu \bar{A}_\rho \bar{A}_\sigma \neq 0\,.$$

Topological charge is determined by the pure gauge field at infinity $\longrightarrow \Lambda(x)|_{r=\infty}$

$$\Lambda : \mathbb{S}^3_\infty \longrightarrow SU(2) \sim \mathbb{S}^3$$

$\Lambda(x)|_{r=\infty}$ characterized by an integer (i.e. Q).

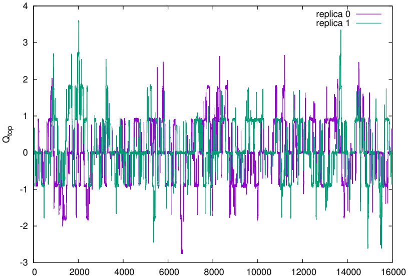
Topology in finite volume

What about gauge fields in $\mathbb{T}^4$?

- $A_\mu(x)$ does not need to be a pure gauge anywhere!
- If the gauge field is periodic $Q = 0$!

$$Q = \frac{1}{16\pi^2} \int_{\mathbb{F}_\mu} d\sigma_\mu \Delta_\mu K_\mu$$

with $\Delta_\mu f(x) = f(x + L\hat{\mu}) - f(x)$



Topology in finite volume

Only Gauge invariant quantities need to be periodic

$$A_\mu(x + L\hat{\nu}) = \Omega_\nu(x) A_\mu(x) \Omega_\nu^\dagger(x) + \Omega_\nu(x)\partial_\mu \Omega_\nu^\dagger(x) = A_\mu^{[\Omega_\nu(x)]}(x).$$

Consistency requires

$$\begin{aligned} A_\mu(x + L\hat{\nu} + L\hat{\rho}) &= A_\mu^{[\Omega_\nu(x+L\hat{\rho})\Omega_\rho(x)]}(x) = A_\mu^{[\Omega_\rho(x+L\hat{\nu})\Omega_\nu(x)]}(x) \\ \Omega_\rho(x+L\hat{\nu})\Omega_\nu(x) &= e^{2\pi \imath n_{\rho\nu}/N}\Omega_\nu(x+L\hat{\rho})\Omega_\rho(x) \end{aligned}$$

- $\Omega_\mu(x)$ are *twist matrices*. They change under gauge transformation.
- $n_{\mu\nu}$ is the *twist tensor*. Invariant under gauge transformations. Encodes physics of the twist.

Abelian twist matrices

$$\Omega_\mu(x) = \exp\{\omega(x)\}; \qquad \omega(x) = \frac{\pi x_\nu}{N L_\nu} W_{\mu\nu}$$

with ($\mathbb{T} = \mathrm{diag}(1, 1, \ldots, 1 - N)$) and $\mathrm{tr}(\mathbb{Q}_{\mu\nu}) = -n_{\mu\nu}$

$$W_{\mu\nu} = n_{\mu\nu}\mathbb{I} + N\mathbb{Q}_{\mu\nu}; \qquad W_{\mu\nu} = n_{\mu\nu}\mathbb{T}$$

Topology in finite volume

K_μ is not gauge invariant, and therefore not periodic

$$Q = -\frac{1}{24\pi^2}\varepsilon_{\mu\nu\rho\sigma}\int d\sigma_\mu \mathrm{tr}\left\{\Omega^\dagger_\mu(\partial_\nu\Omega_\mu)\Omega^\dagger_\mu(\partial_\rho\Omega_\mu)\Omega^\dagger_\mu(\partial_\sigma\Omega_\mu)\right\}$$
$$- \frac{1}{8\pi^2}\varepsilon_{\mu\nu\rho\sigma}\int d\sigma_{\mu\nu}\mathrm{tr}\left\{[\Delta_\rho(\Omega^\dagger_\mu(\partial_\nu\Omega_\mu)) + \Omega^\dagger_\mu(\partial_\nu\Omega_\mu)]\Omega^\dagger_\rho(\partial_\sigma\Omega_\rho)\right\}$$

By using the generic for of the Abelian Twis matrices $W_{\mu\nu} = n_{\mu\nu}\mathbb{I} + N\mathbb{Q}_{\mu\nu}$

$$Q = -\frac{\kappa(n)}{N} + n$$

with $\kappa(n) = n_{\mu\nu}\tilde{n}_{\mu\nu}/4 \in \mathbb{Z}$ **[Van Baal '82]**.

On the lattice $U_\mu(x)$ is periodic

$A_\mu(x + L\hat{\nu}) = A_\mu(x) + 2\pi q$ allows to have all integer topological charges!

Overview

A key tool for non-perturbative studies [Narayanan, Neuberger '06; Lüscher '10]

Main idea

- Add extra (flow) time coordinate $t(\neq x_0)$ with $[t] = \text{length}^2$. Define gauge field $B_\mu(t,x)$

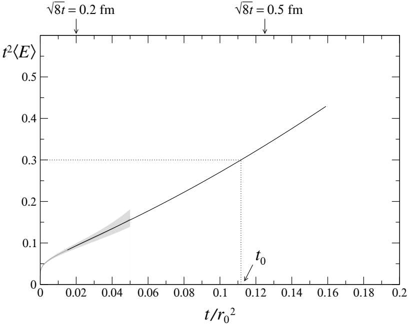
$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t); \quad \left(\sim -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right)$$
$$G_{\nu\mu}(x,t) = \partial_\nu B_\mu(x,t) - \partial_\nu B_\mu(x,t) + [B_\nu(x,t), B_\mu(x,t)],$$

  with initial condition $B_\mu(t,x)|_{t=0} = A_\mu(x)$
- Composite gauge invariant operators are renormalized observables defined at a scale $\mu = 1/\sqrt{8t}$ [M. Lüscher '10; M. Lüscher, P. Weisz '11].
- Example

$$\langle E(t)\rangle = -\frac{1}{2}\mathrm{Tr}\langle G_{\mu\nu}(x,t)G_{\mu\nu}(x,t)\rangle$$

  finite quantity for $t > 0$.
- Continuum limit to be taken at fixed t.

Scale setting [M. Lüscher '10; Borsanyi et. al. '12; R. Sommer Latt. '14]



- $t^2\langle E(t)\rangle$ is dimensionless.
- Depends on scale $\mu = 1/\sqrt{8t}$
- Ideal candidate for scale setting: t_0 [M. Lüscher JHEP 1008 '10].
- Similar quantities: $t_1, w_0, \ldots$.
- Dimensionless ratios of "flow quantities" (i.e. $\sqrt{t_0}/w_0$) are easily computable quantities with continuum limit.

Cheap quantities with continuum limit

With the definition $\mathcal{E}(t) = t^2\langle E(t)\rangle$

$$t_0 \quad : \quad \mathcal{E}(t_0) = 0.3$$

$$w_0 \quad : \quad w_0^2 \frac{d}{dt}\mathcal{E}(t)\Big|_{t=w_0^2} = 0.3$$

we have

$$\sqrt{t_0} \sim 0.15\text{fm} \qquad w_0 \sim 0.18\text{fm}$$

Renormalized couplings [M. Lüscher '10; Z. Fodor et al. '12; P. Fritzsch, A.R. '13; M. Lüscher '14; A.R. '14]

Non-perturbative coupling definition

Perturbative computation

$$t^2\langle E(t)\rangle = \frac{3}{16\pi^2} g^2_{\overline{MS}}(\mu)\left[1 + c_1 g^2_{\overline{MS}}(\mu) + \mathcal{O}(g^4_{\overline{MS}})\right] ,$$

inmediately suggests

$$g^2_{\rm GF}(\mu) = \frac{16\pi^2}{3} t^2\langle E(t)\rangle\Big|_{\mu=1/\sqrt{8t}}$$

- On the lattice, infinite volume $a \ll \sqrt{8t} \ll L$.

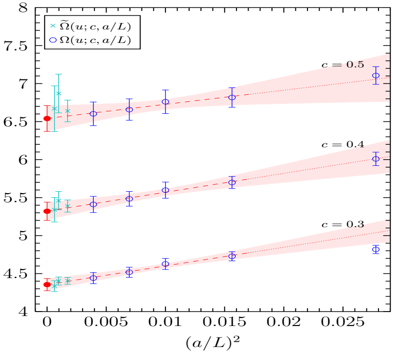
Finite volume renormalization schemes

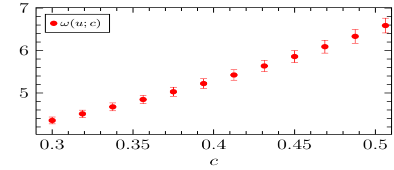
$$g^2_{\rm GF}(\mu) = \mathcal{N}^{-1} t^2\langle E(t)\rangle\Big|_{\mu=1/\sqrt{8t}}$$

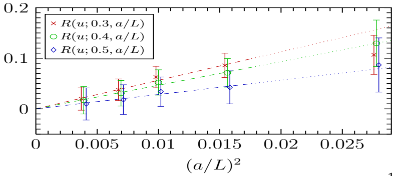
- To avoid the need of a window $a \ll \sqrt{8t} \ll L$, use $\mu = 1/cL$.
- Boundary conditions become relevant, and $\frac{16\pi^2}{3} \to \mathcal{N}^{-1}$:
- Periodic, SF, Twisted, SF-open.

Why is a good choice? $N_f = 2$ and $SU(3)$ simulations

L/a	6	8	10	12	16
β	5.2638	5.4689	5.6190	5.7580	5.9631
$\kappa_{\rm sea}$	0.135985	0.136700	0.136785	0.136623	0.136422
$N_{\rm meas}$	12160	8320	8192	8280	8460
$\bar{g}^2_{\rm SF}(L_1)$	4.423(75)	4.473(83)	4.49(10)	4.501(91)	4.40(10)
$\bar{g}^2_{\rm GF}(\mu)$ $(c = 0.3)$	4.8178(46)	4.7278(46)	4.6269(47)	4.5176(47)	4.4410(53)
$\bar{g}^2_{\rm GF}(\mu)$ $(c = 0.4)$	6.0090(86)	5.6985(86)	5.5976(97)	5.4837(97)	5.410(12)
$\bar{g}^2_{\rm GF}(\mu)$ $(c = 0.5)$	7.106(14)	6.817(15)	6.761(19)	6.658(19)	6.602(24)







Boundary conditions

With periodic b.c., dynamics is dominated by fluctuations constant in space

- Leading order contribution of zero momentum modes is not quadratic **[A. Gonzalez-Arroyo et al. 1983]**

$$S \sim [\tilde{A}_\mu(0), \tilde{A}_\nu(0)]^2.$$

  With periodic b.c. these **are not** gauge degrees of freedom.
- One has to solve these integrals to define a "propagator"

$$\langle A_\mu A_\nu \rangle \sim \int \mathcal{D}A \; A_\mu A_\nu e^{-A_\mu M A_\nu - [\tilde{A}_\mu(0), \tilde{A}_\nu(0)]^2}$$

- Convergence properties of these integrals depend on d and $SU(N)$.
- Difficult (but this running scheme has been proposed **[Z. Fodor et al. 2012. arXiv:1208.1051]**).
- Coupling definition $\alpha_{\rm PT}$ non analytic in α_{MS}.

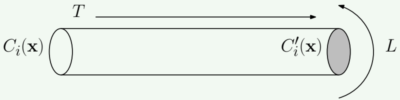
Moral: periodic b.c. in small volume leads to

- Non analytic coupling:
  - $SU(N)$ and $N > 2$: $\alpha = \alpha_{\rm MS}(1 + \sqrt{\alpha_{\rm MS}} + \dots)$
  - $SU(2)$: $\alpha = \alpha_{\rm MS}(1 + \log \alpha_{\rm MS} \dots)$

Boundary conditions

Make constant gauge field configurations incompatible with boundary conditions.

Schrödinger Functional

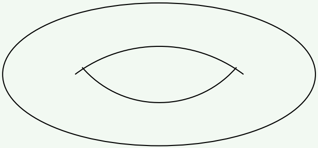


- SF: $L^3 \times T$ box. Dirichlet b.c. in time

$$
\begin{aligned}
A_i(\mathbf{x}, x_0 = 0) &= C_i(\mathbf{x}) \\
A_i(\mathbf{x}, x_0 = T) &= C'_i(\mathbf{x})
\end{aligned}
$$

- If C_i, C'_i chosen wisely $\Rightarrow$ Unique gauge configuration with minimum action.

Twisted boundary conditions



- Gauge field periodic modulo g.t.

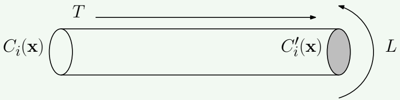
$$
\begin{aligned}
A_\mu(x + L\hat{\nu}) &= \Omega_\nu(x) A_\mu(x) \Omega_\nu^\dagger(x) \\
&+ \Omega_\nu(x) \partial_\mu \Omega_\nu^\dagger(x).
\end{aligned}
$$

- If Ω_μ chosen wisely $\Rightarrow$ Unique gauge configuration with minimum action.

BOUNDARY CONDITIONS

Arbitrary matter content ⇒ Same coupling definition.

Schrödinger Functional

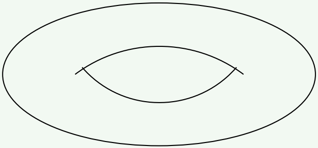


- Manifold with boundary: Boundary Counterterms $\mathcal{O}(a)$
- Chiral symmetry breaking $P_\pm = \frac{1}{2}(1 \pm \gamma_0)$

$$P_+\psi|_{x_0=0} = \rho_+; P_-\psi|_{x_0=T} = \rho_-$$

- Naively larger discretization effects!
- Work hard for $\mathcal{O}(a)$ improvement (PT + numerics): $c_{\rm sw}, c_t, c_{\tilde{t}}$.

Twisted boundary conditions



- Fermions in fundamental rep.

$$\psi(x + L\hat{\mu}) = \Omega_\mu \psi(x)$$

And therefore

$$\begin{aligned} \psi(x + L\hat{1} + L\hat{2}) &= \Omega_1\Omega_2\psi(x) \\ \psi(x + L\hat{2} + L\hat{1}) &= \Omega_2\Omega_1\psi(x) \end{aligned}$$

- Only way out: $SU(N)$ with N_f and $N_f/N \in \mathbb{Z}$ **[Parisi '83]**.

Coupling with twisted bc.

Task in life

Compute in perturbation theory

$$t^2\langle E(t)\rangle = \frac{1}{4}\langle G_{\mu\nu}(t)G_{\mu\nu}(t)\rangle = \mathcal{N}(t)\alpha_0 + \mathcal{O}(\alpha_0^2)$$

1. Compute $B_\mu(x,t)$ to leading order.
2. Compute The observable.

Coupling with twisted bc.

Key equation

$$A_\mu(x+L\hat{\nu}) = \Omega_\nu(x)A_\mu(x)\Omega_\nu^+(x) + \Omega_\nu(x)\partial_\mu\Omega_\nu^+(x) = A_\mu^{[\Omega_\nu(x)]}(x).$$

Our particular setup (similar to “TPL” scheme)

- Use constant twist matrices $\Omega_\mu(x) = \Omega_\mu$.

$$A_\mu(x+L\hat{\nu}) = \Omega_\nu A_\mu(x)\Omega_\nu^+$$

  and $A_\mu(x) = 0$ compatible with bc.
- We choose to twist only the plane $x_1 - x_2$ and $n_{12} = 1$.

$$\Omega_{3,4}(x) = 1; \quad \Omega_1\Omega_2 = e^{2\pi \imath m/N}\Omega_2\Omega_1$$

Coupling with twisted bc.

$$A_\mu(x + L\hat{k}) = \Omega_k A_\mu(x)\Omega_k^+ \,,$$

Define N^2 matrices ($\tilde{p}_i = \frac{2\pi\tilde{n}_i}{NL}$ with $\tilde{n}_i = 0, \ldots, N-1$).

$$\Gamma(\tilde{p}) = e^{\imath\alpha(\tilde{p})}\Omega_1^{-\tilde{n}_2}\Omega_2^{\tilde{n}_1}$$

They are traceless (except $\tilde{p} = 0$), linearly independent and

$$\Omega_i\Gamma(\tilde{p})\Omega_i^+ = e^{iL\tilde{p}_i}\Gamma(\tilde{p})\,.$$

Therefore any gauge connection compatible with bc. can be expanded

$$A_\mu^a(x)T^a = \sum_{\tilde{p}}{}' \hat{A}_\mu(x,\tilde{p})e^{\imath\tilde{p}x}\Gamma(\tilde{p}).$$

with $\hat{A}_\mu(x,\tilde{p})$ (numbers) periodic in x!. $p_\mu = \frac{2\pi n_\mu}{L} \quad (n_\mu \in \mathbb{Z})\,.$

$$A_\mu^a(x)T^a = \frac{1}{L^4}\sum_{\tilde{p}}{}' \tilde{A}_\mu(p,\tilde{p})e^{\imath(p+\tilde{p})x}\Gamma(\tilde{p}) = \frac{1}{L^4}\sum_{P}{}' \tilde{A}_\mu(P)e^{\imath Px}\Gamma(P)\,.$$

“Total” momentum: $P_i = p_i + \tilde{p}_i$, $P_{3,4} = p_{3,4}$. Color dof $\Leftrightarrow$ momentum dof (Large N, reduction, …). Only constant connection: $A_\mu(x) = 0$!

Coupling with twisted BC.

$$\dot{B}_\mu(x,t) = D_\nu G_{\nu\mu}(x,t)\,, \qquad B_\mu(x,0) = A_\mu(x)\,,$$
$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

$B_\mu(x,t)$ has an asymptotic expansion in g_0

$$B_\mu(x,t) = \sum_n B_{\mu,n}(x,t) g_0^n$$

After gauge fixing and to leading order

$$\dot{B}_{\mu,1}(x,t) = \partial_\nu^2 B_{\mu,1}(x,t) \quad (B_{\mu,1}(x,0) = A_\mu(x))$$

with solution

$$B_{\mu,1}(x,t) = \frac{1}{L^4} \sum_{p,\tilde{p}\neq 0} e^{-P^2 t} \tilde{A}_\mu(P) e^{\imath P x} \Gamma(P).$$

And finally $\langle E(t) \rangle = \frac{1}{4} \langle G_{\mu\nu}(t) G_{\mu\nu}(t) \rangle = \mathcal{E}(t) + \mathcal{O}(g_0^4)$

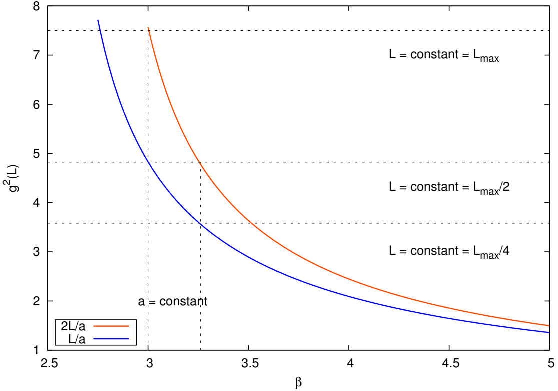
$$\mathcal{E}(t) = \frac{g_0^2(d-1)}{2L^4} \sum_{p,\tilde{p}\neq 0} e^{-P^2 t}$$

Overview

- Motivation
- The Gradient flow
- Applications
  - $SU(2)$ Pure gauge theory
  - $SU(3)$ with $N_f = 12$ flavors
- Improvement of the Gradient flow
- Conclusions

Running coupling

$$\beta \Longleftrightarrow a; \qquad g^2(L) \Longleftrightarrow L \Longleftrightarrow \mu$$



Step scaling function

$$\sigma^{-1}(u,a) = g^2(L/2)\Big|_{g^2(L)=u}$$

Continuum limit

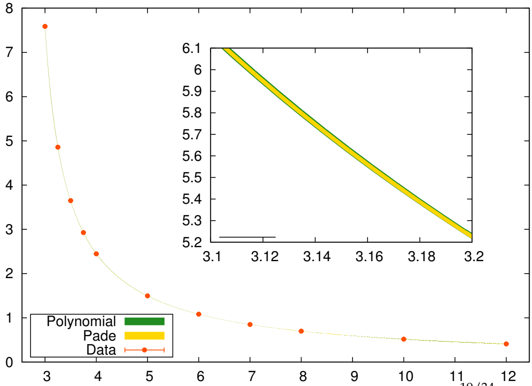
$$\sigma^{-1}(u) = \lim_{a\to 0} \sigma^{-1}(u,a)$$

Simulate several pair of lattices

$SU(2)$ YM running coupling

- Simulations for $L/a = 10, 12, 15, 18, 20, 24, 30, 36$ at $\beta \in [2.75, 12]$.
- Modest statistics: 2048 independent measurements of g^2_{TGF}.
- Between 0.15-0.25% precision in g^2_{TGF} *for all* L/a.
- Padè fit (constrain to PT), 4 parameters, $\chi^2/\text{ndof} = 5.9/7$.
- Example: $L/a = 36$

β	$g^2_{TGF}(L)$
12.0	0.41078(64)
10.0	0.51809(83)
8.0	0.6987(11)
7.0	0.8497(13)
6.0	1.0819(18)
5.0	1.4968(25)
4.0	2.4465(44)
3.75	2.9277(54)
3.5	3.6494(69)
3.25	4.8568(99)
3.0	7.587(20)
2.9	10.610(32)
2.8	16.752(47)
2.75	22.168(59)



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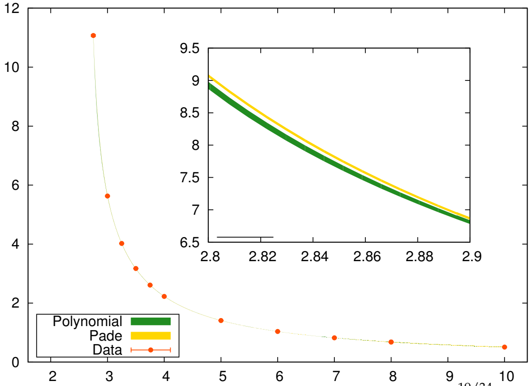
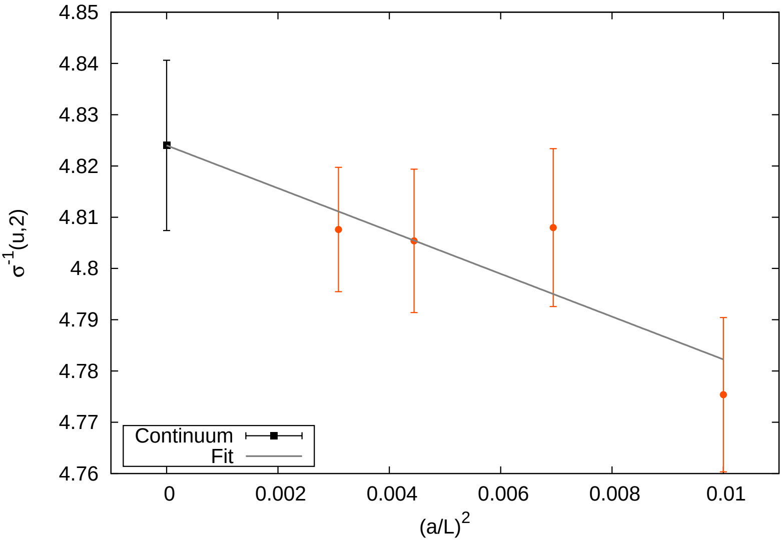


Step scaling function

- Modest cutoff effects. Starting recursion with $u = 7.5$.



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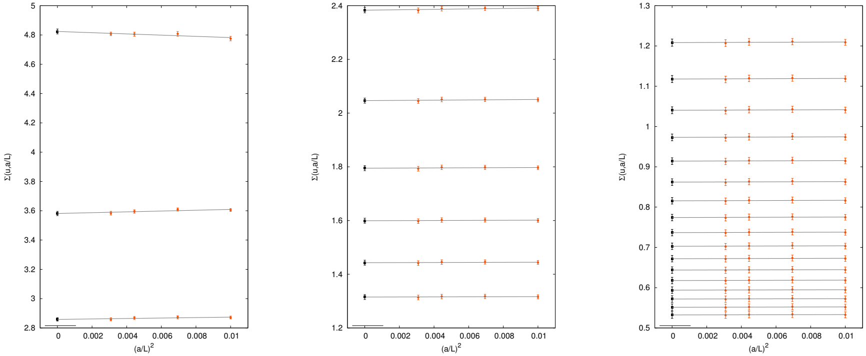
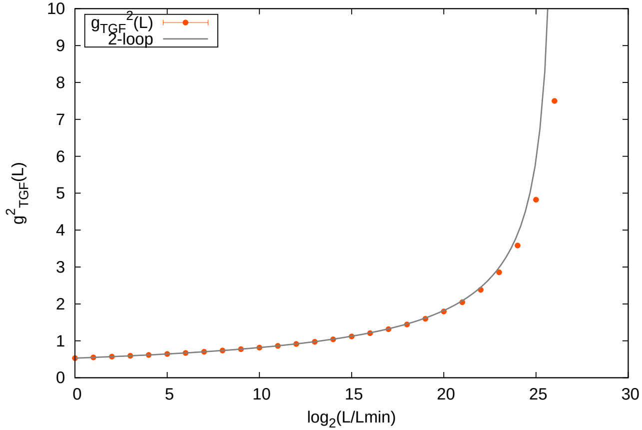


$g^2_{TGF}(L)$ for pure gauge $SU(2)$



Conformal behavior of gauge theories with large N_f.

Looking for conformal behavior (in collaboration with **[C.J. David Lin, K. Ogawa, A.R. '15.]**)

- For small N_f QCD like theory: Confinement, Λ_{QCD}
- If $N_f \gg 1$ No more asymptotic freedom.
- In between *maybe* conformal behavior: No typical scale for your theory at large distances, no mass gap.

- A value of the coupling $g_*^2(L) = g_*^2(2L)$
- This property true for every observable analytically related with the one that defines your scheme (e.g. scheme independent statement).

Conformal behavior of gauge theories with large N_f.

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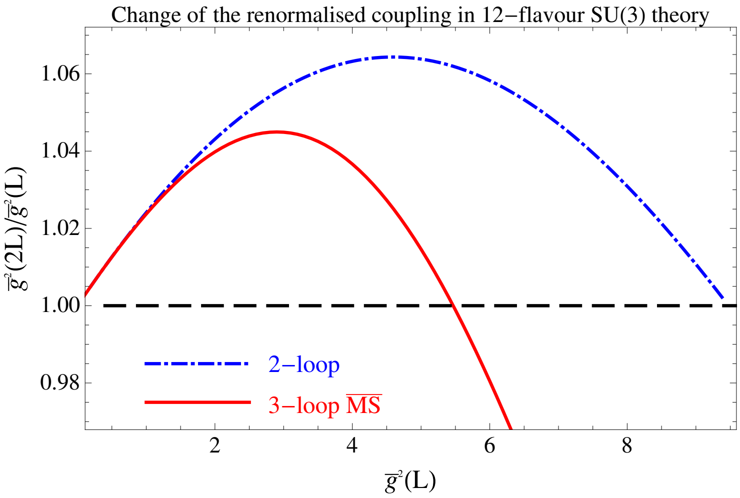
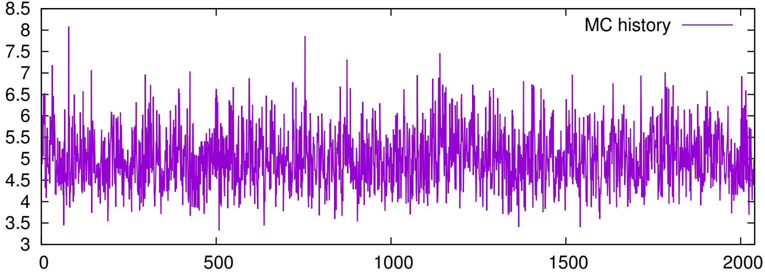


$SU(3)$ WITH $N_f = 12$ FLAVORS

Reuse data generated for the TPL coupling

- Work [C.J. David Lin, K. Ogawa, H. Ohki, E. Shintani.; JHEP 1208 (2012) 096]:
- Already $\mathcal{O}(200.000 - 1000.000)$ configurations for TPL coupling.
- Choose $\mathcal{O}(1000)$ of them and win an order of magnitude in statistics.
- A worst case example $L/a = 16$, $\beta = 4.38$.



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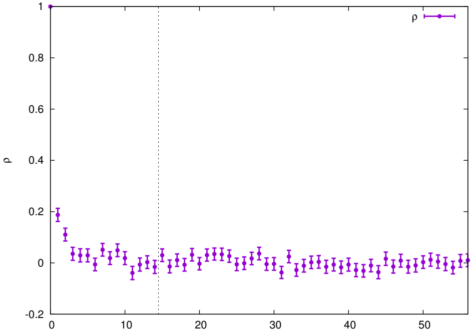
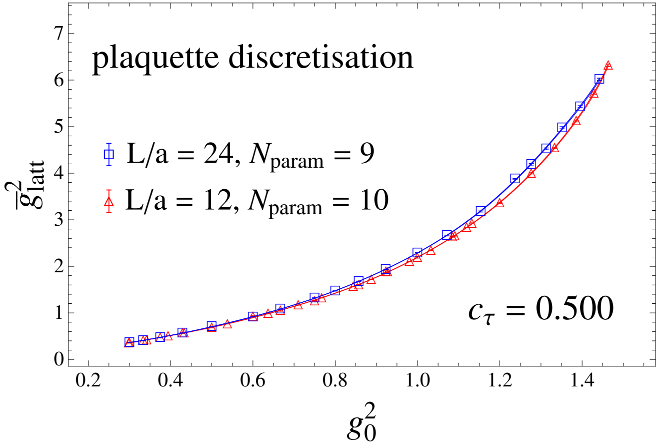
- Choose $\mathcal{O}(1000)$ of them and win an order of magnitude in statistics.
- A worst case example $L/a = 16$, $\beta = 4.38$.



$SU(3)$ with $N_f = 12$ flavors

$$L/a = 6, 8, 10, 12 \longrightarrow 12, 16, 20, 24$$



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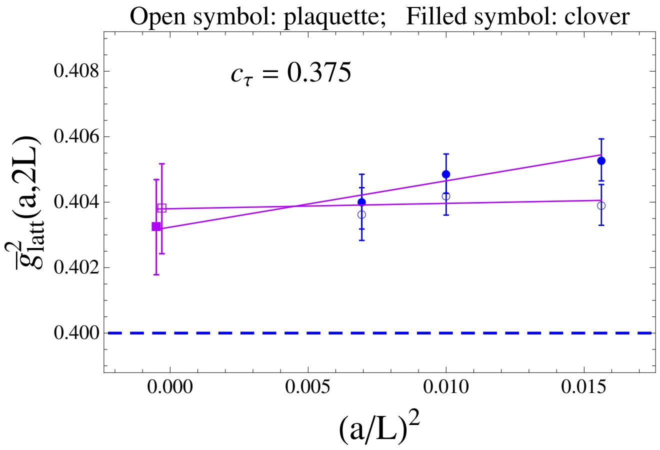

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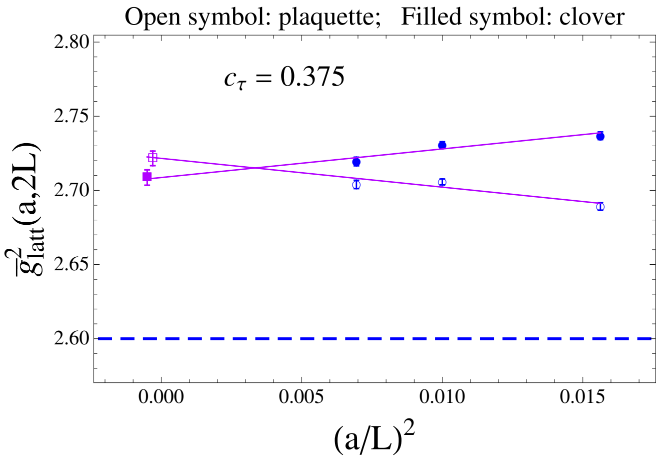

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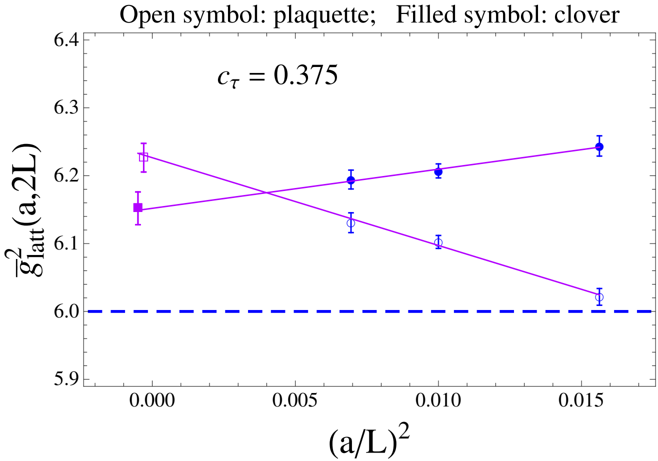

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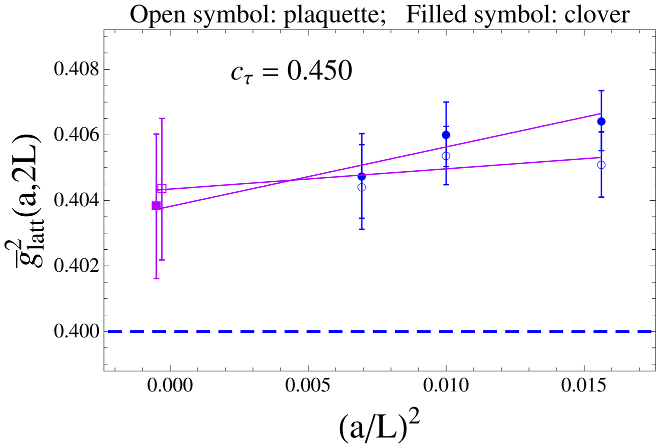

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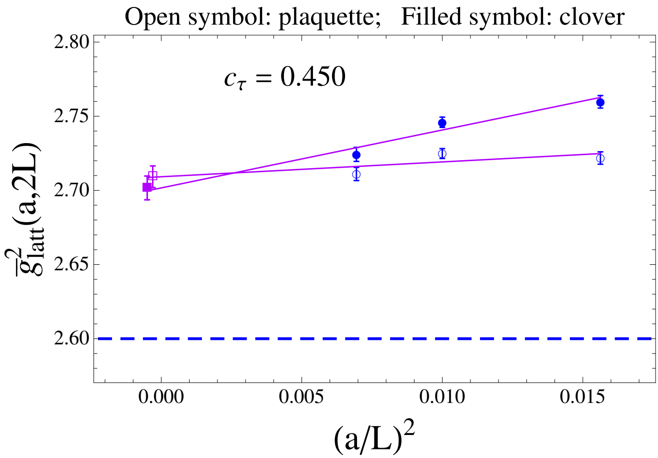

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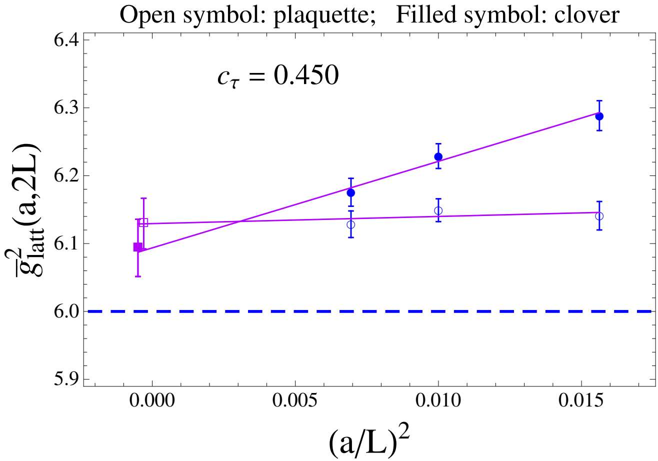

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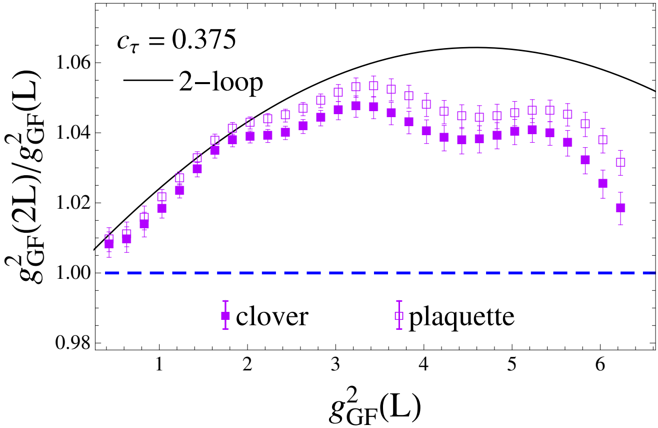

$SU(3)$ WITH $N_f = 12$ FLAVORS

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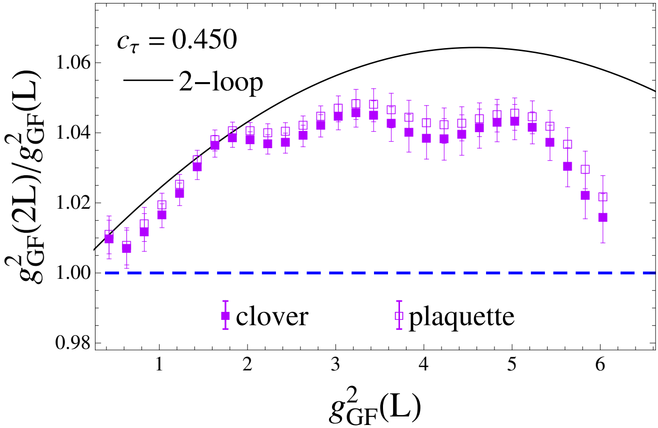


Overview

Motivation

The Gradient flow

Applications

Improvement of the Gradient flow
- The symanzik effective action
- $\mathcal{O}(a^2)$ cutoff effects to leading order in PT

Conclusions

Solving the flow equation on the lattice

The continuum equation

$$\frac{dB_\mu(x,t)}{dt} = D_\nu G_{\nu\mu}(x,t); \quad \left(\sim -g_0^2 \frac{\delta S_{\rm YM}[B]}{\delta B_\mu}\right)$$

How do the links $V_\mu(x,t)$ change with the t?

$$a^2 \frac{d}{dt} V_\mu(x,t) = -g_0^2 \frac{\delta S^{\rm latt}[V]}{\delta V_\mu(x,t)} V_\mu(x,t)$$

- Is this the best option?
- Which lattice action $S^{\rm latt}$?

The Zeuthen flow

$$a^2 \frac{d}{dt} V_\mu(x,t) = -g_0^2 \left(1 + \frac{a^2}{12} D_\mu D_\mu^*\right) \frac{\delta S^{\rm LW}[V]}{\delta V_\mu(x,t)} V_\mu(x,t)$$

This equation is **the result of a computation**.

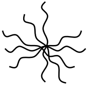
Lattice people hate discovering "new physics"

S = Standard model + Quantum Gravity

At low energies ($\ll M_{\rm pl}$)
$\langle O\rangle = \langle O\rangle_{SM} + \mathcal{O}(1/M_{\rm pl})$

We simulate

$S = \sum_{x,\mu\neq\nu} \mathrm{Tr}(1 - U_\mu(x)U_\nu(x+\mu)\cdots)$



Multi gluon interactions:
6,8,10,12,... gluon vertices.
at energy scales $1/a$

⟹ UNIVERSALITY
(Symmetries, dimensions, ...)

We obtain QCD

$S = -\frac{1}{2}\int \mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu}) + \ldots$

At low energies ($\ll 1/a$)
$\langle O\rangle_{\rm latt} = \langle O\rangle_{\rm QCD} + \mathcal{O}(a^2)$

Symanzik improvement program

Fine tune (i.e. cook) a lattice action $S^{\rm latt}$ such that the efective theory at energy scales much smaller than the cutoff looks as close as possible to the continuum.

The Symanzik effective action

Use pure gauge as an example

- Any lattice action S^{latt} can be described by an effective continuum action

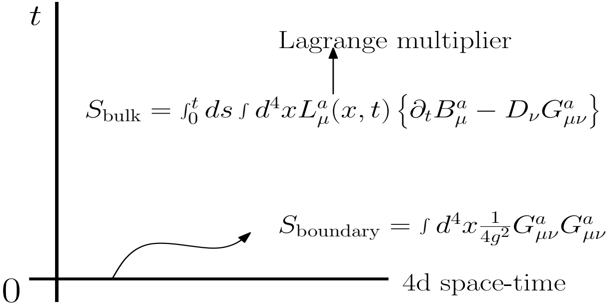
$$S_{\text{eff}} = S_{\text{YM}} + a^2 S^{(2)} + a^4 S^{(4)} + \ldots$$

- An action is "improved" if some of the terms are absent in the effective low energy description (i.e. $S^{(2)} = 0$).
- The effective action is constructed by a linear combination of all the operators compatible with the lattice symmetries. Dimensional analysis helps in the classification ($J_{\mu\nu\rho} = D_\mu F_{\nu\rho}$)

$$S^{(2)} = \alpha_1 \int d^4x \, J_{\mu\nu\rho}(x) J_{\mu\nu\rho}(x) + \alpha_2 \int d^4x \, J_{\mu\mu\rho}(x) J_{\nu\nu\rho}(x) + \alpha_3 \int d^4x \, J_{\mu\mu\rho}(x) J_{\mu\mu\rho}(x)$$

- Even if some of this terms "is not" initially in the lattice action, it will be generated radiatively (i.e. by loops).

The symanzik effective action for the Gradient flow



The symanzik effective action for the Gradient flow

Action composed of bulk part and boundary part

$$S_{\text{bndry}} = -\frac{1}{2g_0^2}\int d^4x\,\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) + \sum_i \alpha_i \int d^4x\, O_i^{\mathrm{d}=6}(x)$$

$$S_{\text{bulk}} = -2\int_0^\infty dt \int d^4x\,\mathrm{Tr}\left\{L_\mu(x,t)[\partial_t B_\mu(x,t) - D_\mu G_{\mu\nu}]\right\} + \sum_i \int_0^\infty dt \int d^4x\, O_i^{\mathrm{d}=8}(x,t)$$

Possible boundary counterterms: ($J_{\mu\nu\rho} = D_\mu F_{\nu\rho}$)

$$O_1^{\mathrm{d}=6} = \mathrm{Tr}\left\{J_{\mu\nu\rho}(x)J_{\mu\nu\rho}(x)\right\}$$

$$O_2^{\mathrm{d}=6} = \mathrm{Tr}\left\{J_{\mu\mu\rho}(x)J_{\nu\nu\rho}(x)\right\}$$

$$O_3^{\mathrm{d}=6} = \mathrm{Tr}\left\{J_{\mu\mu\rho}(x)J_{\mu\mu\rho}(x)\right\}$$

$$O_4^{\mathrm{d}=6} = \mathrm{Tr}\left\{L_\mu(0,x)J_{\nu\mu\nu}(x)\right\}$$

$$O_5^{\mathrm{d}=6} = \mathrm{Tr}\left\{L_\mu(0,x)L_\mu(0,x)\right\}$$

The symanzik effective action for the Gradient flow

Action composed of bulk part and boundary part

$$S_{\text{bndry}} = -\frac{1}{2g_0^2}\int d^4x\,\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) + \sum_i \alpha_i \int d^4x\, O_i^{\mathrm{d}=6}(x)$$

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Possible bulk counterterms

- Remember: No loops in the bulk ⇒ No new counterterms are generated.
- Classical improvement in the bulk is equivalent to **non-perturbative** improvement.

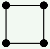
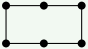
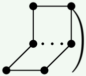
Classical expansion of the flow equation

Lattice flow action

$$S_{\text{bulk}}(c_i) = -2\int_0^\infty dt \sum_x \text{Tr}\left\{L_\mu(x,t)\left[a^2\partial_t V_\mu(x,t)V_\mu^{-1}(x,t) - g_0^2\partial_{x,\mu}S_{\text{latt}}(V)\right]\right\}$$

with a general S_{latt}

$$S_{\text{latt}}(c_i) = \frac{1}{g_0^2}\sum_x \text{Tr}\Big(1 - c_0 \; \square \; - c_1 \; \square\!\square \; - c_2 \; \ldots\Big)$$

$$\left(a^2\partial_t V_\mu\right)V_\mu^{-1} = a^3\partial_t B_\mu + \frac{1}{2}a^4 D_\mu\partial_t B_\mu + \frac{1}{6}a^5 D_\mu^2\partial_t B_\mu + \mathcal{O}(a^6)$$

$$-\partial_{x,\mu}\left[g_0^2 S_{\text{lat}}(V)\right] = \sum_{\nu=0}^{3}\Big\{a^3 D_\nu G_{\nu\mu} + \frac{1}{2}a^4 D_\mu D_\nu G_{\nu\mu}$$
$$+ \frac{1}{12}a^5\Big[(1+12(c_1-c_2))\left(2D_\nu D_\mu^2 + D_\nu^3\right) - 12(c_1-c_2)D_\mu^2 D_\nu$$
$$+ 12c_2\sum_{\rho=0}^{3}\left(3D_\rho^2 D_\nu - 4D_\rho D_\nu D_\rho + 2D_\nu D_\rho^2\right)\Big]G_{\nu\mu}\Big\} + \mathcal{O}(a^6)$$

Classical expansion of the flow equation

$$\left(a^2\partial_t V_\mu\right) V_\mu^{-1} = a^3\partial_t B_\mu + \frac{1}{2}a^4 D_\mu \partial_t B_\mu + \frac{1}{6}a^5 D_\mu^2 \partial_t B_\mu + \mathcal{O}(a^6)$$

$$-\partial_{x,\mu}\left[g_0^2 S_{\text{lat}}(V)\right] = \sum_{\nu=0}^{3}\Big\{a^3 D_\nu G_{\nu\mu} + \frac{1}{2}a^4 D_\mu D_\nu G_{\nu\mu} + \frac{1}{12}a^5\Big[(1+12(c_1-c_2))\left(2D_\nu D_\mu^2 + D_\nu^3\right) - 12(c_1-c_2)D_\mu^2 D_\nu + 12c_2\sum_{\rho=0}^{3}\left(3D_\rho^2 D_\nu - 4D_\rho D_\nu D_\rho + 2D_\nu D_\rho^2\right)\Big]G_{\nu\mu}\Big\} + \mathcal{O}(a^6)$$

Some conclusions

- Correct continuum flow equation

$$\partial_t B_\mu = D_\nu G_{\nu\mu}$$

- $\mathcal{O}(a)$ corrections cancel.
- No value of c_1, c_2 for which the $\mathcal{O}(a^2)$ corrections cancel!

Classical expansion of the flow equation

$$\left(a^2\partial_t V_\mu\right) V_\mu^{-1} = a^3\partial_t B_\mu + \frac{1}{2}a^4 D_\mu\partial_t B_\mu + \frac{1}{6}a^5 D_\mu^2\partial_t B_\mu + \mathcal{O}(a^6)$$

$$-\partial_{x,\mu}\left[g_0^2 S_{\rm lat}(V)\right] = \sum_{\nu=0}^{3}\Big\{a^3 D_\nu G_{\nu\mu} + \frac{1}{2}a^4 D_\mu D_\nu G_{\nu\mu} + \frac{1}{12}a^5\Big[(1+12(c_1-c_2))\left(2D_\nu D_\mu^2 + D_\nu^3\right) - 12(c_1-c_2)D_\mu^2 D_\nu + 12c_2\sum_{\rho=0}^{3}\left(3D_\rho^2 D_\nu - 4D_\rho D_\nu D_\rho + 2D_\nu D_\rho^2\right)\Big]G_{\nu\mu}\Big\} + \mathcal{O}(a^6)$$

the Symanzik/LW flow ($c_1 = -1/12$, $c_2 = 0$), is “almost” $\mathcal{O}(a^2)$ improved

$$\partial_t B_\mu = \sum_{\nu=0}^{3}\left\{D_\nu G_{\nu\mu}(x,t) - \frac{1}{12}a^2 D_\mu^2 D_\nu G_{\nu\mu} + \mathcal{O}(a^3)\right\}$$

The Zeuthen flow

$$\left(a^2\partial_t V_\mu(x,t)\right) V_\mu(x,t)^{-1} = -g_0^2\left(1 + \frac{1}{12}a^2 D_\mu^* D_\mu\right)\partial_{x,\mu}\left[g_0^2 S_{\rm LW}(V)\right]$$

$$aD_\mu F(x) = V_\mu(x,t)F(x+a\hat{\mu})V_\mu(x,t)^\dagger - F(x), \ldots$$

Testing the Zeuthen flow

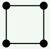
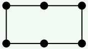
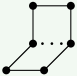
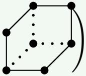
Precise predictions

- If one uses the Zeuthen flow and a classically improved definition of the observable, only $\mathcal{O}(a^2)$ cutoff effects should come from the boundary (i.e. action + boundary terms).
- Quantities computed with the Zeuthen flow are similar to spectral quantities: Cutoff effects directly related to scaling properties of the action.
- If moreover one uses a tree-level improved action there should remain no $\mathcal{O}(g_0^2 a^2)$ cutoff effects. (Testable in PT).

Anatomy of $\mathcal{O}(g_0^2 a^2)$ cutoff effects [Z. Fodor '14; A.R., S. Sint '14]

Three contributions to cutoff effects

1. Action

$$S(c_i^{(a)}) = \frac{1}{g_0^2}\sum_x \mathrm{Tr}\Big(1 - c_0^{(a)}$$  $$- c_1^{(a)}$$  $$- c_2^{(a)}$$  $$- c_3^{(a)}$$  $$\Big)$$

2. Flow discretization

$$\frac{\mathrm{d}}{\mathrm{d}t}V_\mu(x,t) = -g_0^2 \frac{\delta S(c_i^{(f)})}{\delta V_\mu(x,t)} V_\mu(x,t) \quad \Big(+\mathcal{O}(a^p)\Big)$$

3. Observable discretization (i.e. for $\langle E(t)\rangle$)

$$-\frac{1}{2}\mathrm{Tr}G_{\mu\nu}(x,t)G_{\mu\nu}(x,t) = S(c^{(o)})$$

Any action can be written to leading order

$$S(c)[A] = \int_p \tilde{A}_\mu(-p)K_{\mu\nu}(p,\lambda;c)\tilde{A}_\nu(p) + \mathcal{O}(g_0)$$

Anatomy of $\mathcal{O}(g_0^2 a^2)$ cutoff effects [Z. Fodor '14; A.R., S. Sint '14]

Master formula

flow equation is solved by

$$\tilde{B}_\mu^a(t,p) = H_{\mu\nu}^{(f)}(t,p;\lambda)\tilde{A}_\nu^a(p);$$

with the heat kernel

$$H_{\mu\nu}^{(f)}(t,p;\lambda) = \exp\left(-tK^{(f)}(p;\lambda)\right)_{\mu\nu} .$$

We also define the propagator

$$D_{\mu\nu}^{(a)}(p)K_{\nu\rho}^{(a)}(q) = \delta_{\mu\rho}\delta^{(4)}(p+q) .$$

The total contribution to the energy density

$$\begin{aligned} t^2\langle E(t,x)\rangle &= g_0^2 t^2 \mathcal{E}_0(t) + \mathcal{O}(g_0^3) \\ t^2\mathcal{E}_o(t) &= 4\int_p \left\{ H_{\mu\sigma}^{(f)}(t,-p)K_{\mu\nu}^{(o)}(p)H_{\nu\rho}^{(f)}(t,p)D_{\rho\sigma}^{(a)}(p)\right\} , \end{aligned}$$

Anatomy of $\mathcal{O}(g_0^2 a^2)$ cutoff effects [Z. Fodor '14; A.R., S. Sint '14]

Expanding to order a^2

$$t^2\mathcal{E}_0(t) = t^2\mathcal{E}_0^{\rm cont}(t)\left\{1 + \frac{a^2}{t}\left[(d_1^{(o)} - d_1^{(a)})\mathcal{J}_{4,-2} + (d_2^{(o)} - d_2^{(a)})\mathcal{J}_{2,0} - 2d_1^{(f)}\mathcal{J}_{4,0} - 2d_2^{(f)}\mathcal{J}_{2,2}\right]\right\},$$

with

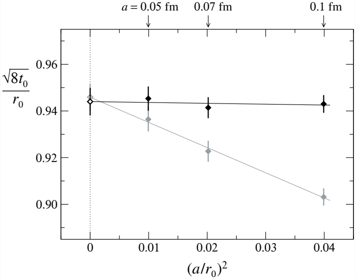
$$\mathcal{J}_{i,j} = t^{(i+j)/2}\frac{\int_p e^{-2tp^2}\,(p^i)(p^j)}{\int_p e^{-2tp^2}},$$

Some lessons

- The observable competes in cutoff effects with the action and the flow.
- Numerically, the flow produces $3\times$ more cutoff effects than the action or the observable (i.e. $\mathcal{J}_{2,2} = 3\mathcal{J}_{2,0}$, $\mathcal{J}_{4,0} = 3\mathcal{J}_{4,-2}$)

Anatomy of $\mathcal{O}(g_0^2 a^2)$ cutoff effects [Z. Fodor '14; A.R., S. Sint '14]

An urban legend: The clover observable has smaller cutoff effects



- In [M. Lüscher '10] never stated that "clover is better".
- This plot only shows that the Wilson action (pure gauge), with Wilson flow and clover observable produce smaller cutoff effects in $\sqrt{8t_0}/r_0$
- But different sources of cutoff effects can be responsible of this behavior.
- This is an accidental cancellation.

Observable	Action	Flow	Total
Clover	Wilson	Wilson	
15	-3	-9	3
Clover	Lüscher-Weisz	Symanzik	
15	1	3	19

Anatomy of $\mathcal{O}(g_0^2 a^2)$ cutoff effects [Z. Fodor '14; A.R., S. Sint '14]

Is the Zeuthen flow improved? YES!

Discretization	d_1	d_2
Plaquette	1/2	0
Lüscher-Weisz	1/6	-1/6
Clover	3/2	-1/2
Zeuthen	0	0

In infinite volume there are many ways to cancel a single $\mathcal{O}(a^2)$ effect, but in a finite volume the numbers $\mathcal{J}_{i,j}$ become functions of the dimensionless ratio $c = \sqrt{8t}/L$

$$\mathcal{J}_{i,j}(c) = \left(\frac{\pi}{\sqrt{2}}\right)^{i+j} c^{i+j} \sum_n{}' \exp\{-c^2\pi^2(n_\mu+\tilde{n}_\mu)^2\}\, n^i n^j \Big/ \sum_n{}' \exp\{-c^2\pi^2(n_\mu+\tilde{n}_\mu)^2\}\,.$$

The Zeuthen flow produces improved observables for all values of the parameter c!.

- Also checked the correlation function

$$\langle E(t,x)E(s,y)\rangle$$

- Other boundary conditions: SF (as long as one remains far from the SF boundaries).

Overview

Conclusions

Twisted boundary conditions

- Essential to understand topology in finite volume
- Together with gradient flow nice FV renormalization scheme
  - Absence of zero-modes in PT
  - Large c to reduce cutoff effects in $g^2_{\rm GF}$
  - Shown some results for $SU(2)$ pure gauge and $SU(3)$ with $N_f = 12$.
- Nice analytic properties for perturbative studies: invariance under translations and absence of zero modes.
- Linked to many more interesting phenomena: Large N volume reduction, Fractional semi-classical solutions, etc…