Three topics with twisted boundary conditions

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February 8, 2016
Overview

Motivation

Understanding topology on a finite volume

The Gradient flow

Applications

Improvement of the Gradient flow

Conclusions
**Topology in infinite volume**

*SU(N)* Yang-Mils theory: Field configurations with finite action

\[
S = - \frac{1}{2g^2} \int d^4x \, \text{tr} \left( F_{\mu\nu} F_{\mu\nu} \right)
\]

\[
Q = \frac{1}{16\pi^2} \int d^4x \, \text{tr} \left( F_{\mu\nu} \tilde{F}_{\mu\nu} \right)
\]

with

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad \text{and} \quad \tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}
\]

Since

\[
\text{tr} \left( F_{\mu\nu} \tilde{F}_{\mu\nu} \right) = \partial_\mu K_\mu = 2\partial_\mu \varepsilon_{\mu\nu\rho\sigma} \text{tr} \left( A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma \right)
\]

Topological charge is the integral of a total divergence

\[
Q = \int_{S^3_\infty} d\sigma_\mu K_\mu = 0(??)
\]
**Topology in infinite volume**

Chern-Simons does not vanish at infinity!

In spherical coordinates $x = (r, \theta, \phi, \varphi)$, at $r = \infty$ the action density must be zero

$$F_{\mu\nu}(x)|_{r=\infty} = 0 \implies A_\mu(x)|_{r=\infty} = \Lambda \partial_\mu \Lambda^\dagger$$

But Chern-Simons is not gauge invariant $A_\mu \to \Lambda A_\mu \Lambda^\dagger + \Lambda \partial_\mu \Lambda^\dagger$

$$K_\mu[A_\mu] \longrightarrow K_\mu[A_\mu] + \frac{2}{3} \varepsilon_{\mu\nu\rho\sigma} \bar{A}_\nu \bar{A}_\rho \bar{A}_\sigma + 2 \varepsilon_{\mu\nu\rho\sigma} \partial_\rho \text{tr}(\bar{A}_\nu A_\sigma),$$

with $\bar{A}_\mu = \Lambda \partial_\mu \Lambda^\dagger$.

$$Q = \frac{1}{16\pi^2} \int_{S_3^\infty} d\sigma_\mu \frac{2}{3} \varepsilon_{\mu\nu\rho\sigma} \bar{A}_\nu \bar{A}_\rho \bar{A}_\sigma \neq 0.$$  

Topological charge is determined by the pure gauge field at infinity $\Lambda(x)|_{r=\infty}$

$$\Lambda : S_3^\infty \longrightarrow SU(2) \sim S^3$$

$\Lambda(x)|_{r=\infty}$ characterized by an integer (i.e. $Q$).
What about gauge fields in $\mathbb{T}^4$?

- $A_\mu(x)$ does not need to be a pure gauge anywhere!
- If the gauge field is periodic $Q = 0!$

$$Q = \frac{1}{16\pi^2} \int_{\mathbb{F}_\mu} d\sigma_\mu \Delta_\mu K_\mu$$

with $\Delta_\mu f(x) = f(x + L\hat{\mu}) - f(x)$
**Topology in finite volume**

Only Gauge invariant quantities need to be periodic

\[
A_\mu(x + L\hat{\nu}) = \Omega_\nu(x)A_\mu(x)\Omega_\nu^\dagger(x) + \Omega_\nu(x)\partial_\mu\Omega_\nu^\dagger(x) = A_\mu^{[\Omega_\nu(x)]}(x) .
\]

Consistency requires

\[
A_\mu(x + L\hat{\nu} + L\hat{\rho}) = A_\mu^{[\Omega_\nu(x+L\hat{\rho})\Omega_\rho(x)]}(x) = A_\mu^{[\Omega_\rho(x+L\hat{\nu})\Omega_\nu(x)]}(x)
\]

\[
\Omega_\rho(x + L\hat{\nu})\Omega_\nu(x) = e^{2\pi in_{\rho\nu}/N} \Omega_\nu(x + L\hat{\rho})\Omega_\rho(x)
\]

- \(\Omega_\mu(x)\) are *twist matrices*. They change under gauge transformation.
- \(n_{\mu\nu}\) is the *twist tensor*. Invariant under gauge transformations. Encodes physics of the twist.

**Abelian twist matrices**

\[
\Omega_\mu(x) = \exp\{\omega(x)\}; \quad \omega(x) = \frac{\pi x_\nu}{NL_\nu} W_{\mu\nu}
\]

with \((\mathbb{T} = \text{diag}(1, 1, \ldots, 1 - N))\) and \(\text{tr}(Q_{\mu\nu}) = -n_{\mu\nu}\)

\[
W_{\mu\nu} = n_{\mu\nu} \mathbb{I} + NQ_{\mu\nu}; \quad W_{\mu\nu} = n_{\mu\nu} \mathbb{T}
\]
**Topology in finite volume**

\[ Q = -\frac{1}{24\pi^2} \epsilon_{\mu\nu\rho\sigma} \int d\sigma_{\mu} \text{tr} \left\{ \Omega_{\mu}^\dagger (\partial_{\nu} \Omega_{\mu}) \Omega_{\mu}^\dagger (\partial_{\rho} \Omega_{\mu}) \Omega_{\mu}^\dagger (\partial_{\sigma} \Omega_{\mu}) \right\} 
- \frac{1}{8\pi^2} \epsilon_{\mu\nu\rho\sigma} \int d\sigma_{\mu\nu} \text{tr} \left\{ [\Delta_{\rho} (\Omega_{\mu}^\dagger (\partial_{\nu} \Omega_{\mu})) + \Omega_{\mu}^\dagger (\partial_{\nu} \Omega_{\mu})] \Omega_{\rho}^\dagger (\partial_{\sigma} \Omega_{\rho}) \right\} \]

By using the generic form of the Abelian Twist matrices \( W_{\mu\nu} = n_{\mu\nu} I + NQ_{\mu\nu} \)

\[ Q = -\frac{\kappa(n)}{N} + n \]

with \( \kappa(n) = n_{\mu\nu} \tilde{n}_{\mu\nu} / 4 \in \mathbb{Z} \) [Van Baal ‘82].

**On the lattice** \( U_{\mu}(x) \) is periodic

\( A_{\mu}(x + L\hat{\nu}) = A_{\mu}(x) + 2\pi q \) allows to have all integer topological charges!
Overview

Motivation

The Gradient flow
   Main idea behind the Gradient flow
   Uses of the Gradient flow
   Twisted coupling

Applications

Improvement of the Gradient flow

Conclusions
A key tool for non-perturbative studies [Narayanan, Neuberger ‘06; Lüscher ‘10]

Main idea

- Add extra (flow) time coordinate $t(\neq x_0)$ with $[t] = \text{length}^2$. Define gauge field $B_\mu(t, x)$

$$\frac{dB_\mu(x, t)}{dt} = D_\nu G_{\nu\mu}(x, t); \quad \left( \sim - \frac{\delta S_{YM}[B]}{\delta B_\mu} \right)$$

$$G_{\nu\mu}(x, t) = \partial_\nu B_\mu(x, t) - \partial_\nu B_\mu(x, t) + [B_\nu(x, t), B_\mu(x, t)],$$

with initial condition $B_\mu(t, x)|_{t=0} = A_\mu(x)$

- Composite gauge invariant operators are renormalized observables defined at a scale $\mu = 1/\sqrt{8t}$ [M. Lüscher ‘10; M. Lüscher, P. Weisz ‘11].

- Example

$$\langle E(t) \rangle = -\frac{1}{2} \text{Tr}\langle G_{\mu\nu}(x, t)G_{\mu\nu}(x, t) \rangle$$

finite quantity for $t > 0$.

- Continuum limit to be taken at fixed $t$. 
Scale setting [M. Lüscher '10; Borsanyi et. al. '12; R. Sommer Latt. '14]

- $t^2\langle E(t) \rangle$ is dimensionless.
- Depends on scale $\mu = 1/\sqrt{8t}$
- Ideal candidate for scale setting: $t_0$ [M. Lüscher JHEP 1008 '10].
- Similar quantities: $t_1, w_0, \ldots$
- Dimensionless ratios of “flow quantities” (i.e. $\sqrt{t_0}/w_0$) are easily computable quantities with continuum limit.

Cheap quantities with continuum limit

With the definition $\mathcal{E}(t) = t^2\langle E(t) \rangle$

$t_0 : \quad \mathcal{E}(t_0) = 0.3$
$w_0 : \quad w_0^2 \frac{d}{dt} \mathcal{E}(t) \bigg|_{t=w_0^2} = 0.3$

we have

$\sqrt{t_0} \sim 0.15 \text{fm} \quad w_0 \sim 0.18 \text{fm}$
Renormalized couplings \[\text{[M. Lüscher ‘10; Z. Fodor et al. ‘12; P. Fritzsch, A.R. ‘13; M. Lüscher ‘14; A.R. ‘14]}\]

Non-perturbative coupling definition

Perturbative computation

\[
t^2 \langle E(t) \rangle = \frac{3}{16\pi^2} g_{\text{MS}}^2(\mu) \left[ 1 + c_1 g_{\text{MS}}^2(\mu) + \mathcal{O}(g_{\text{MS}}^4) \right],
\]

immediately suggests

\[
g_{\text{GF}}^2(\mu) = \frac{16\pi^2}{3} t^2 \langle E(t) \rangle \bigg|_{\mu=1/\sqrt{8t}}
\]

- On the lattice, infinite volume \( a \ll \sqrt{8}t \ll L \).

Finite volume renormalization schemes

\[
g_{\text{GF}}^2(\mu) = N^{-1} t^2 \langle E(t) \rangle \bigg|_{\mu=1/\sqrt{8t}}
\]

- To avoid the need of a window \( a \ll \sqrt{8}t \ll L \), use \( \mu = 1/cL \).
- Boundary conditions become relevant, and \( \frac{16\pi^2}{3} \rightarrow N^{-1} \):
- Periodic, SF, Twisted, SF-open.
**Why is a good choice? \( N_f = 2 \) and \( SU(3) \) simulations**

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<th>( L/a )</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>16</th>
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<td>0.136700</td>
<td>0.136785</td>
<td>0.136623</td>
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<tr>
<td>( N_{\text{meas}} )</td>
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<td>8320</td>
<td>8192</td>
<td>8280</td>
<td>8460</td>
</tr>
</tbody>
</table>

| \( \tilde{\delta}^2_{\text{SF}}(L_1) \) | 4.423(75) | 4.473(83) | 4.49(10) | 4.501(91) | 4.40(10) |
| \( \tilde{\delta}^2_{\text{GF}}(\mu) \) (\( c = 0.3 \)) | 4.8178(46) | 4.7278(46) | 4.6269(47) | 4.5176(47) | 4.4410(53) |
| \( \tilde{\delta}^2_{\text{GF}}(\mu) \) (\( c = 0.4 \)) | 6.0090(86) | 5.6985(86) | 5.5976(97) | 5.4837(97) | 5.410(12) |
| \( \tilde{\delta}^2_{\text{GF}}(\mu) \) (\( c = 0.5 \)) | 7.106(14) | 6.817(15) | 6.761(19) | 6.658(19) | 6.602(24) |
Boundary conditions

With periodic b.c., dynamics is dominated by fluctuations constant in space

- Leading order contribution of zero momentum modes is not quadratic [A. Gonzalez-Arroyo et al. 1983]
  \[ S \sim [\hat{A}_\mu(0), \hat{A}_\nu(0)]^2. \]

  With periodic b.c. these are not gauge degrees of freedom.

- One has to solve these integrals to define a "propagator"
  \[ \langle A_\mu A_\nu \rangle \sim \int DA A_\mu A_\nu e^{-A_\mu MA_\nu - [\hat{A}_\mu(0), \hat{A}_\nu(0)]^2} \]

- Convergence properties of these integrals depend on \( d \) and SU(\( N \)).
- Difficult (but this running scheme has been proposed [Z. Fodor et al. 2012. arXiv:1208.1051]).
- Coupling definition \( \alpha_{PT} \) non analytic in \( \alpha_{MS} \).

Moral: periodic b.c. in small volume leads to

- Non analytic coupling:
  - \( SU(N) \) and \( N > 2 \): \( \alpha = \alpha_{MS}(1 + \sqrt{\alpha_{MS}} + \ldots) \)
  - \( SU(2) \): \( \alpha = \alpha_{MS}(1 + \log \alpha_{MS} \ldots) \)
**Boundary conditions**

Make constant gauge field configurations incompatible with boundary conditions.

**Schrödinger Functional**

- SF: $L^3 \times T$ box. Dirichlet b.c. in time

\[
A_i(x, x_0 = 0) = C_i(x) \\
A_i(x, x_0 = T) = C'_i(x)
\]

- If $C_i, C'_i$ chosen wisely $\Rightarrow$ Unique gauge configuration with minimum action.

**Twisted boundary conditions**

- Gauge field periodic modulo g.t.

\[
A_\mu(x + L\hat{\nu}) = \Omega_\nu(x)A_\mu(x)\Omega_\nu^\dagger(x) + \Omega_\nu(x)\partial_\mu\Omega_\nu^\dagger(x).
\]

- If $\Omega_\mu$ chosen wisely $\Rightarrow$ Unique gauge configuration with minimum action.
**Boundary conditions**

Arbitrary matter content ⇒ Same coupling definition.

- **Manifold with boundary:** Boundary Counterterms $O(a)$
- **Chiral symmetry breaking**
  \[ P_{\pm} = \frac{1}{2} (1 \pm \gamma_0) \]
  \[ P_+ \psi|_{x_0=0} = \rho_+; P_- \psi|_{x_0=T} = \rho_- \]
- **Naively larger discretization effects!**
- **Work hard for $O(a)$ improvement** (PT + numerics): $c_{SW}, c_t, c_{\tilde{t}}$.

**Schrödinger Functional**

- Fermions in fundamental rep.
  \[ \psi(x + L\hat{\mu}) = \Omega_{\mu} \psi(x) \]
  And therefore
  \[ \psi(x + L\hat{1} + L\hat{2}) = \Omega_1 \Omega_2 \psi(x) \]
  \[ \psi(x + L\hat{2} + L\hat{1}) = \Omega_2 \Omega_1 \psi(x) \]
- **Twisted boundary conditions**
- **Only way out:** $SU(N)$ with $N_f$ and $N_f/N \in \mathbb{Z}$ [Parisi '83].
**Coupling with twisted bc.**

(Task in life)

Compute in perturbation theory

\[ t^2 \langle E(t) \rangle = \frac{1}{4} \langle G_{\mu\nu}(t)G_{\mu\nu}(t) \rangle = \mathcal{N}(t)\alpha_0 + \mathcal{O}(\alpha_0^2) \]

1. Compute \( B_\mu(x, t) \) to leading order.
2. Compute The observable.
**Coupling with twisted bc.**

**Key equation**

\[ A_\mu(x + L\hat{\nu}) = \Omega_\nu(x)A_\mu(x)\Omega_\nu^+(x) + \Omega_\nu(x)\partial_\mu \Omega_\nu^+(x) = A_\mu^{[\Omega_\nu(x)]}(x). \]

**Our particular setup (similar to “TPL” scheme)**

- Use constant twist matrices \( \Omega_\mu(x) = \Omega_\mu \).

\[ A_\mu(x + L\hat{\nu}) = \Omega_\nu A_\mu(x)\Omega_\nu^+ \]

and \( A_\mu(x) = 0 \) compatible with bc.

- We choose to twist only the plane \( x_1 - x_2 \) and \( n_{12} = 1 \).

\[ \Omega_{3,4}(x) = 1; \quad \Omega_1\Omega_2 = e^{2\pi im/N} \Omega_2\Omega_1 \]
Coupling with twisted bc.

\[ A_{\mu}(x + L\hat{k}) = \Omega_k A_{\mu}(x)\Omega^+_k, \]

Define \( N^2 \) matrices \( (\tilde{p}_i = \frac{2\pi\tilde{n}_i}{NL} \text{ with } \tilde{n}_i = 0, \ldots, N - 1) \).

\[ \Gamma(\tilde{p}) = e^{i\alpha(\tilde{p})} \Omega_1^{-\tilde{n}_2} \Omega_2^{\tilde{n}_1} \]

They are traceless (except \( \tilde{p} = 0 \)), linearly independent and

\[ \Omega_i \Gamma(\tilde{p}) \Omega^+_i = e^{iL\tilde{p}_i} \Gamma(\tilde{p}) . \]

Therefore any gauge connection compatible with bc. can be expanded

\[ A^a_{\mu}(x) T^a = \sum_{\tilde{p}} \hat{A}_{\mu}(x, \tilde{p}) e^{i\tilde{p}x} \Gamma(\tilde{p}). \]

with \( \hat{A}_{\mu}(x, \tilde{p}) \) (numbers) periodic in \( x! \). \( p_{\mu} = \frac{2\pi n_{\mu}}{L} \) \( (n_{\mu} \in \mathbb{Z}) \).

\[ \frac{1}{L^4} \sum_{\tilde{p}} \hat{A}_{\mu}(p, \tilde{p}) e^{i(p+\tilde{p})x} \Gamma(\tilde{p}) = \frac{1}{L^4} \sum_P \hat{A}_{\mu}(P) e^{iPx} \Gamma(P) . \]

“Total” momentum: \( P_i = p_i + \tilde{p}_i, P_{3,4} = p_{3,4} \). Color dof \( \leftrightarrow \) momentum dof (Large \( N \), reduction, \( \ldots \)). Only constant connection: \( A_{\mu}(x) = 0! \)
Coupling with twisted bc.

\[
\dot{B}_\mu(x,t) = D_\nu G_{\nu\mu}(x,t), \quad B_\mu(x,0) = A_\mu(x),
\]

\[
G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]
\]

\(B_\mu(x,t)\) has an asymptotic expansion in \(g_0\)

\[
B_\mu(x,t) = \sum_n B_{\mu,n}(x,t)g_0^n
\]

After gauge fixing and to leading order

\[
\dot{B}_{\mu,1}(x,t) = \partial^2_\nu B_{\mu,1}(x,t) \quad (B_{\mu,1}(x,0) = A_\mu(x))
\]

with solution

\[
B_{\mu,1}(x,t) = \frac{1}{L^4} \sum_{p,\tilde{p} \neq 0} e^{-p^2 t} \tilde{A}_\mu(P) e^{iP x} \Gamma(P).
\]

And finally

\[
\langle E(t) \rangle = \frac{1}{4} \langle G_{\mu\nu}(t) G_{\mu\nu}(t) \rangle = \mathcal{E}(t) + \mathcal{O}(g_0^4)
\]

\[
\mathcal{E}(t) = \frac{g_0^2(d - 1)}{2L^4} \sum_{p,\tilde{p} \neq 0} e^{-p^2 t}
\]
Overview

Motivation

The Gradient flow

Applications

- $SU(2)$ Pure gauge theory
- $SU(3)$ with $N_f = 12$ flavors

Improvement of the Gradient flow

Conclusions
Running coupling

\[ \beta \leftrightarrow a; \quad g^2(L) \leftrightarrow L \leftrightarrow \mu \]

Step scaling function

\[ \sigma^{-1}(u, a) = \left. g^2 \left( \frac{L}{2} \right) \right|_{g^2(L) = u} \]

Continuum limit

\[ \sigma^{-1}(u) = \lim_{a \to 0} \sigma^{-1}(u, a) \]

Simulate several pair of lattices
**SU(2) YM running coupling**

- Simulations for $L/a = 10, 12, 15, 18, 20, 24, 30, 36$ at $\beta \in [2.75, 12]$.
- Modest statistics: 2048 independent measurements of $g_{TGF}^2$.
- Between 0.15-0.25% precision in $g_{TGF}^2$ for all $L/a$.
- Padé fit (constrain to PT), 4 parameters, $\chi^2/\text{ndof} = 5.9/7$.
- Example: $L/a = 36$

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<th>$g_{TGF}^2(L)$</th>
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<td>12.0</td>
<td>0.41078(64)</td>
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<td>10.0</td>
<td>0.51809(83)</td>
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<tr>
<td>8.0</td>
<td>0.6987(11)</td>
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<td>7.0</td>
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<td>4.0</td>
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<td>2.9</td>
<td>10.610(32)</td>
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![Graph showing simulations and fit](image-url)
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Step scaling function

- Modest cutoff effects. Starting recursion with $u = 7.5$. 

![Graph showing the relationship between $\sigma^{-1}(u,2)$ and $(a/L)^2$. The graph includes a continuum fit and error bars for each data point.]
Step scaling function

- Modest cutoff effects. Starting recursion with $u = 7.5$. 
$g^2_{TGF}(L)$ FOR PURE GAUGE $SU(2)$
Conformal behavior of gauge theories with large $N_f$.

Looking for conformal behavior (in collaboration with [C.J. David Lin, K. Ogawa, A.R. '15.])

- For small $N_f$ QCD like theory: Confinement, $\Lambda_{\text{QCD}}$
- If $N_f \gg 1$ No more asymptotic freedom.
- In between *maybe* conformal behavior: No typical scale for your theory at large distances, no mass gap.

- A value of the coupling $g^2_*(L) = g^2_*(2L)$
- This property true for every observable analytically related with the one that defines your scheme (e.g. scheme independent statement).
CONFORMAL BEHAVIOR OF GAUGE THEORIES WITH LARGE $N_f$.

Looking for conformal behavior (in collaboration with [C.J. David Lin, K. Ogawa, A.R. '15.])

Change of the renormalised coupling in 12-flavour SU(3) theory
SU(3) with $N_f = 12$ flavors

Reuse data generated for the TPL coupling

  - Already $O(200,000 - 1000,000)$ configurations for TPL coupling.
  - Choose $O(1000)$ of them and win an order of magnitude in statistics.
  - A worst case example $L/a = 16, \beta = 4.38$. 

![MC history graph]
**SU(3) with \( N_f = 12 \) flavors**

Reuse data generated for the TPL coupling

- Already \( \mathcal{O}(200,000 - 1000,000) \) configurations for TPL coupling.
- Choose \( \mathcal{O}(1000) \) of them and win an order of magnitude in statistics.
- A worst case example \( L/a = 16, \beta = 4.38 \).
$SU(3)$ with $N_f = 12$ flavors

$L/a = 6, 8, 10, 12 \rightarrow 12, 16, 20, 24$

Plaquette discretisation

$L/a = 24, \ N_{\text{param}} = 9$

$L/a = 12, \ N_{\text{param}} = 10$

$c_T = 0.500$
SU(3) with $N_f = 12$ flavors

$L/a = 6, 8, 10, 12 \rightarrow 12, 16, 20, 24$

Open symbol: plaquette; Filled symbol: clover

$c_\tau = 0.375$
**SU(3) with N_f = 12 Flavors**

\[ L/a = 6, 8, 10, 12 \rightarrow 12, 16, 20, 24 \]

- Open symbol: plaquette; Filled symbol: clover
- \( c_\tau = 0.375 \)
SU(3) with $N_f = 12$ flavors

$L/a = 6, 8, 10, 12 \rightarrow 12, 16, 20, 24$

Open symbol: plaquette; Filled symbol: clover

$c_T = 0.375$
**SU(3) with $N_f = 12$ Flavors**

$L/a = 6, 8, 10, 12 \rightarrow 12, 16, 20, 24$

![Graph showing $g_{\text{latt}}^2(a,2L)$ vs. $(a/L)^2$ with open and filled symbols for plaquette and clover, respectively. The horizontal line at $0.400$ and the equation $c_\tau = 0.450$.](image)
**SU(3) with N_f = 12 Flavors**

$L/a = 6, 8, 10, 12 \rightarrow 12, 16, 20, 24$

Open symbol: plaquette; Filled symbol: clover

$c_\tau = 0.450$
$SU(3)$ with $N_f = 12$ flavors

$L/a = 6, 8, 10, 12 \rightarrow 12, 16, 20, 24$

Open symbol: plaquette; Filled symbol: clover

$c_T = 0.450$
SU(3) with $N_f = 12$ flavors

$L/a = 6, 8, 10, 12 \rightarrow 12, 16, 20, 24$

c$_\tau = 0.375$

$\frac{g^2_{\text{GF}}(2L)}{g^2_{\text{GF}}(L)}$

$g^2_{\text{GF}}(L)$

clover

plaquette
$SU(3)$ with $N_f = 12$ flavors

$L/a = 6, 8, 10, 12 \rightarrow 12, 16, 20, 24$

![Graph showing $g_{GF}(L)$ vs. $g_{GF}(2L)/g_{GF}(L)$ with error bars for clover and plaquette methods. The graph also shows $c_\tau = 0.450$ and a 2-loop correction.]
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The Gradient flow

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Improvement of the Gradient flow
   The symanzik effective action
      $O(a^2)$ cutoff effects to leading order in PT

Conclusions
SOLVING THE FLOW EQUATION ON THE LATTICE

The continuum equation

\[ \frac{d B_\mu(x, t)}{dt} = D_\nu G_{\nu \mu}(x, t); \quad \left( \sim -g_0^2 \frac{\delta S_{\text{YM}}[B]}{\delta B_\mu} \right) \]

How do the links \( V_\mu(x, t) \) change with the \( t \)?

\[ a^2 \frac{d}{dt} V_\mu(x, t) = -g_0^2 \frac{\delta S_{\text{latt}}[V]}{\delta V_\mu(x, t)} V_\mu(x, t) \]

- Is this the best option?
- Which lattice action \( S_{\text{latt}} \)?

The Zeuthen flow

\[ a^2 \frac{d}{dt} V_\mu(x, t) = -g_0^2 \left( 1 + \frac{a^2}{12} D_\mu D_\mu^* \right) \frac{\delta S_{\text{LS}}[V]}{\delta V_\mu(x, t)} V_\mu(x, t) \]

This equation is the result of a computation.
**Lattice people hate discovering “new physics”**

\[ S = \text{Standard model + Quantum Gravity} \]

We simulate

\[ S = \sum_{x,\mu \neq \nu} \text{Tr}(1 - U_\mu(x)U_\nu(x + \mu) \cdots) \]

Multi gluon interactions:

6, 8, 10, 12, \ldots gluon vertices.

at energy scales \(1/a\)

---

At low energies (\(\ll M_{pl}\))

\[ \langle O \rangle = \langle O \rangle_{SM} + \mathcal{O}(1/M_{pl}) \]

---

We obtain QCD

\[ S = -\frac{1}{2} \int \text{Tr}(F_{\mu\nu}F_{\mu\nu}) + \cdots \]

At low energies (\(\ll 1/a\))

\[ \langle O \rangle_{\text{latt}} = \langle O \rangle_{\text{QCD}} + \mathcal{O}(a^2) \]

---

**Symanzik improvement program**

Fine tune (i.e. cook) a lattice action \(S^{\text{latt}}\) such that the effective theory at energy scales much smaller than the cutoff looks as close as possible to the continuum.
The Symanzik effective action

Any lattice action $S^{\text{latt}}$ can be described by an effective continuum action

$$S_{\text{eff}} = S_{\text{YM}} + a^2 S^{(2)} + a^4 S^{(4)} + \ldots$$

An action is “improved” if some of the terms are absent in the effective low energy description (i.e. $S^{(2)} = 0$).

The effective action is constructed by a linear combination of all the operators compatible with the lattice symmetries. Dimensional analysis helps in the classification $(J_{\mu\nu\rho} = D_{\mu}F_{\nu\rho})$

$$S^{(2)} = \alpha_1 \int d^4x J_{\mu\nu\rho}(x)J_{\mu\nu\rho}(x) + \alpha_2 \int d^4x J_{\mu\mu\rho}(x)J_{\nu\nu\rho}(x) + \alpha_3 \int d^4x J_{\mu\mu\rho}(x)J_{\mu\mu\rho}(x)$$

Even if some of this terms “is not” initially in the lattice action, it will be generated radiatively (i.e. by loops).
The symanzik effective action for the Gradient flow

\[ S_{\text{bulk}} = \int_0^t ds \int d^4x L^a_\mu(x, t) \left\{ \partial_t B^a_\mu - D_\nu G^a_{\mu\nu} \right\} \]

\[ S_{\text{boundary}} = \int d^4x \frac{1}{4g^2} G^a_{\mu\nu} G^a_{\mu\nu} \]

4d space-time
**The Symanzik effective action for the Gradient flow**

Action composed of bulk part and boundary part

\[
S_{\text{bndry}} = -\frac{1}{2g_0^2} \int d^4 x \, \text{Tr}(F_{\mu\nu}F_{\mu\nu}) + \sum_i \alpha_i \int d^4 x \, O_{i=6}^d(x)
\]

\[
S_{\text{bulk}} = -2 \int_0^\infty dt \int d^4 x \, \text{Tr} \{L_\mu(x, t)[\partial_t B_\mu(x, t) - D_\mu G_{\mu\nu}]\}
\]

\[
+ \sum_i \int_0^\infty dt \int d^4 x \, O_{i=8}^d(x, t)
\]

Possible boundary counterterms: \((J_{\mu\nu\rho} = D_\mu F_{\nu\rho})\)

\[
O_{1=6}^d = \text{Tr} \{J_{\mu\nu\rho}(x)J_{\mu\nu\rho}(x)\} \quad O_{4=6}^d = \text{Tr} \{L_\mu(0, x)J_{\nu\mu\nu}(x)\}
\]

\[
O_{2=6}^d = \text{Tr} \{J_{\mu\mu\rho}(x)J_{\nu\nu\rho}(x)\} \quad O_{5=6}^d = \text{Tr} \{L_\mu(0, x)L_\mu(0, x)\}
\]

\[
O_{3=6}^d = \text{Tr} \{J_{\mu\rho}(x)J_{\mu\rho}(x)\}
\]
The symanzik effective action for the Gradient flow

Action composed of bulk part and boundary part

\[ S_{\text{bndry}} = -\frac{1}{2g^2_0} \int d^4x \, \text{Tr}(F_{\mu\nu}F_{\mu\nu}) + \sum_i \alpha_i \int d^4x \, O^{d=6}_i(x) \]

\[ S_{\text{bulk}} = -2 \int_0^\infty dt \int d^4x \, \text{Tr} \{ L_\mu(x, t)[\partial_t B_\mu(x, t) - D_\mu G_{\mu\nu}] \} \]

\[ + \sum_i \int_0^\infty dt \int d^4x \, O^{d=8}_i(x, t) \]

Possible bulk counterterms

- Remember: No loops in the bulk ⇒ No new counterterms are generated.
- Classical improvement in the bulk is equivalent to non-perturbative improvement.
**Classical expansion of the flow equation**

Lattice flow action

\[ S_{\text{bulk}}(c_i) = -2 \int_0^\infty dt \sum_x \text{Tr} \left\{ L_{\mu}(x, t) \left[ a^2 \partial_t V_\mu(x, t) V^{-1}_\mu(x, t) - g_0^2 \partial_{x, \mu} S_{\text{latt}}(V) \right] \right\} \]

with a general \( S_{\text{latt}} \)

\[ S_{\text{latt}}(c_i) = \frac{1}{g_0^2} \sum_x \text{Tr} \left( 1 - c_0 - c_1 - c_2 \right) \]

\[
\left( a^2 \partial_t V_\mu \right) V^{-1}_\mu = a^3 \partial_t B_\mu + \frac{1}{2} a^4 D_{\mu} \partial_t B_\mu + \frac{1}{6} a^5 D^2_{\mu} \partial_t B_\mu + \mathcal{O}(a^6) \\
- \partial_{x, \mu} \left[ g_0^2 S_{\text{latt}}(V) \right] = \sum_{\nu=0}^3 \left\{ a^3 D_{\nu} G_{\nu \mu} + \frac{1}{2} a^4 D_{\mu} D_{\nu} G_{\nu \mu} \right. \\
+ \frac{1}{12} a^5 \left[ (1 + 12(c_1 - c_2)) \left( 2D_{\nu} D^2_{\mu} + D^3_{\nu} \right) - 12(c_1 - c_2) D^2_{\mu} D_{\nu} \\
+ 12 c_2 \sum_{\rho=0}^3 \left( 3D^2_{\rho} D_{\nu} - 4D_{\rho} D_{\nu} D_{\rho} + 2D_{\nu} D^2_{\rho} \right) \right] G_{\nu \mu} \right\} + \mathcal{O}(a^6)
\]
Classical expansion of the flow equation

\[
\left( a^2 \partial_t V_\mu \right) V_\mu^{-1} = a^3 \partial_t B_\mu + \frac{1}{2} a^4 D_\mu \partial_t B_\mu + \frac{1}{6} a^5 D^2_\mu \partial_t B_\mu + \mathcal{O}(a^6)
\]

\[
- \partial_{x,\mu} \left[ g_0^2 S_{\text{lat}}(V) \right] = \sum_{\nu=0}^{3} \left\{ a^3 D_\nu G_{\nu\mu} + \frac{1}{2} a^4 D_\mu D_\nu G_{\nu\mu} + \frac{1}{12} a^5 \left[ (1 + 12(c_1 - c_2)) \left( 2D_\nu D^2_\mu + D^3_\nu \right) - 12(c_1 - c_2) D^2_\mu D_\nu \right. \right.
\]
\[
+ \left. 12 c_2 \sum_{\rho=0}^{3} \left( 3D^2_\rho D_\nu - 4D_\rho D_\nu D_\rho + 2D_\nu D^2_\rho \right) \right] G_{\nu\mu} \} + \mathcal{O}(a^6)
\]

Some conclusions

- Correct continuum flow equation
  \[
  \partial_t B_\mu = D_\nu G_{\nu\mu}
\]

- \( \mathcal{O}(a) \) corrections cancel.

- No value of \( c_1, c_2 \) for which the \( \mathcal{O}(a^2) \) corrections cancel!
Classical expansion of the flow equation

\[
\left(a^2 \partial_t V_\mu \right) V_\mu^{-1} = a^3 \partial_t B_\mu + \frac{1}{2} a^4 D_\mu \partial_t B_\mu + \frac{1}{6} a^5 D_\mu^2 \partial_t B_\mu + O(a^6)
\]

\[
- \partial_{x,\mu} \left[ g_0^2 S_{\text{lat}}(V) \right] = \sum_{\nu=0}^{3} \left\{ a^3 D_\nu G_\nu \mu + \frac{1}{2} a^4 D_\mu D_\nu G_\nu \mu 
+ \frac{1}{12} a^5 \left[ \left( 1 + 12(c_1 - c_2) \right) \left( 2D_\nu D_\mu^2 + D_\nu^3 \right) - 12(c_1 - c_2) D_\mu^2 D_\nu 
+ 12c_2 \sum_{\rho=0}^{3} \left( 3D_\rho^2 D_\nu - 4D_\rho D_\nu D_\rho + 2D_\nu D_\rho^2 \right) \right] G_\nu \mu \right\} + O(a^6)
\]

the Symanzik/LW flow \((c_1 = -1/12, c_2 = 0)\), is “almost” \(O(a^2)\) improved

\[
\partial_t B_\mu = \sum_{\nu=0}^{3} \left\{ D_\nu G_\nu \mu(x, t) - \frac{1}{12} a^2 D_\mu^2 D_\nu G_\nu \mu + O(a^3) \right\}
\]

The Zeuthen flow

\[
\left(a^2 \partial_t V_\mu(x, t) \right) V_\mu(x, t)^{-1} = -g_0^2 \left( 1 + \frac{1}{12} a^2 D_\mu^* D_\mu \right) \partial_{x,\mu} \left[ g_0^2 S_{\text{LW}}(V) \right]
\]

\[
aD_\mu F(x) = V_\mu(x, t) F(x + a\hat{\mu}) V_\mu(x, t)^\dagger - F(x), ... 
\]
Testing the Zeuthen flow

Precise predictions

- If one uses the Zeuthen flow and a classically improved definition of the observable, only $O(a^2)$ cutoff effects should come from the boundary (i.e. action + boundary terms).
- Quantities computed with the Zeuthen flow are similar to spectral quantities: Cutoff effects directly related to scaling properties of the action.
- If moreover one uses a tree-level improved action there should remain no $O(g_0^2a^2)$ cutoff effects. (Testable in PT).
ANATOMY OF $\mathcal{O}(g_0^2a^2)$ CUTOFF EFFECTS [Z. Fodor '14; A.R., S. Sint '14]

Three contributions to cutoff effects

1. Action

$$S(c_i^{(a)}) = \frac{1}{g_0^2} \sum_x \text{Tr} \left( 1 - c_{0}^{(a)} - c_{1}^{(a)} - c_{2}^{(a)} \right)$$

2. Flow discretization

$$\frac{d}{dt} V_\mu(x,t) = -g_0^2 \frac{\delta S(c_i^{(f)})}{\delta V_\mu(x,t)} V_\mu(x,t) \left( + \mathcal{O}(a^p) \right)$$

3. Observable discretization (i.e. for $\langle E(t) \rangle$)

$$-\frac{1}{2} \text{Tr} G_{\mu\nu}(x,t) G_{\mu\nu}(x,t) = S(c^{(o)})$$

Any action can be written to leading order

$$S(c)[A] = \int P \tilde{A}_\mu(-p)K_{\mu\nu}(p, \lambda; c)\tilde{A}_\nu(p) + \mathcal{O}(g_0)$$
**Anatomy of $O(g_0^2a^2)$ cutoff effects** [Z. Fodor ‘14; A.R., S. Sint ‘14]

**Master formula**

Flow equation is solved by

$$\tilde{B}^a_\mu(t,p) = H^{(f)}_{\mu\nu}(t,p;\lambda)\tilde{A}^a_\nu(p);$$

with the heat kernel

$$H^{(f)}_{\mu\nu}(t,p;\lambda) = \exp\left(-tK^{(f)}(p;\lambda)\right)_{\mu\nu}.$$

We also define the propagator

$$D^{(a)}_{\mu\nu}(p)K^{(a)}_{\nu\rho}(q) = \delta_{\mu\rho}\delta^{(4)}(p+q).$$

The total contribution to the energy density

$$t^2\langle E(t,x) \rangle = g_0^2t^2\mathcal{E}_0(t) + O(g_0^3)$$

$$t^2\mathcal{E}_0(t) = 4\int_p \left\{ H^{(f)}_{\mu\sigma}(t,-p)K^{(0)}_{\mu\nu}(p)H^{(f)}_{\nu\rho}(t,p)D^{(a)}_{\rho\sigma}(p) \right\},$$
Motivation

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**Anatomy of $O(g_0^2a^2)$ Cutoff Effects** [Z. Fodor ’14; A.R., S. Sint ’14]

Expanding to order $a^2$

$$t^2\mathcal{E}_0(t) = t^2\mathcal{E}_0^{\text{cont}}(t) \left\{ 1 + \frac{a^2}{t} \left[ (d_1^{(o)} - d_1^{(a)}) \mathcal{J}_{4,-2} + (d_2^{(o)} - d_2^{(a)}) \mathcal{J}_{2,0} - 2d_1^{(f)} \mathcal{J}_{4,0} - 2d_2^{(f)} \mathcal{J}_{2,2} \right] \right\},$$

with

$$\mathcal{J}_{i,j} = t^{(i+j)/2} \frac{\int_p e^{-2tp^2} (p^i)(p^j)}{\int_p e^{-2tp^2}},$$

Some lessons

- The observable competes in cutoff effects with the action and the flow.
- Numerically, the flow produces $3 \times$ more cutoff effects than the action or the observable (i.e. $\mathcal{J}_{2,2} = 3\mathcal{J}_{2,0}$, $\mathcal{J}_{4,0} = 3\mathcal{J}_{4,-2}$)
**Anatomy of $O(g^2a^2)$ cutoff effects** [Z. Fodor ’14; A.R., S. Sint ’14]

An urban legend: The clover observable has smaller cutoff effects

- In [M. Lüscher ’10] never stated that “clover is better”.
- This plot only shows that the Wilson action (pure gauge), with Wilson flow and clover observable produce smaller cutoff effects in $\sqrt{8t_0/r_0}$
- But different sources of cutoff effects can be responsible of this behavior.
- This is an accidental cancellation.

<table>
<thead>
<tr>
<th>Observable</th>
<th>Action</th>
<th>Flow</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clover</td>
<td>Wilson</td>
<td>Wilson</td>
<td>3</td>
</tr>
<tr>
<td>15</td>
<td>-3</td>
<td>-9</td>
<td>3</td>
</tr>
<tr>
<td>Clover</td>
<td>Lüscher-Weisz</td>
<td>Symanzik</td>
<td>19</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>
**Anatomy of $O(g_0^2a^2)$ cutoff effects** [Z. Fodor ’14; A.R., S. Sint ’14]

Is the Zeuthen flow improved? YES!

<table>
<thead>
<tr>
<th>Discretization</th>
<th>$d_1$</th>
<th>$d_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plaquette</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>Lüscher-Weisz</td>
<td>1/6</td>
<td>-1/6</td>
</tr>
<tr>
<td>Clover</td>
<td>3/2</td>
<td>-1/2</td>
</tr>
<tr>
<td>Zeuthen</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In infinite volume there are many ways to cancel a single $O(a^2)$ effect, but in a finite volume the numbers $\mathcal{J}_{i,j}$ become functions of the dimensionless ratio $c = \sqrt{8t/L}$

$$\mathcal{J}_{i,j}(c) = \left(\frac{\pi}{\sqrt{2}}\right)^{i+j} c^{i+j} \sum' \exp\{-c^2\pi^2(n_\mu + \tilde{n}_\mu)^2\} n^i n^j / \sum' \exp\{-c^2\pi^2(n_\mu + \tilde{n}_\mu)^2\} .$$

The Zeuthen flow produces improved observables for all values of the parameter $c$.

- Also checked the correlation function
  $$\langle E(t, x) E(s, y) \rangle$$

- Other boundary conditions: SF (as long as one remains far from the SF boundaries).
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**Conclusions**

- Essential to understand topology in finite volume
- Together with gradient flow nice FV renormalization scheme
  - Absence of zero-modes in PT
  - Large $c$ to reduce cutoff effects in $g^2_{GF}$
  - Shown some results for $SU(2)$ pure gauge and $SU(3)$ with $N_f = 12$.
- Nice analytic properties for perturbative studies: invariance under translations and absence of zero modes.
- Linked to many more interesting phenomena: Large $N$ volume reduction, Fractional semi-classical solutions, etc…