The spectrum of SU($N$) gauge theories at large $N$

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DESY Zeuthen, 11th January 2016
1. The large $N$ limit
2. The spectrum at fixed UV cutoff
3. Towards the continuum limit
4. Comparison with analytic approaches
5. Conclusions
In QCD some phenomena (confinement, chiral symmetry breaking) are non-perturbative.

Lattice calculations are successful in this regime, but more effective if some analytic guidance is available.

If we embed QCD in a larger context (SU($N$) gauge theory with $N_f$ quark flavours), the theory simplify in the limit $N \to \infty$, $N_f$ and $g^2N = \lambda$ fixed. Yet, it is non-trivial. Perhaps QCD is physically close to this limit?

However large $N$ QCD is still complicated enough that an analytic solution has not been found.

Lattice calculations can shed light on the existence of the limiting theory and on the proximity of QCD to it.
SU(N) gauge theory (possibly enlarged with $N_f$ fermions in the fundamental representation)

A sensible, non-trivial large $N$ limit can be defined by keeping fixed $\lambda = g^2 N$, with $N_f$ fixed ('t Hooft)
Consider connected vacuum diagrams, e.g.

In the large $N$ limit only those diagrams survive that can be drawn in a plane without crossing of the lines (planar graphs).

planar graph

non-planar graph
Double line representation

Regarding the flow of the colour indices a gluon propagator is equivalent to a quark and an antiquark propagators
Diagrammatic at large $N$

For each vertex: N
For each propagator: $1/N$
For each loop: N

$\Rightarrow \langle \rangle \propto N^{F-E+V}$

Euler characteristic $\chi = F - E + V = 2 - 2H - B$

1. The leading connected vacuum-to-vacuum graphs are of order $N^2$ (planar graphs made of gluons only)
2. The leading connected vacuum-to-vacuum graphs with quark lines are of order $N$ (planar graphs with just one quark loop at the boundary)
3. Corrections down by factors of $1/N^2$ in the gauge theory and by factors of $1/N$ in the theory with fermions
Phenomenology and large $N$

- Quark loop effects $\propto 1/N \Rightarrow$ The $N = \infty$ limit is quenched
- Mixing glueballs-mesons $\propto 1/\sqrt{N} \Rightarrow$ No mixing between glueballs and mesons at $N = \infty$
- Meson decay widths $\propto 1/N \Rightarrow$ mesons do not decay at $N = \infty$
- OZI rule $\propto 1/N \Rightarrow$ OZI rule exact at $N = \infty$

$\hookrightarrow$ The simpler large $N$ phenomenology can explain features of SU(3) phenomenology
Large $N$ limit on the lattice

The lattice approach allows us to go beyond perturbative and diagrammatic arguments. For a given observable:

1. **Continuum extrapolation**
   - Determine its value at fixed $a$ and $N$
   - Extrapolate to the continuum limit
   - Extrapolate to $N \to \infty$ using a power series in $1/N^2$

2. **Fixed lattice spacing**
   - Choose $a$ in such a way that its value in physical units is common to the various $N$
   - Determine the value of the observable for that $a$ at any $N$
   - Extrapolate to $N \to \infty$ using a power series in $1/N^2$

Study performed for various observables both at zero and finite temperature for $2 \leq N \leq 8$ (and $N = 17$!)
$m_\rho$ vs. $m_\pi^2$ at $N = \infty$

The large $N$ limit

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SU($N$) Gauge Theories on the lattice

$m_\pi$ vs. $m_\rho$ - fixing $\sigma$

Figure 5: Fits of $m_\rho/\sqrt{\sigma}$ as functions of $m_\pi^2/\sigma$, eq. (3.6), after different $N$.

Table 7: The fit parameters of eq. (3.6) for each $N$, with reduced $\chi^2$-values.

In chiral perturbation theory, as well as in the heavy quark limit, $m_\rho$ depends linearly on the quark mass $m_q$. Within our range of pion masses $m_\pi/\sqrt{\sigma} \approx 1.3...2.6$ we find $m_\pi^2$ to linearly depend on $\kappa^{-1}$ and therefore to be proportional to the quark mass. Hence, we can fit our $\rho$ masses to the parametrization,

$$m_\rho/\sqrt{\sigma} = m_\rho(0)/\sqrt{\sigma} + B m_\pi^2/\sigma,$$

(3.6)

Computing the lowest-lying isotriplet spectrum

1. Aims: determining the mesonic spectrum (including some excitations) and decay constants

2. Calculations performed for $2 \leq N \leq 7$ and $N = 17$

3. $\beta$ fixed across the various $n$ by imposing $a\sqrt{\sigma} = 0.2093$, implying $a \approx 0.093 \text{fm}$ (or $a^{-1} \approx 2.1 \text{ GeV}$)

4. Range of $\kappa$ down to $m_\pi \approx 0.5\sqrt{\sigma}$ for $N \geq 5$ and $m_\pi \approx 0.75\sqrt{\sigma}$ for $N \leq 4$

5. Size $24^3 \times 48$ for $N \neq 17$, $12^3 \times 24$ for $N = 17$ (finite size effects negligible at large $N$)

6. 200 configurations (80 configurations for $N = 17$)

Fermionic operators

For isotriplet states (flavour index $\alpha \neq \beta$):

<table>
<thead>
<tr>
<th>Particle</th>
<th>Bilinear</th>
<th>$J^{PC}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>$\bar{\psi}<em>\alpha \psi</em>\beta$</td>
<td>0++</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$\bar{\psi}<em>\alpha \gamma_5 \psi</em>\beta$, $\bar{\psi}<em>\alpha \gamma_0 \gamma_5 \psi</em>\beta$</td>
<td>0--</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$\bar{\psi}<em>\alpha \gamma_i \psi</em>\beta$, $\bar{\psi}<em>\alpha \gamma_0 \gamma_i \psi</em>\beta$</td>
<td>1--</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$\bar{\psi}<em>\alpha \gamma_5 \gamma_i \psi</em>\beta$</td>
<td>1++</td>
</tr>
<tr>
<td>$b_1$</td>
<td>$\bar{\psi}<em>\alpha \gamma_i \gamma_j \psi</em>\beta$</td>
<td>1+-</td>
</tr>
</tbody>
</table>

Flavour singlet states more difficult to study
The PCAC mass
Putting together large-$N$ and $\chi$PT predictions

\[ a m_{\text{PCAC}} = \frac{Z_P}{Z_A Z_S} \left( 1 + A a m_{\text{PCAC}} \right) \frac{1}{2} \left( \frac{1}{\kappa} - \frac{1}{\kappa_c} \right), \]

with $A$, $K$ and $\frac{Z_P}{Z_A Z_S}$ expected to have $1/N^2$ corrections in the large-$N$ limit:

\[ \frac{Z_P}{Z_A Z_S} = 0.8291(20) - \frac{0.699(45)}{N^2}, \]

\[ A = 0.390(13) + \frac{2.73(26)}{N^2}, \]

\[ \kappa_c = 0.1598555(33)(447) - \frac{0.028242(68)(394)}{N^2} \]
The pseudoscalar
Fit results for the pseudoscalar

Ansatz:

\[(am_\pi)^2 = A (am_q)^{\frac{1}{1+\delta}} + B (am_q)^2\]

Results:

\[
A = 11.99(0.10) - \frac{8.7(1.6)}{N^2}
\]

\[
B = 2.05(0.13) + \frac{5.0(2.2)}{N^2}
\]

\[
\delta = \frac{0.056(19)}{N} + \frac{0.94(21)}{N^3}
\]
The vector

The large $N$ limit

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SU($N$) Gauge Theories on the lattice
Fit results for the vector

Ansatz:

\[ m_\rho = A + B m_1^{1/2} + C m_q + D m_3^{3/2} \]

Results:

\[
\begin{align*}
A &= 1.504(51) + \frac{2.19(75)}{N^2} \\
B &= -\frac{2.47(94)}{N} \\
C &= 3.08(53) + \frac{16.8(8.2)}{N^2} \\
D &= -0.84(31) - \frac{9.4(4.8)}{N^2}
\end{align*}
\]
Other states

\[
\frac{m_{a_1}}{\sqrt{\sigma}} = \left( 2.860(21) + \frac{0.84(36)}{N^2} \right) + \left( 2.289(35) - \frac{2.02(61)}{N^2} \right) \frac{m_{\text{PCAC}}}{\sqrt{\sigma}}
\]

\[
\frac{m_{b_1}}{\sqrt{\sigma}} = \left( 2.901(23) + \frac{1.07(40)}{N^2} \right) + \left( 2.273(38) - \frac{2.83(72)}{N^2} \right) \frac{m_{\text{PCAC}}}{\sqrt{\sigma}}
\]

\[
\frac{m_{a_0}}{\sqrt{\sigma}} = \left( 2.402(34) + \frac{4.25(62)}{N^2} \right) + \left( 2.721(53) - \frac{6.84(96)}{N^2} \right) \frac{m_{\text{PCAC}}}{\sqrt{\sigma}}
\]
Excited states

\[
\frac{m_\pi^*}{\sqrt{\sigma}} = \left( 3.392(57) + \frac{1.0(1.1)}{N^2} \right) + \left( 2.044(80) - \frac{1.2(1.6)}{N^2} \right) \frac{m_{\text{PCAC}}}{\sqrt{\sigma}}
\]

\[
\frac{m_\rho^*}{\sqrt{\sigma}} = \left( 3.696(54) + \frac{0.23(55)}{N^2} \right) + \left( 1.782(67) - \frac{1.30(54)}{N^2} \right) \frac{m_{\text{PCAC}}}{\sqrt{\sigma}}
\]

\[
\frac{m_\alpha^*_0}{\sqrt{\sigma}} = \left( 4.356(65) + \frac{1.8(1.4)}{N^2} \right) + \left( 1.902(98) - \frac{2.9(2.1)}{N^2} \right) \frac{m_{\text{PCAC}}}{\sqrt{\sigma}}
\]

\[
\frac{m_\alpha^*_1}{\sqrt{\sigma}} = \left( 4.587(75) + \frac{1.2(1.2)}{N^2} \right) + \left( 1.76(12) - \frac{2.1(19)}{N^2} \right) \frac{m_{\text{PCAC}}}{\sqrt{\sigma}}
\]

\[
\frac{m_\beta^*_1}{\sqrt{\sigma}} = \left( 4.609(99) + \frac{1.7(1.5)}{N^2} \right) + \left( 1.87(15) - \frac{2.5(2.2)}{N^2} \right) \frac{m_{\text{PCAC}}}{\sqrt{\sigma}}
\]
Pion decay constant

Fit \( f_{\pi}^{\text{lat}} / \sqrt{N\sigma} = A + B \cdot m_{\text{PCAC}} / \sqrt{\sigma} + C \cdot m_{\text{PCAC}}^2 / \sigma \)
$\rho$ decay constant

\[ \text{Fit } f_{\rho}^{\text{lat}} / \sqrt{N\sigma} = A + B \cdot m_{\text{PCAC}} / \sqrt{\sigma} + C \cdot m_{\text{PCAC}}^2 / \sigma \]
Approaching $N = \infty$
The large $N$ limit
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SU($N$) Gauge Theories on the lattice

Chiral condensate

\[ \langle \bar{\psi} \psi \rangle_{\text{lat}} / (N \sigma^{3/2}) = A + B \cdot m_{\text{PCAC}} / \sqrt{\sigma} \]
The $A^{++}$ glueball channel

Lattice spacing fixed by requiring $aT_c = 1/6$
The glueball spectrum at $aT_c = 1/6$

![Graph showing the spectrum at $N = \infty$. The yellow boxes represent the large $N$ extrapolation of masses obtained in ref. [38].](image)


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**Figure 21**: Chew-Frautschi plot of the glueball spectrum.
Continuum meson spectrum – SU(7)
Comparison with QCD

\[ \sqrt{\sigma} \text{ fixed from the condition } \hat{F}_\infty = 85.9 \text{ MeV, } m_{ud} \text{ from } m_\pi = 138 \text{ MeV} \]
The strange meson spectrum

\[ m_q = m_s \]

\[ m_s \text{ fixed from } m_\pi(m_s) = \left( m_{K^\pm}^2 + m_{K^0}^2 - m_\pi^2 \right)^{1/2} = 686.9 \text{ MeV} \]
What do we learn?

- Lowest-lying mesons broadly compatible with QCD, excitations off by 20% (however, excitations less controlled in our calculation)

- The calculated large-$N$ masses $m_\rho = 753(14)$ MeV and of the $m_\phi = 981(44)$ MeV are remarkably close to their experimental values $m_\rho = 775$ and $m_\phi = 1019$ MeV

- Observed degeneracy of $(\rho, a_0)$ and $(a_1, \pi^*)$ (predicted by $\chi$PT)
Glueballs in the continuum limit

Example

Glueball masses in SU(4)

Large-$N$ extrapolation of glueball masses

Glueball masses at $N = \infty$

\[ 0^{++} \quad \frac{m}{\sqrt{\sigma}} = 3.28(8) + \frac{2.1(1.1)}{N^2} \]

\[ 0^{++*} \quad \frac{m}{\sqrt{\sigma}} = 5.93(17) - \frac{2.7(2.0)}{N^2} \]

\[ 2^{++} \quad \frac{m}{\sqrt{\sigma}} = 4.78(14) + \frac{0.3(1.7)}{N^2} \]

Accurate $N = \infty$ value, small $O(1/N^2)$ correction
Topological observables

Topological charge

\[ Q = \frac{1}{32\pi^2} \int \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \, d^4 x = \int q(x) \, d^4 x \]

Topological susceptibility

\[ \chi_t = \frac{\langle Q^2 \rangle - \langle Q \rangle^2}{V} \]

Witten-Veneziano formula (large-\( N \) result)

\[ \chi_t = \frac{m^2_{\eta'} f_{\eta'}^2}{2N_f} \]

\( \chi_t \) finite at large \( N \)

From lattice calculations

\[ \frac{\chi_t}{\sigma^{1/4}} = 0.397(7) + \frac{0.35(13)}{N^2} - \frac{1.32(41)}{N^4} \]

Large-\(N\) limit of \(\chi_t\) – An example

[BL and M. Teper, JHEP 0106 (2001) 050]
Monte Carlo history of $Q - SU(3)$

$V=16^4$, $\beta = 6.0$
Monte Carlo history of $Q - SU(5)$

$V=16^4, \beta = 17.45$
Decorrelating the topological charge

- At large $N$ small size instantons are suppressed and this explains why the topology does not change.
- Open Boundary Conditions (OBC) in time have been proposed as a way to push instantons into the lattice.
- OBC have shown to be better than Periodic Boundary Conditions (PBC) in time at decorrelating topology in QCD.

Does this mechanism also work at large $N$?
Periodic vs Open BC: a case study

- Gauge group: SU(7)
- Lattice spacing: $a \sqrt{\sigma} \sim 0.21$
- Lattice sizes: $16^3 \times N_t$, $N_t = 32, 48, 64$
- Statistics: $\sim 500$ configurations, separated by 200 composite sweeps (1 composite sweep is 1 hb + 4 overrelax)
- Purpose: comparing results with PBC and OBC at fixed lattice parameters in a case where there is a severe ergodicity problem
- Observables: $Q$, gluonic correlators in the $0^{++}$ and $0^{--}$ channels and instantons
- Both cooling and Wilson Flow used to filter UV modes
Monte Carlo history of $Q$

[A. Amato, G. Bali and B. Lucini, arXiv:1512.00806 and in progress]

Significantly better decorrelation for OBC
Distribution of $Q$

OBC seem to cure the problem of ergodicity
Meson masses from gauge-string duality

The spectrum from the topological string

Figure 2. The glueball and meson spectrum of large-$N$ massless $QCD$: The points in black, and the straight Regge trajectories labelled by the internal quantum numbers, $k$ for glueballs and $n$ for mesons, represent the spectrum implied by the laws Eqs. (1.1).

Good progress has been made in computing the spectrum of SU($N$) gauge theories in the large-$N$ limit, with precise results available at fixed lattice spacing.

Extrapolation to the continuum limit still in progress.

First results show little (and controlled) lattice spacing dependency on most groundstate masses (with noticeable exceptions, e.g. $a_0$).

Slow topological modes produce a systematic error that is hard to quantify.

OBC are being explored for a more controlled continuum extrapolation.
Lattice action

Path integral

\[ Z = \int (\mathcal{D}U_\mu(i)) (\det M(U_\mu))^N e^{-S_g(U_{\mu\nu}(i))} \]

with

\[ U_\mu(i) = P\exp \left( ig \int_i^{i+a_\mu} A_\mu(x) dx \right) \]

\[ U_{\mu\nu}(i) = U_\mu(i) U_\nu(i + \hat{\mu}) U_\mu^\dagger(i + \hat{\nu}) U_\nu^\dagger(i) \]

Gauge part

\[ S_g = \beta \sum_{i,\mu} \left( 1 - \frac{1}{N} \text{Re Tr}(U_{\mu\nu}(i)) \right) \]

with \( \beta = 2N/g_0^2 \)

Invariance under SU(N) gauge transformations

\[ \tilde{U}_\mu(i) = G^\dagger(i) U_\mu(i) G(i + \hat{\mu}) \]
Wilson fermions

Take the naive Dirac fermions and add an irrelevant term that goes like the Laplacian

\[ M_{\alpha\beta}(ij) = (m + 4r)\delta_{ij}\delta_{\alpha\beta} \]

\[ - \frac{1}{2} \left[ (r - \gamma_\mu)_{\alpha\beta} U_\mu(i)\delta_{i,j+\mu} + (r + \gamma_\mu)_{\alpha\beta} U^\dagger_\mu(j)\delta_{i,i-\mu} \right] \]

This formulation breaks explicitly chiral symmetry

Define the hopping parameter

\[ \kappa = \frac{1}{2(m + 4r)} \]

Chiral symmetry recovered in the limit \( \kappa \to \kappa_C \) (\( \kappa_C \) to be determined numerically)
Quenched approximation

For an observable $\mathcal{O}$

$$
\langle \mathcal{O} \rangle = \frac{\int (\mathcal{D}U_\mu(i)) (\det M(U_\mu))^{N_f} f(M) e^{-S_g(U_{\mu\nu}(i))}}{\int (\mathcal{D}U_\mu(i)) (\det M(U_\mu))^{N_f} e^{-S_g(U_{\mu\nu}(i))}}
$$

Assume $\det M(U_\mu) \simeq 1$ i.e. fermions loops are removed from the action

The approximation is exact in the $m \rightarrow \infty$ and $N \rightarrow \infty$ limit ($g^2 N$ is fixed) $\leftrightarrow$ the large $N$ spectrum is quenched for $m \neq 0$

As $N$ increases, unquenching effects are expected for smaller quark masses
Extracting the spectrum

Trial operators $\Phi_1(t), \ldots, \Phi_n(t)$ with the quantum numbers of the state of interest

\[
C_{ij}(t) = \langle 0 | (\Phi_i(0))^\dagger \Phi_j(t) | 0 \rangle \\
= \langle 0 | (\Phi_i(0))^\dagger e^{-Ht} \Phi_j(0) e^{Ht} | 0 \rangle \\
= \sum_n \langle 0 | (\Phi_i(0))^\dagger | n \rangle \langle n | e^{-Ht} \Phi_j(0) e^{Ht} | 0 \rangle \\
= \sum_n e^{-\Delta E_n t} \langle 0 | (\Phi_i(0))^\dagger | n \rangle \langle n | \Phi_j(0) | 0 \rangle = \sum_n c_{in}^* c_{jn} e^{-\Delta E_n t}
\]

Masses can be extracted from diagonal correlators

\[
C_{ii} = \sum_n |c_{in}|^2 e^{-a m_n t} \quad \rightarrow \quad |c_{i1}|^2 e^{-a m_1 t}
\]
Variational calculations

Using the information of the whole correlation matrix allows us to better control the groundstate mass and to determine masses of excitations with a variational procedure

1. Find the eigenvector $v$ that minimises

$$am_1(t_d) = -\frac{1}{t_d} \log \frac{v_i^* C_{ij}(t_d)v_j}{v_i^* C_{ij}(0)v_j}$$

for some $t_d$

2. Fit $v(t)$ with the law $Ae^{-m_1 t}$ to extract $m_1$

3. Find the complement to the space generated by $v(t)$

4. Repeat 1-3 to extract $m_2, \ldots, m_n$

Need a good overlap with the state of interest
Glueball operators

- Using traced Wilson loops of various shapes, build operators $O_i(t)$ that are eigenstates of $P$ and $C$ and under the (discrete) group of rotations transform according to irreducible representations.

- Examples of basic contours used

- Continuous spin reconstructed by looking the decomposition of the irreducible representations of the dihedric group in irreps of SO(3)
String tension

Confining potential: \( V = \sigma R \)

Polyakov loop

\[
P_k(i) = \frac{1}{N} \text{Tr} \prod_{j=0}^{L} U_k(i + j\hat{k})
\]

\[
P(t) = \sum_{\vec{n}, k} P_k(\vec{n}, t)
\]

\[
C(t) = \langle (P(0))^\dagger P(t) \rangle = \sum_j |c_j|^2 e^{-a m_j t} \quad t \to \infty \quad |c_l|^2 e^{-a m_l t}
\]

\[
am_l \simeq a^2 \sigma L - \frac{\pi (D-2)}{6L}
\]
Building extended operators

Starting from links, we can build extended loop operators via

- Smearing

\[ \text{Smearing} \quad \Rightarrow \quad \text{Smearing} \]

- Blocking

\[ \text{Blocking} \quad \Rightarrow \quad \text{Blocking} \]

For mesons, extended operators can be built by smearing source and sink operators over a few lattice spacings with a weight function (e.g. Gaussian smearing, wall smearing etc.)