Volume (in-)dependence in $SU(N)$ gauge theories with twisted boundary conditions

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1 Introduction
   - Large-$N$ volume independence
   - Finite-$N$ volume (in-)dependence
   - $x$-scaling conjecture

2 Lattice calculation

3 Results
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3 Results
History of volume reduction (Eguchi, Kawai, 1982)

Consider two pure gauge lattice theories:

- \( S_{\text{gauge}} = \beta \sum_{x, \mu < \nu} (1 - \frac{1}{N} \text{ReTr} U_{x, \mu \nu}^\square) \)
- \( S_{\text{EK}} = \beta \sum_{\mu < \nu} (1 - \frac{1}{N} \text{ReTr} U_{\mu \nu}^\square) \)
Large-\(N\) volume independence

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Large-$N$ volume independence

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  - $S_{\text{EK}} = \beta \sum_{\mu<\nu} (1 - \frac{1}{N} \text{Re} \text{Tr} U_{\mu} U_{\nu} U_{\mu}^\dagger U_{\nu}^\dagger)$
- They satisfy the same loop equations in $N \to \infty \iff$ no spontaneous breaking of the center symmetry
- Center symmetry of EK model: $(\mathbb{Z}_N)^4 \xrightarrow[N\to\infty]{\sim} U(1)^4$
- Polyakov loop $P_\mu \equiv \text{Tr} \prod_i U_{x,x+i\mu}$ – order parameter for the center symmetry ($P_\mu \equiv \text{Tr} U_\mu$ in the single-site model).
Large-$N$ volume independence: Failure of EK reduction

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- Perturbative explanation (large $\beta$):
  - $U_\mu = V_\mu^\dagger \Lambda_\mu V_\mu$, where $\Lambda_\mu = \text{diag}[e^{i\theta_1^\mu}, \ldots, e^{i\theta_N^\mu}]$
  - Assume $V_\mu = \exp(i A_\mu)$ with small $A_\mu$
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  - Assume \(V_\mu = \exp(iA_\mu)\) with small \(A_\mu\)
  - \(Z_{EK} = \int \prod_{\mu,i} \frac{d\vartheta^i_\mu}{2\pi} \exp\{-F_{EK}(\vartheta)\}\), where
    \[
    F_{EK} \xrightarrow{\beta \to \infty} (d - 2) \sum_{i<j} \log \left( \sum_\mu \sin^2 \frac{\vartheta^i_\mu - \vartheta^j_\mu}{2} \right)
    \]
  - For \(d > 2\) eigenvalues attract \(\Rightarrow P_\mu \neq 0\)
  - Note: eigenvalues \(\vartheta\) play similar role as momenta in ordinary lattice calculation.
Large-$N$ volume independence: Possible fixes

- Bhanot, Heller, Neuberger, 1982: Quenched EK (fails)
- Gonzalez-Arroyo, Okawa, 1983: Twisted EK “1.0” (fails)
- Narayanan, Neuberger, 2003: Partial reduction (in $L^4$ box with big enough $L$; works, but $L \to \infty$ in the continuum limit)
- Kovtun, "Unsal, Yaffe, 2003-07: Trace-deformed EK (e.g. with adjoint fermions) (in principle works, but . . . )

Add twisted boundary conditions to the model:

$$S_{TEK} = \beta \sum_{\mu < \nu} (1 - \frac{1}{N} z_{\mu\nu} \text{Re} \text{Tr} U_{\mu} U_{\nu} U_{\mu}^\dagger U_{\nu}^\dagger),$$

where $z_{\mu\nu} = \exp(i \epsilon_{\mu\nu} 2 \pi k N)$, $k$ – magnetic flux.

Can be used to calculate Wilson loop expectation values, as well as meson correlators in momentum space.
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Gonzalez-Arroyo, Okawa, 2010: Twisted EK “2.0” (works)
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\[
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\]

where \(z_{\mu\nu} = \exp\left(i \epsilon_{\mu\nu} \frac{2\pi k}{N}\right)\), \(k\) – magnetic flux

- Can be used to calculate Wilson loop expectation values, as well as meson correlators in momentum space.
Large-$N$ volume independence: Plaquette

Comparison of TEK and p.b.c. lattice calculations (Gonzalez-Arroyo, Okawa, 2014)
Large-$N$ volume independence: Mesons

Comparison of TEK and p.b.c. lattice calculations (Gonzalez-Arroyo, Okawa, 2015)
Finite-\(N\) volume (in-)dependence

- Large-\(N\) volume independence strictly true only in the limit \(N \to \infty\).
- What is the situation when working with \(N < \infty\)? Can we quantitatively define some effective system size \(L_{\text{eff}}(N)\)?
Finite-$N$ volume (in-)dependence

Adjoint Eguchi-Kawai model (Bringoltz, MK, Sharpe, 2011):
Log of $1 \times T$ Wilson loops vs $T$ for various values of $N$
Finite-$N$ volume (in-)dependence

- Large-$N$ volume independence strictly true only in the limit $N \rightarrow \infty$.
- What is the situation when working with $N < \infty$? Can we quantitatively define some effective system size $L_{\text{eff}}(N)$?
- Yes, when using twisted boundary conditions!
- Twisted PT allows interchanging of $L$ and $N$, physics depends on the product $N^{2/d}L$, $d$ being the number of compact twisted dimensions.
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Twisted PT allows interchanging of $L$ and $N$, physics depends on the product $N^{2/d}L$, $d$ being the number of compact twisted dimensions.

We have three interesting possibilities:

- 2+1 dim, spatial dimensions compact, $\propto NL$
- 3+1 dim, all dimensions compact, $\propto \sqrt{NL}$
- 3+1 dim, two spatial dimensions compact, $\propto NL$
Twisted boundary conditions ('t Hooft, 1980):

- Theory: pure $SU(N)$ gauge theory on a spatial two-torus of size $L$; one can choose constant twist matrices:

$$A_i(x + Lj) = \Gamma_j A_i(x) \Gamma_j^\dagger,$$

where $\Gamma_1 \Gamma_2 = e^{i\pi k/N} \Gamma_2 \Gamma_1$.
Finite-$N$ volume (in-)dependence: 2+1 dimensions

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  \[ A_i(x + Lj) = \Gamma_j A_i(x) \Gamma_j^\dagger, \text{ where } \Gamma_1 \Gamma_2 = e^{i\pi k/N} \Gamma_2 \Gamma_1 \]

- If $k$ and $N$ co-prime $\Rightarrow$ unique solution “irreducible twist”: construct $N \times N$ matrices: $\hat{\Gamma}(p)$, with $p_i = \frac{2\pi n_i}{NL}$ such that:

  \[ T^a A^a_i(x) = \sum'_p e^{ip \cdot x} \tilde{A}(p) \hat{\Gamma}(p) \]

- Garcia, Gonzalez-Arroyo, Okawa 2013, 14: Now do PT in these variables, need also structure constants:

  \[ [\hat{\Gamma}(p), \hat{\Gamma}(q)] = iF(p, q, -p - q)\hat{\Gamma}(p + q), \]

  \[ F(p, q, -p - q) \propto \sin \left( \epsilon_{\mu\nu} p_\mu q_\nu \tilde{\theta}(NL)^2 / 8\pi^2 \right) \]
x-scaling conjecture

- Two dimensionless parameters:
  1. Scaling variable \( x = \frac{NL}{4\pi b} \), where \( b = \frac{1}{g^2 N} \) is the inverse 't Hooft coupling (dimensionful in 2+1!)
  2. Angle \( \tilde{\theta} = \frac{2\pi \bar{k}}{N} \), with integer \( \bar{k} \) defined as: \( k\bar{k} = 1 \pmod{N} \).
The $x$-scaling conjecture

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- Can be thought of as a strong form of TEK-like volume independence also valid at finite $N$.
- Caveat: $2\pi\tilde{\theta}$ – unique rational number for every $k$ and $N$ $\Rightarrow$ equivalence up to unknown (outside PT) function of $\tilde{\theta}$. 

Physical quantities in the theory depend only on $x$ and the angle $\tilde{\theta}$ given by the parameters of the twist.
Garcia, Gonzalez-Arroyo, Okawa 2013, 14: conjecture satisfied in PT for the non-zero electric flux sector ($\propto k$-string tensions), also non-perturbative lattice confirmation.

Can avoid tachyonic instabilities by suitably scaling $k, \bar{k} \propto N$, analogous to the Twisted Eguchi-Kawai model “1.0” → “2.0”.

What about the zero electric flux sector ($\propto$ glueballs/torelons)?

Here $1/N^2$ corrections can arise in higher orders of PT.

Also known to be only approximate at large $x$ (large volume), but $1/N^2$ coefficients “remarkably small” (Teper et al. 2015)
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2 Lattice calculation

3 Results
Numerical calculation

- Goal: numerically verify the conjecture, particularly in the glueball sector.
- For numerical investigation: lattice model with Wilson action:

\[ S = Nb \sum_{n \in \mathbb{Z}^3_{(L,L,T)}} \sum_{\mu \neq \nu} \left( N - z_{\mu \nu}^*(n) U_{\mu \nu}^\Box (n) \right), \]

where \( z_{\mu \nu}(n) = \exp \left( i \epsilon_{\mu \nu} \frac{2 \pi k}{N} \right) \) at corner plaquettes in each (1,2)-plane, and 1 everywhere else.
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- **Numerical agenda:**
  1. Take theories with \( N = 5, 7, 11, 17 \) and approx. matching \( NL \).
  2. Take all values of \( k \) and wide range of couplings, ranging from small-volume perturbative regime (small \( x \)), to large-volume non-perturbative one (large \( x \)).
  3. Calculate lightest scalar [and tensor] glueball masses, as well as electric flux energies.
Numerical calculation: details

- Electric flux energies: find energies from plateaux of the Polyakov loop correlators with winding number $\bar{k}$.
- Glueballs: variational analysis, use basis of rectangular Wilson loops and moduli of multi-winding Polyakov loops $|\text{Tr } P^n|^2$, with 3 different levels of APE smearing.
- Instead of blocking, use Wilson of large extent trying to follow the physical size of the glueball (including loops larger than $L$ for small and moderate $x$)
\[ C_{RR}(4) \] is the (normalized) correlator of \( W(R, R) \) at time distance 4a, \( N = 5, L = 14, \bar{k} = 2 \).
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- Instead of blocking, use Wilson of large extent trying to follow the physical size of the glueball (including loops larger than $L$ for small and moderate $x$)
- Construct $C_{ij}(t) = \sum_{t'} \langle O_i(t'+t)O_j(t') \rangle - \langle O_i(t'+t) \rangle \langle O_j(t') \rangle$ and solve GEVP to find improved plateaux.
Numerical calculation: details contd.

GEVP:

1. “Old-school” way:

\[ C(t_1)v = C(t_0)\lambda v \]

to find \( v \), use them to change the basis \( C(t) \rightarrow \tilde{C}(t) \ \forall t \) and fit to diagonal elements of \( \tilde{C}(t) \) (after finding the plateau).

2. ALPHA-way: Narrow down the basis to 3-4 operators (thinning/pruning), solve:

\[ C_{thin}(t)v = C_{thin}(\lceil t/2 \rceil)\lambda(t)v, \]

use \( \lambda(t) \) directly to produce effective mass plots:

\[ E_{n}^{\text{eff}}(t) = -\partial_t \log(\lambda_n(t)). \]

Better theoretical behaviour but more subject to noise. In practice, use only as cross-check of method 1.

Volume dependence in \( SU(N) \) gauge theories with twisted b.c.
Numerical calculation: details cntd.

- Technicalities: use approx. 12 – 25 operators for $C_{ij}(t)$, estimate if basis allows reliable GEVP by first solving it on non-symmetrized $C(t)$, use quad precision for analysis.
- Plateau extraction: following Ryan, 14: “fit histogram”: correlated fits to all possible $t$ ranges, select one with best quality (function of $\chi^2$, interval length, uncertainty size), look at stability.
- Small $x \Rightarrow$ lattice spacings very small $\Rightarrow$ look out for finite-temperature effects from wrap-around gluons!
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Volume dependence in $SU(N)$ gauge theories with twisted b.c.
Glueball masses, $x = 0.199 \ (b \approx 28 @ NL = 70)$

Lightest scalar glueball mass as function of $\tilde{\theta}$ for $N = 5, 7, 11, 17$, all values of $k$. The values of $L$ are 14, 10, 6, 4 respectively.
Overlap of operators, $x = 0.199$

State admixture (= absolute value of the eigenvector) in the GEVP for $N = 17$ in the perturbative region. Black numbers on squares denote the PT expectation.
Comparison of lightest scalar glueball mass for $N = 5, \tilde{\theta} = 2.513$ and $N = 17, \tilde{\theta} = 2.587$, across a wide range of $x$. 
Glueball masses, $x = 2.785$ ($b \approx 2 \otimes NL = 70$)

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Glueball masses, $x = 2.785$ ($b \approx 2$ @ $NL = 70$)

Lightest scalar glueball mass as function of $\tilde{\theta}$ for $N = 5, 7, 11, 17$, all values of $k$. Also values for double $L$ (and double $b$) are shown.
Electric flux energy, $x = 2.785$

Lowest electric flux energy as function of $\tilde{\theta}$ for $N = 5, 7, 11, 17$, all values of $k$. 
Glueball masses, $x = 5.570$ ($b \approx 2 \circledast NL = 140$)

Lightest scalar glueball mass as function of $\tilde{\theta}$ for $N = 5, 7, 11, 17$, all values of $k$. 
Glueball masses, $x = 5.570 \ (b \approx 2 \ @ \ NL = 140)$

Lightest scalar glueball mass as function of $\tilde{\theta}$ for $N = 17$, all values of $k$. 
Why so small mass for $N = 17$, $\tilde{k} = 6$?

- One expects that:
  1. At small $x$ the lightest “glueball” corresponds to a weakly interacting two-gluon state.
  2. At larger $x$ the large-volume glueball becomes the lightest state.
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Electric flux energies as function of $\tilde{\theta}$ for $N = 5, 7, 11, 17$, all values of $k$. 

$N=5$, $kbar=2$, $e=2$  
$N=17$, $kbar=6$, $e=6$  
$1/2$ of large V glueball mass, Teper, 98
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- Can one work around it? Fine tune $\bar{k}$ to avoid unexpected light states or restrict to selected eigenstates...
Summary & outlook

- $x$-scaling conjecture: in a 2+1 dimensional Yang-Mills theory with twisted boundary conditions physics is governed by product $NL\lambda$.
- $x$-scaling with strong confirmation in the non-zero electric flux sector.
- In the glueball sector: conjecture satisfied for small and moderate $x$, approach to large volume shows significant deviations from a smooth $\tilde{\theta}$ dependence.
- More understanding required to see if one can “save” the conjecture in a restricted form.
- In principle, straightforward to generalize to 3+1 dimensions (cf. Keegan, Ramos, 15).
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Thank you for your attention!