Volume (in-)dependence in $SU(N)$ gauge theories with twisted boundary conditions

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Joint Lattice Seminar, Zeuthen, 16-11-2015

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- History of volume reduction (Eguchi, Kawai, 1982)
- Consider two pure gauge lattice theories:
  - $S_{gauge} = \beta \sum_{x,\mu<\nu}(1 - \frac{1}{N}\mathrm{Re}\mathrm{Tr}\, U^{\square}_{x,\mu\nu})$
  - $S_{EK} = \beta \sum_{\mu<\nu}(1 - \frac{1}{N}\mathrm{Re}\mathrm{Tr}\, U^{\square}_{\mu\nu})$

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- They satisfy the same loop equations in $N \to \infty \Leftrightarrow$ no spontaneous breaking of the center symmetry
- Center symmetry of EK model: $(\mathbb{Z}_N)^4 \xrightarrow[N\to\infty]{} U(1)^4$
- Polyakov loop $P_\mu \equiv \mathrm{Tr} \prod_i U_{x,x+i\mu}$ – order parameter for the center symmetry ($P_\mu \equiv \mathrm{Tr}\, U_\mu$ in the single-site model).

Large-N volume independence: Failure of EK reduction

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- Perturbative explanation (large β):
  - $U_\mu = V_\mu^\dagger \Lambda_\mu V_\mu$, where $\Lambda_\mu = \mathrm{diag}\left[e^{i\vartheta_\mu^1}, \dots, e^{i\vartheta_\mu^N}\right]$
  - Assume $V_\mu = \exp(iA_\mu)$ with small A_μ

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  - Assume $V_\mu = \exp(iA_\mu)$ with small A_μ
  - $Z_{EK} = \int \prod_{\mu,i} \frac{d\vartheta_\mu^i}{2\pi} exp\{-F_{EK}(\vartheta)\}$, where
    $$F_{EK} \xrightarrow{\beta\to\infty} (d-2)\sum_{i<j} \log\left(\sum_\mu \sin^2 \frac{\vartheta_\mu^i - \vartheta_\mu^j}{2}\right)$$
  - For $d > 2$ eigenvalues attract $\Rightarrow P_\mu \neq 0$
- Note: eigenvalues ϑ play similar role as momenta in ordinary lattice calculation.

Large-N volume independence: Possible fixes

- Bhanot, Heller, Neuberger, 1982: Quenched EK (fails)
- Gonzalez-Arroyo, Okawa, 1983: Twisted EK "1.0" (fails)
- Narayanan, Neuberger, 2003: Partial reduction (in L^4 box with big enough L; works, but $L \to \infty$ in the continuum limit)
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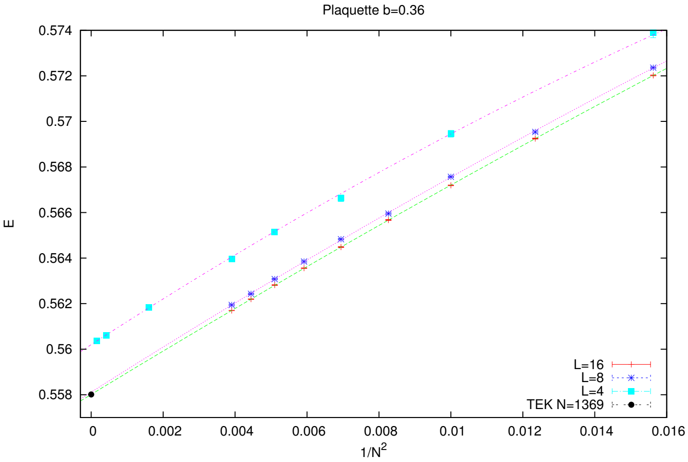
  - Add twisted boundary conditions to the model:

    $S_{TEK} = \beta \sum_{\mu<\nu}(1 - \frac{1}{N} z_{\mu\nu} \mathrm{ReTr}\, U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger),$

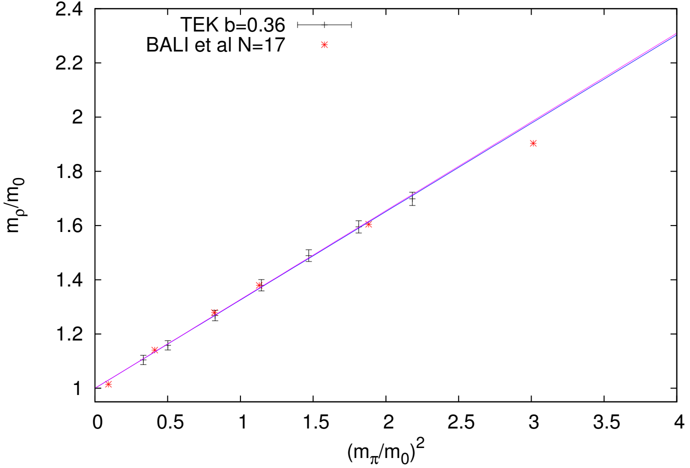
    where $z_{\mu\nu} = \exp\left(i\epsilon_{\mu\nu}\frac{2\pi k}{N}\right)$, k – magnetic flux
  - Can be used to calculate Wilson loop expectation values, as well as meson correlators in momentum space.

Large-N volume independence: Plaquette



Comparison of TEK and p.b.c. lattice calculations (Gonzalez-Arroyo, Okawa, 2014)

Large-N volume independence: Mesons



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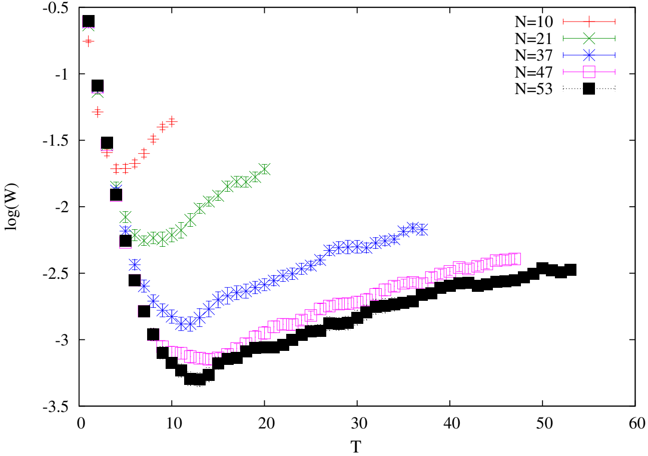
Finite-N volume (in-)dependence

- Large-N volume independence strictly true only in the limit $N \to \infty$.
- What is the situation when working with $N < \infty$? Can we quantitatively define some effective system size $L_{\mathrm{eff}}(N)$?

Finite-N volume (in-)dependence



Adjoint Eguchi-Kawai model (Bringoltz, MK, Sharpe, 2011):
Log of $1 \times T$ Wilson loops vs T for various values of N

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- What is the situation when working with $N < \infty$? Can we quantitatively define some effective system size $L_{\text{eff}}(N)$?
- Yes, when using twisted boundary conditions!
- Twisted PT allows interchanging of L and N, physics depends on the product $N^{2/d}L$, d being the number of compact twisted dimensions.

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- Yes, when using twisted boundary conditions!
- Twisted PT allows interchanging of L and N, physics depends on the product $N^{2/d}L$, d being the number of compact twisted dimensions.
- We have three interesting possibilities:
  - 2+1 dim, spatial dimensions compact, $\propto NL$
  - 3+1 dim, all dimensions compact, $\propto \sqrt{N}L$
  - 3+1 dim, two spatial dimensions compact, $\propto NL$

Finite-N volume (in-)dependence: 2+1 dimensions

Twisted boundary conditions ('t Hooft, 1980):

- Theory: pure $SU(N)$ gauge theory on a spatial two-torus of size L; one can choose constant twist matrices:

$$A_i(x + L\hat{j}) = \Gamma_j A_i(x) \Gamma_j^\dagger, \text{ where } \Gamma_1 \Gamma_2 = e^{i\pi k/N} \Gamma_2 \Gamma_1$$

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- If k and N co-prime $\Rightarrow$ unique solution "irreducible twist": construct $N \times N$ matrices: $\hat{\Gamma}(p)$, with $p_i = \frac{2\pi n_i}{NL}$ such that:

$$T^a A_i^a(x) = \sum_p' e^{ip\cdot x} \tilde{A}(p) \hat{\Gamma}(p)$$

- Garcia, Gonzalez-Arroyo, Okawa 2013, 14: Now do PT in these variables, need also structure constants:

$$[\hat{\Gamma}(p), \hat{\Gamma}(q)] = iF(p, q, -p-q) \hat{\Gamma}(p+q),$$
$$F(p, q, -p-q) \propto \sin\left(\epsilon_{\mu\nu} p_\mu q_\nu \tilde{\theta} (NL)^2 / 8\pi^2\right)$$

x-scaling conjecture

- Two dimensionless parameters:
  1. Scaling variable $x = \frac{NL}{4\pi b}$, where $b = \frac{1}{g^2 N}$ is the inverse 't Hooft coupling (dimensionful in 2+1!)
  2. Angle $\tilde{\theta} = \frac{2\pi \bar{k}}{N}$, with integer $\bar{k}$ defined as: $k\bar{k} = 1 \pmod{N}$.

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x-scaling conjecture

Physical quantities in the theory depend only on x and the angle $\tilde{\theta}$ given by the parameters of the twist.

- Can be thought of as a strong form of TEK-like volume independence also valid at finite N.
- Caveat: $2\pi\tilde{\theta}$ – unique rational number for every k and N $\Rightarrow$ equivalence up to unknown (outside PT) function of $\tilde{\theta}$.

x-scaling conjecture cntd.

- Garcia, Gonzalez-Arroyo, Okawa 2013, 14: conjecture satisfied in PT for the non-zero electric flux sector ($\propto$ k-string tensions), also non-perturbative lattice confirmation.
- Can avoid tachyonic instabilities by suitably scaling $k, \bar{k} \propto N$, analogous to the Twisted Eguchi-Kawai model "1.0" $\rightarrow$ "2.0".
- What about the zero electric flux sector ($\propto$ glueballs/torelons)?
- Here $1/N^2$ corrections can arise in higher orders of PT.
- Also known to be only approximate at large x (large volume), but $1/N^2$ coefficients "remarkably small" (Teper et al. 2015)

Contents

Numerical calculation

- Goal: numerically verify the conjecture, particularly in the glueball sector.
- For numerical investigation: lattice model with Wilson action:

$$S = Nb \sum_{n \in \mathbb{Z}^3_{(L,L,T)}} \sum_{\mu \neq \nu} \left(N - z^*_{\mu\nu}(n) U^{\square}_{\mu\nu}(n) \right),$$

where $z_{\mu\nu}(n) = \exp\left(i\epsilon_{\mu\nu}\frac{2\pi k}{N}\right)$ at corner plaquettes in each (1,2)-plane, and 1 everywhere else.

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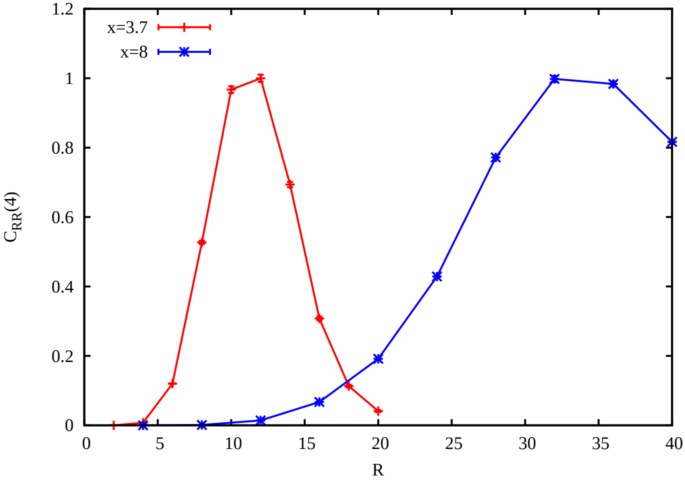
- Numerical agenda:
  1. Take theories with $N = 5, 7, 11, 17$ and approx. matching NL.
  2. Take all values of k and wide range of couplings, ranging from small-volume perturbative regime (small x), to large-volume non-perturbative one (large x).
  3. Calculate lightest scalar [and tensor] glueball masses, as well as electric flux energies.

Numerical calculation: details

- Electric flux energies: find energies from plateaux of the Polyakov loop correlators with winding number $\bar{k}$.
- Glueballs: variational analysis, use basis of rectangular Wilson loops and moduli of multi-winding Polyakov loops $|\mathrm{Tr}\, P^n|^2$, with 3 different levels of APE smearing.
- Instead of blocking, use Wilson of large extent trying to follow the physical size of the glueball (including loops larger than L for small and moderate x)

Numerical calculation: details



$C_{RR}(4)$ is the (normalized) correlator of $W(R, R)$ at time distance 4a,
$N = 5$, $L = 14$, $\bar{k} = 2$.

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- Instead of blocking, use Wilson of large extent trying to follow the physical size of the glueball (including loops larger than L for small and moderate x)
- Construct $C_{ij}(t) = \sum_{t'} \langle O_i(t'+t) O_j(t') \rangle - \langle O_i(t'+t) \rangle \langle O_j(t') \rangle$ and solve GEVP to find improved plateaux.

Numerical calculation: details cntd.

GEVP:

1. "Old-school" way:

$$C(t_1)v = C(t_0)\lambda v$$

to find v, use them to change the basis $C(t) \to \tilde{C}(t)\ \forall t$ and fit to diagonal elements of $\tilde{C}(t)$ (after finding the plateau).

2. ALPHA-way: Narrow down the basis to 3-4 operators (thinning/pruning), solve:

$$C_{thin}(t)v = C_{thin}(\lceil t/2 \rceil)\lambda(t)v,$$

use $\lambda(t)$ directly to produce effective mass plots:

$$E_n^{eff}(t) = -\partial_t \log(\lambda_n(t)).$$

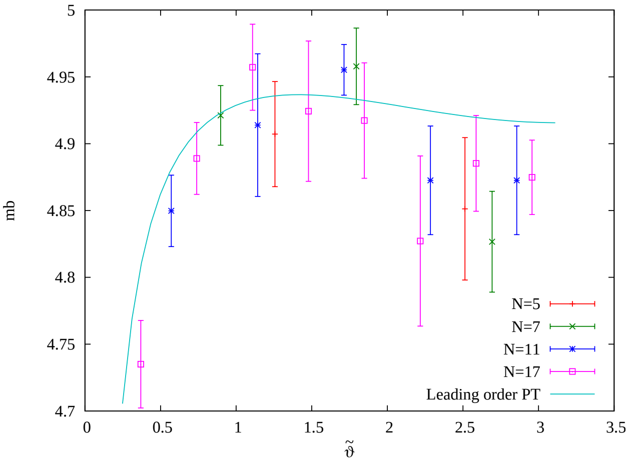
Better theoretical behaviour but more subject to noise. In practice, use only as cross-check of method 1.

Numerical calculation: details cntd.

- Technicalities: use approx. $12 - 25$ operators for $C_{ij}(t)$, estimate if basis allows reliable GEVP by first solving it on *non-symmetrized* $C(t)$, use quad precision for analysis.
- Plateau extraction: following Ryan, 14: "fit histogram": correlated fits to all possible t ranges, select one with best quality (function of χ^2, interval length, uncertainty size), look at stability.
- Small $x \Rightarrow$ lattice spacings very small $\Rightarrow$ look out for finite-temperature effects from wrap-around gluons!

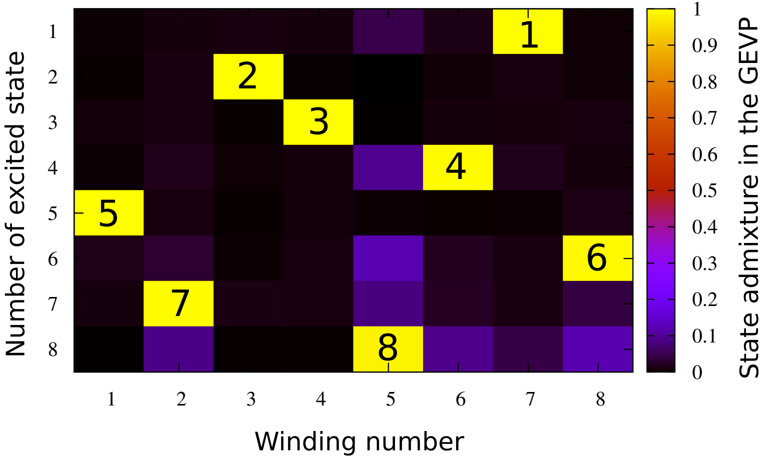
Contents

Glueball masses, $x = 0.199$ ($b \approx 28$ @ $NL = 70$)



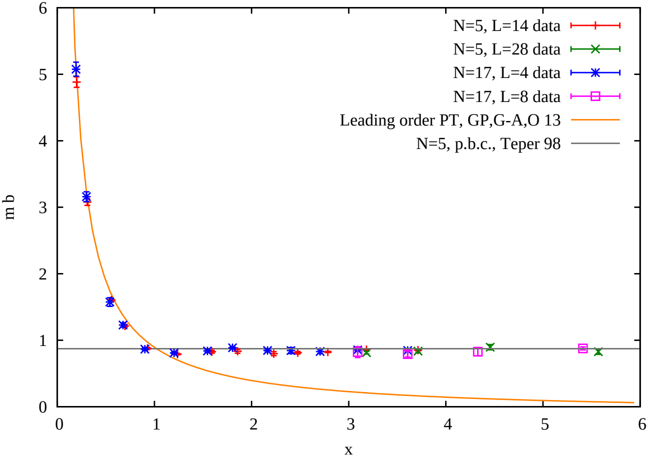
Lightest scalar glueball mass as function of $\tilde{\theta}$ for $N = 5, 7, 11, 17$, all values of k.
The values of L are $14, 10, 6, 4$ respectively.

Overlap of operators, $x = 0.199$



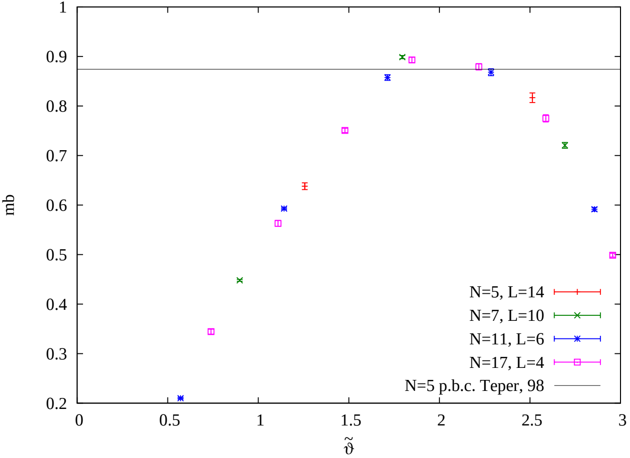
State admixture (= absolute value of the eigenvector) in the GEVP for $N = 17$ in the perturbative region. Black numbers on squares denote the PT expectation.

Glueball masses, scan in x



Comparison of lightest scalar glueball mass for $N = 5$, $\tilde{\theta} = 2.513$ and $N = 17$, $\tilde{\theta} = 2.587$, across a wide range of x.

Glueball masses, $x = 2.785$ ($b \approx 2$ @ $NL = 70$)



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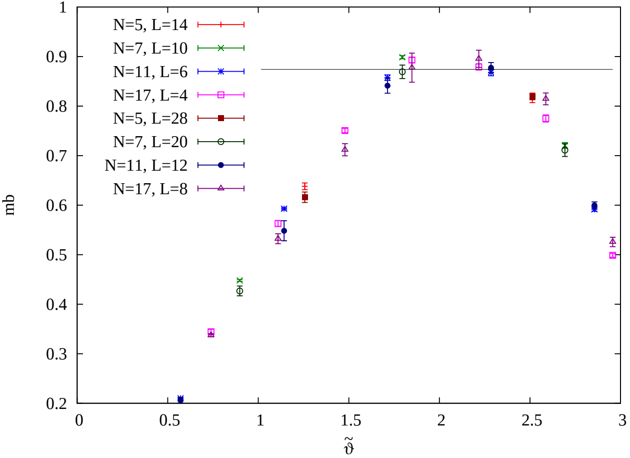
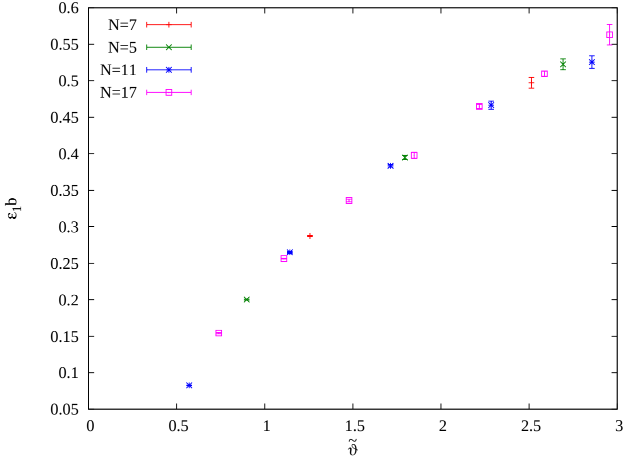


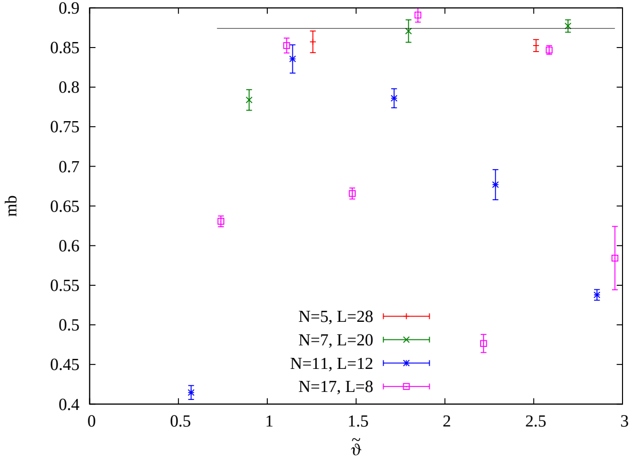
Lightest scalar glueball mass as function of $\tilde{\theta}$ for $N = 5, 7, 11, 17$, all values of k.
Also values for double L (and double b) are shown.

Electric flux energy, $x = 2.785$



Lowest electric flux energy as function of $\tilde{\theta}$ for $N = 5, 7, 11, 17$, all values of k.

Glueball masses, $x = 5.570$ ($b \approx 2$ @ $NL = 140$)



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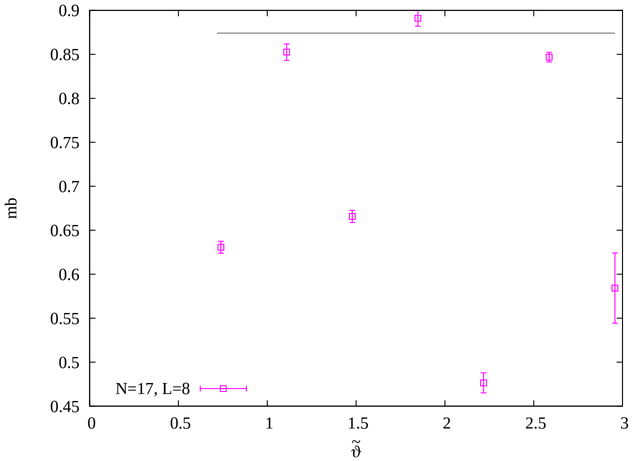

Lightest scalar glueball mass as function of $\tilde{\theta}$ for $N = 17$, all values of k.

Why so small mass for $N = 17$, $\bar{k} = 6$?

- One expects that:
  1. At small x the lightest "glueball" corresponds to a weakly interacting two-gluon state.
  2. At larger x the large-volume glueball becomes the lightest state.

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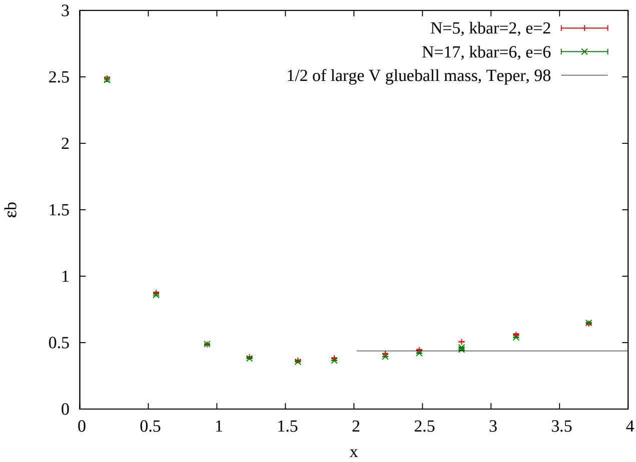

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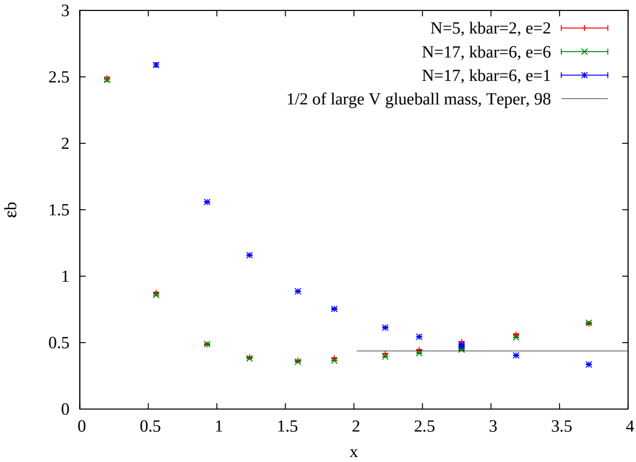

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- Can one work around it? Fine tune $\bar{k}$ to avoid unexpected light states or restrict to selected eigenstates...

Summary & outlook

- x-scaling conjecture: in a 2+1 dimensional Yang-Mills theory with twisted boundary conditions physics is governed by product $NL\lambda$
- x-scaling with strong confirmation in the non-zero electric flux sector.
- In the glueball sector: conjecture satisfied for small and moderate x, approach to large volume shows significant deviations from a smooth $\tilde{\theta}$ dependence.
- More understanding required to see if one can "save" the conjecture in a restricted form.
- In principle, straightforward to generalize to $3+1$ dimensions (cf. Keegan, Ramos, 15).

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Thank you for your attention!