$D_s$ to $\eta/\eta'$ semi-leptonic decay form factors from the lattice

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Outline

- Introduction: form factors
- Details of ensembles
- Determining \( \eta, \eta' \) on the lattice
- Results for masses and mixing angle(s)
- Finite volume effects
- \( D_s \rightarrow \eta(\eta') \) decay
- Results for the scalar form factor
- Phenomenological relevance
- Outlook
**$D_s$ semi-leptonic decays**

20% of decays to leptons.

- **Leptonic decays**, $D_s \rightarrow \ell^+\nu$, $\langle 0|A_\mu|D_s\rangle = p_\mu f_{D_s}$. Well measured in expt. and on the lattice. FLAG report [1310.8555] $f_{D_s} = 248.6 \pm 2.7$ MeV for $N_f = 2 + 1$, used to determine $V_{cs}$. Expt: $f_{D_s} = 257.5(4.6)$ MeV PDG (2013) (using $V_{cs}$).

- **Semi-leptonic decay** $D_s \rightarrow \phi \ell^+\nu$. Helicity functions measured in expt. On the lattice only HPQCD [1311.6669].

- **Semi-leptonic decay** $D_s \rightarrow \eta^{(')} \ell^+\nu$. Only branching fractions measured by CLEO [0903.0601].

$D_s \rightarrow \eta \ell^+ \nu, \ D_s \rightarrow \eta' \ell^+ \nu$

\[
A = \frac{G_F}{\sqrt{2}} V_{cs} \bar{c} \gamma_{\mu} (1 - \gamma_5) \nu \langle \eta^{(')} | \bar{c} \gamma_{\mu} (1 - \gamma_5) s | D_s \rangle
\]

\(\bar{c} \gamma_{\mu} \gamma_5 s\) doesn't contribute due to parity.

\[
\langle \eta^{(')} | V_{\mu} | D_s \rangle = f_+(q^2) \left( p_{D_s \mu} + p_{\eta^{(')} \mu} - \frac{m_{D_s}^2 - m_{\eta^{(')}}^2}{q^2} q_{\mu} \right) + f_0(q^2) \frac{m_{D_s}^2 - m_{\eta^{(')}}^2}{q^2} q_{\mu}
\]

Kinematical constraint at \(q^2 = 0\): \(f_+(0) = f_0(0)\)

\[
\frac{d\Gamma}{dq^2} = \frac{G_F^2 |V_{cs}|^2}{24\pi^3} \frac{(q^2 - m_{\ell}^2)^2 \sqrt{E_{\eta^{(')}}^2 - m_{\eta^{(')}}^2}}{q^4 m_{D_s}^2} \left[ \left( 1 + \frac{m_{\ell}^2}{2q^2} \right) m_{D_s}^2 (E_{\eta^{(')}}^2 - m_{\eta^{(')}}^2) |f_+(q^2)|^2 + \frac{3m_{\ell}^2}{8q^2} (m_{D_s}^2 - m_{\eta^{(')}}^2)^2 |f_0(q^2)|^2 \right]
\]

If set \(m_{\ell} = 0\), only \(f_+(q^2)\) contributes.
Only branching fraction \( Br = \frac{\Gamma(D_s \rightarrow \eta^{(i)})}{\Gamma} \) is measured so far in expt.

CLEO collaboration [0903.0601]

\[
Br(D_s^+ \rightarrow \eta e^+ \nu_e) = (2.48 \pm 0.29 \pm 0.13)\%
\]
\[
Br(D_s^+ \rightarrow \eta' e^+ \nu_e) = (0.91 \pm 0.33 \pm 0.05)\%
\]

To compare, lattice results for \( f_+(q^2) \) are needed.

However, we make use of PCVC relation to avoid renormalisation of \( V_\mu \) operator (HPQCD [1008.4562])

\[
q^\mu \langle V_\mu \rangle = (m_c - m_s)\langle S \rangle + O(a^2)
\]

which leads to

\[
f_0(q^2) = \frac{m_c - m_s}{M_{D_s}^2 - M_{\eta^{(i)}}^2} \langle \eta^{(i)} | S | D_s \rangle + O(a^2)
\]

Only predict \( f_0(0) = f_+(0) \).

Will make use of a parameterisation for \( f_+(q^2) \) to predict \( Br \).
Flavour singlets

SU(3) flavour symmetry \((u,d,s)\): for mesons, \(\bar{q}q\), we have \(3 \otimes \bar{3} = 8 \oplus 1\)

\[
\text{octet: } \pi^0, \pi^\pm, K^\pm, K^0, \bar{K}^0, \eta, \quad \text{singlet: } \eta' \\
\eta = \eta_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}), \quad \eta' = \eta_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})
\]

Chiral symmetry \((m_q = 0)\): \(SU_A(3)\) symmetry spontaneously broken
\(\pi^0, \pi^\pm, K^\pm, K^0, \bar{K}^0, \eta\) Goldstone bosons.

\(U_A(1)\) symmetry anomalously broken
\[
\partial_\mu J_{\mu 5} = 2N_f \rho(x), \quad \rho(x) = \frac{1}{32\pi^2} \epsilon^{\alpha\beta\mu\nu} \text{Tr}(F_{\alpha\beta}F_{\mu\nu})
\]

\[
Q = \sum_x \rho(x) \in \mathbb{Z}
\]

\(\eta'\) heavier than octet mesons.

Physical \(\eta\) and \(\eta'\) mixtures of \(\eta_8\) and \(\eta_1\).
Technically challenging

Quark line diagrams for studying $\eta$ and $\eta'$. 

For $D_s \to \eta(\eta')$

- Disconnected diagrams which may give a large contribution due to the anomaly also due to sum over $l = u, d, s$.
- Sensitivity to the topology of the gauge field configurations.

First step, determine the physical basis for $\eta/\eta'$. 

\[ \sum_{l=u,d,s} \left( \eta, \eta' \right) \]
Ensembles, lattice details

- $N_f = 2 + 1$, QCDSF configurations, Stout Link Non-perturbatively improved Clover (SLiNC) fermions, $O(a^2)$ discretisation errors in $f_0(q^2)$.
- Simulate along $\bar{m} = \frac{1}{3}(m_s + 2m_{u/d}) = \text{const.} \propto (X^\text{phys}_\pi)^2 = \frac{1}{3}(2M_K^2 + M_\pi^2)$
- In practice, $X_\pi$ 60 MeV heavier due to change in $a \sim 0.083$ fm from average octet baryon mass QCDSF [1003.1114] to $a \sim 0.075$ fm.
- Two ensembles with $V = 24^3 \times 48$
  - Symmetric ($m_s = m_l$): $M_\pi = M_K = 471$ MeV, 939 configs., $LM_\pi = 4.3$
  - Asymmetric ($m_s > m_l$): $M_\pi = 370$ MeV, $M_K = 509$ MeV, 239 configs., $LM_\pi = 3.3$
Extracting $\eta$ and $\eta'$ physical states

Start from the SU(3) basis

$$\eta_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}), \quad \eta_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

[flavour basis commonly used: $\eta_l = \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d)$ and $\eta_s = \bar{s}s$]

Construct correlation matrix

$$\langle C_{2pt}(t, p) \rangle = \begin{pmatrix} \langle C_{2pt}^{88}(t, p) \rangle & \langle C_{2pt}^{81}(t, p) \rangle \\ \langle C_{2pt}^{18}(t, p) \rangle & \langle C_{2pt}^{11}(t, p) \rangle \end{pmatrix} \equiv \begin{pmatrix} \langle O_8(t; p)O_8^\dagger(0) \rangle & \langle O_8(t; p)O_1^\dagger(0) \rangle \\ \langle O_1(t; p)O_8^\dagger(0) \rangle & \langle O_1(t; p)O_1^\dagger(0) \rangle \end{pmatrix},$$

Optimised smeared operators. Solve the generalised eigenvalue problem

$$\langle C_{2pt}(t_0, p) \rangle^{-\frac{1}{2}} \langle C_{2pt}(t, p) \rangle \nu_\alpha(t, p) = \lambda_\alpha(t, p) \langle C_{2pt}(t_0, p) \rangle^{\frac{1}{2}} \nu_\alpha(t, p),$$

Parameterise the eigenvectors:

$$\nu_\eta(t, p) = (\cos \theta(t, p), -\sin \theta(t, p))^T, \quad \nu_{\eta'}(t, p) = (\sin \theta'(t, p), \cos \theta'(t, p))^T.$$ 

Arrive at the physical basis

$$O_\eta = \cos \theta(p)O_8 - \sin \theta(p)O_1, \quad O_{\eta'} = \sin \theta'(p)O_8 + \cos \theta'(p)O_1.$$
Correlation functions of the physical states.

\[
\langle C^{\eta}_{2\text{pt}}(t, p) \rangle = \langle O^{\eta}_{\eta'}(t; p)O^{\dagger}_{\eta'}(0) \rangle = A_{\eta'}(p) \left( e^{-tE_{\eta'}(p)} + e^{-(T-t)E_{\eta'}(p)} \right)
\]

For masses, \( p = 0 \) sufficient. \( p \neq 0 \) needed for \( D_s \to \eta^{(')} \).

However, use finite momentum data to improve \( M_{\eta^{(')}} \). Fit to continuum \( ((aE(p))^2 = (aM)^2 + (ap)^2) \) and lattice dispersion relation:

\[
2 \cosh(aE(p)) = 2 \cosh(aM) + \sum_{i=1}^{3} 4 \sin^2 \frac{ap_i}{2},
\]

\( aM \) is a free parameter. Example, symmetric point, \( \eta' \).
Effective masses of $\eta$, $\eta'$ and $\pi$

Symmetric ensemble, $M_\pi = M_K = 471$ MeV,

Asymmetric ensemble, $M_\pi = 370$ MeV, $M_K = 509$ MeV.

Final masses:

<table>
<thead>
<tr>
<th></th>
<th>$M_\eta$ [MeV]</th>
<th>$M_{\eta'}$ [MeV]</th>
<th>$N_{\text{conf}}$</th>
<th>$N_{\text{bin}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sym.</td>
<td>470.5 (1.8)</td>
<td>1032 (27)</td>
<td>939</td>
<td>5 (25 traj.)</td>
</tr>
<tr>
<td>Asym</td>
<td>542.8 (6.2)</td>
<td>946 (65)</td>
<td>239</td>
<td>2 (20 traj.)</td>
</tr>
</tbody>
</table>

Expt. $M_\eta = 547.8$ MeV $M_{\eta'} = 957.8$ MeV. \( t/a = 10 \) corresponds to \( t = 0.75 \) fm.
Comparison with other determinations

**ETMC [1310.1207]**

- $LM_\pi = 5.2$, $24^3 \times 48$, $a \sim 0.09 - 0.10$ fm, $M_\pi = 475 - 427$ MeV.
- $N_{conf} \approx 2500$ with $N_{bin} = 10$. $t/a = 7 \rightarrow t = 0.67$ fm.
- Final results obtained by subtracting off excited states using g.s. determined from connected twopt fn.

**HSC [1309.2608]**

- $LM_\pi = 5.7$, $24^3 \times 128$, $a_s \sim 0.12$ fm, $a_t^{-1} \sim 5.6$ GeV, $M_\pi = 391$ MeV.
- $N_{conf} = 553$ with $N_{bin} = 10$. Distillation method. $t/a = 10 \rightarrow t = 0.42$ fm.
Comparison with other mass determinations

Our work: flavour average quark mass fixed. Approach physical point $m_\pi \downarrow, m_K \uparrow, \eta \uparrow$.

Consistency with other lattice determinations.

Chiral extrapolations not shown, e.g. ETMC: $M_\eta = 551(8)(6)\text{ MeV}$ and $M_{\eta'} = 1006(54)(31)(61)\text{ MeV}$.
Mixing angle(s)

$\eta$ and $\eta'$ are mixtures of the SU(3) basis.

Use pseudoscalar matrix elements (leading order distribution amplitudes)

$$
\begin{pmatrix}
A_{8\eta} & A_{1\eta} \\
A_{8\eta'} & A_{1\eta'}
\end{pmatrix}
=
\begin{pmatrix}
\langle 0|O_8|\eta \rangle & \langle 0|O_1|\eta \rangle \\
\langle 0|O_8|\eta' \rangle & \langle 0|O_1|\eta' \rangle
\end{pmatrix}
=
\begin{pmatrix}
\cos \theta_8 & -\sin \theta_1 \\
\sin \theta_8 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix}
Z_8 & 0 \\
0 & Z_1
\end{pmatrix}
$$

with local (unsmeread) operators [Alternatively use decay constants - $A_8/A_1$, diff. angles]

$$
O_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}), \quad O_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})
$$

If two states are enough $\theta_8 = \theta_1$. Renormalisation cancels in the ratios:

$$
\frac{A_{8\eta'}}{A_{8\eta}} = \tan \theta_8, \quad \frac{A_{1\eta}}{A_{1\eta'}} = -\tan \theta_1, \quad \tan^2 \bar{\theta} = \tan \theta_8 \tan \theta_1.
$$

Extract from fits to twopt correlators

$$
\langle 0|O_j^{\text{local}}(t)O^\dagger_{\eta(\nu)}(0)|0 \rangle
\rightarrow \frac{A_{j\eta(\nu)} Z^S_{\eta(\nu)}}{2M_{\eta(\nu)}} \left(\exp[-M_{\eta(\nu)} t] + \exp[-M_{\eta(\nu)} (T - t)]\right) + \text{add. terms}
$$

Smeared amplitude: $Z^S_{\eta(\nu)} = \langle \eta(\nu)|O^\dagger_{\eta(\nu)}|0 \rangle$
Sym. ensemble: $\theta_8 = \theta_1 = 0$

Asym. ensemble: $\theta_8 = -10.9(1.5)(0.5)$, $\theta_1 = -5.5(1.5)(1.2)$, $\bar{\theta} = -7.7(0.9)(0.8)$

Two angles are needed to describe $\eta$ and $\eta'$. $\eta$ mostly octet, $\eta'$ mostly singlet.

$|\theta_j|$ likely to become larger for physical $m_s/m_l$.

** with non-local operators.

ETMC: results consistent with a single angle.

Most other lattice studies use flavour basis: $\bar{\theta} = \alpha - 54.7^\circ$. 
Calculational techniques

\[ \langle O^i(t; p) O^{j \dagger}(0) \rangle \] contain contributions, \( l, l' = u/d, s \)

Connected 2pt function.

**Low mode averaging (LMA)** [DeGrand and Schäfer hep-lat/0401011, Giusti et al. hep-lat/0402002]:

Use eigenvalues and eigenvectors:  
\[ \gamma_5 M | \lambda_i(x) \rangle = \lambda_i | \lambda_i(x) \rangle \]

\[
M_{x,y}^{-1}\big|_{\text{low}} = \sum_{i=1}^{n_{\text{low}}} \frac{1}{\lambda_i} | \lambda_i(x) \rangle \langle \lambda_j(y) | \gamma_5
\]

\[
C_{\text{low}}^{2\text{pt}}(t) = \sum_{x,y} \sum_{i,j=1}^{n_{\text{low}}} \frac{1}{\lambda_i \lambda_j} \langle \lambda_i(x, t_0) | \gamma_5 \Gamma^\dagger | \lambda_j(x, t_0) \rangle \langle \lambda_j(y, t + t_0) | \gamma_5 \Gamma | \lambda_i(y, t + t_0) \rangle
\]

\[
C_{\text{LMA}}^{2\text{pt}}(t) = C_{\text{pa}}^{2\text{pt}}(t; x_0) - C_{\text{low,pa}}^{2\text{pt}}(t; x_0) + C_{\text{low}}^{2\text{pt}}(t)
\]
Disconnected loops, $\text{Tr}(M^{-1}\gamma_5)$, using stochastic estimates (complex $Z_2$).

$$\frac{1}{N_s} \sum_{s=1}^{N_s} |\eta_s\rangle\langle\eta_s| = 1 + O\left(\frac{1}{\sqrt{N_s}}\right), \quad M|s_s\rangle = |\eta_s\rangle, \quad M^{-1} \approx \frac{1}{N_s} \sum_s |s_s\rangle\langle\eta_s|$$

Low modes: $M^{-1} = M^{-1}|_{\text{low}} + M^{-1}|_{\text{high}}$, stochastically estimate $M^{-1}|_{\text{high}}$.

Partitioning [Bernardson et al. '93]:
$|\eta_s\rangle$ non-zero on every 4th timeslice + spin partitioning.

Hopping parameter acceleration [Thron et al. hep-lat/9707001, Bali et al. hep-lat/0505012]:
Finite volume effects on the $\eta$ and $\eta'$ masses

Symmetric ensemble ($\eta' = \eta_1$, $\eta = \eta_8$): $\eta'$ twopt fn. decays to a small non-zero value. Only for $p = 0$.

Due to a finite volume effect coupled to insufficient sampling of $Q$ sectors.

939 configs separated by 5 trajectories.
One point loops used for disconnected twopt fn. are $\propto Q_f$

Disconnected twopt fn $D_{ab}(t)$:

$$D_{ab}(t) = \frac{a^4}{V_4} \sum_{t_0/a=0}^{T/a-1} C_{1pt}^a(t + t_0) C_{1pt}^b(t_0)$$

Onept loop, $p = 0$:

$$\sum_t C_{1pt}(t) = \sum_t \sum_x C_{1pt}(t, x) = \sum_t \sum_x \text{Tr}[\gamma_5 M^{-1}] = \alpha Q_f \approx \alpha Q,$$

Reminder:

**Fermionic:** $Q_f = -\frac{Z_p}{Z_s} m \sum_x \text{tr}[\gamma_5 M_{x,x}^{-1}]$. **Gluonic:** $Q = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr}[F_{\mu\nu}(x)F_{\rho\sigma}(x)].$

Topological charge density: $\rho(t, x) \sim C_{1pt}(t, x)/(\alpha a^4)$. 
For fixed $Q$ [Aoki et al. 0707.0396]:

$$\langle \rho(x)\rho(0) \rangle_Q \rightarrow \frac{1}{V_4} \left( \frac{Q^2}{V_4} - \chi_t - \frac{c_4}{2\chi_t V_4} \right) + \cdots,$$

Constant term only for $p = 0$. $c_4 \sim 0$ for our ensembles.

Estimate of $\eta'$ twopt function.

$$C_{2pt}^{\eta'}(t, p = 0) = C_{\text{conn.}}(t, p = 0) - 3 \frac{\alpha^2 a^{12}}{V_4} \sum_{t_0/a=0} \sum_{x,x_0} \rho(t + t_0, x)\rho(t_0, x_0).$$

At large times

$$\langle C_{2pt}^{\eta'}(t, p = 0) \rangle_Q \rightarrow \frac{3\alpha^2 a^5}{T} \left( \chi_t - \frac{Q^2}{V_4} \right), \quad \langle C_{2pt}^{\eta'}(t, p = 0) \rangle_{Q=0} \rightarrow \frac{3\alpha^2 a^5 \chi_t}{T}$$
Topological charge of the symmetric ensembles

Gluonic $Q$ measured using improved $F_{\mu\nu}$ with stout smearing on gauge fields.

$\langle Q \rangle = 0$ within errors. $\chi_t = \langle Q^2 \rangle / V_4 = 9.1(0.4) / V_4$

$n - 0.5 \leq |Q| < n + 0.5$ configurations denoted $|Q| = n$
Large time behaviour of $\langle C_{2pt}(t) \rangle$ for $p = 0$ for fixed $Q$ depends on $|Q|$.

Fit to a constant, $\langle C_{2pt}(t = \infty) \rangle = \frac{3\alpha^2 a^5}{T} \left( \chi_t - \frac{Q^2}{V_4} \right)$.

- Linear dependence on $Q^2$ expected.
- $Q = 0$ prediction using measured $\chi_t = \langle Q^2 \rangle / V_4 = 9.1(0.4) / V_4$ and $\alpha$ from fitting $\sum_t \frac{1}{3} \left[ 2C_{1pt}^l(t) + C_{1pt}^s(t) \right] = \alpha Q$.
- $Q$ distribution too narrow due to insufficient sampling, $\langle Q^2 \rangle \approx 13$. 

![Graph](image-url)
Other studies

(left) Observation of the $|Q|$ dependence of $\eta'$ SESAM and $T\chi_L$ [hep-lat/0102002]

(right) Fixed topology: JLQCD and TWQCD [0910.4648]

+ other literature, e.g. Brower et al. [hep-lat/0302005], Aoki et al. [0909.0396]...
Improvement

Could remove constant by adding a constant term to the fit fn. for $\langle C_{\eta'}^{2\text{pt}}(t) \rangle$.

Alternatively add term which cancels $Q^2$ dependence (normalizing to $Q = 0$) and then fit including a constant.

$$D_{ab}(t) = \frac{a^4}{V_4} \sum_{t_0/a=0}^{T/a-1} C_{1\text{pt}}^a(t+t_0) C_{1\text{pt}}^b(t_0) \rightarrow \tilde{D}_{ab}(t) \equiv D_{ab}(t) - \frac{a^5}{V_4 T} \sum_{t_1/a, t_2/a=0}^{T/a-1} C_{1\text{pt}}^a(t_1) C_{1\text{pt}}^b(t_2)$$

Subtraction only affects constant part of twopt fn.

$$\langle \tilde{C}_{2\text{pt}}^{\eta'}(t) \rangle = A_{\eta'} \left( \exp[-E_{\eta'} t] + \exp[-E_{\eta'} (T - t)] \right) + \beta_{\eta'}$$

SU(3) symmetric case $\eta = \eta_8 = \pi$ and does not contain disconnected contributions and $\beta_{\eta} = 0$.

Improvement also applied to asymmetric ensemble, for which $\langle C_{2\text{pt}}^{\eta'}(t = \infty) \rangle$ consistent with zero.

Procedure is more involved, $\eta$ and $\eta'$ both have a constant term.
Effect of improvement

Improved method gives best signal and clear plateaus also for the asymmetric ensemble.

In limit of large statistics improved and naive results will agree.
$D_s \rightarrow \eta(\eta') \ell \nu$ threept function

\[
\left\langle C^{D_s \rightarrow \eta(\nu)}_{3pt} (t, \vec{p}, \vec{k}; t_{\text{sep}}) \right\rangle = \left\langle \mathcal{O}_{\eta(\nu)} (\vec{k}, t_{\text{sep}}) S(0, t) \mathcal{O}^\dagger_{D_s} (\vec{p}, 0) \right\rangle
\]

- Connected threept fn. uses stochastic estimates for $c$ (all-to-all) propagator. Average over rotationally equivalent momenta.
- Recycle one-point loops from $\eta/\eta'$ twopt fns.
- In addition, LMA for $D_s$ part of disconnected threept fn.

Relative error for $\left\langle C^{3pt, l}_{\text{disc}} (t, t_{\text{sep}}, \vec{p}, \vec{k}) \right\rangle$ for symmetric ensemble. $t_{\text{sep}} = 10a$, $D_s$ meson at $t = 0$ with $\vec{p} = (1, 0, 0)$, one-point loop is at $t/a = 10$ with $\vec{k} = (1, 0, 0)$. 
Comparison of connected and disconnected contributions to the threept fn.

Asymmetric ensemble, $M_\pi = 370$ MeV, $M_K = 509$ MeV (left) $\eta$, (right) $\eta'$.

Disconnected contribution is significant, in particular for $\eta'$.

Also dominates error on total threept fn.
Extracting the $D_s \rightarrow \eta(\eta')\ell\nu$ matrix element

$t_{\text{sep}}$ kept small due to noise.

Even with optimised smearing at source and sink, excited state contributions are significant.

Vary $t_{\text{sep}}$ and take excited states into account.

Spectral decomposition: ($D_s^*$, $\eta^{(i)*}$ refer to excited states.)

\[
\left\langle C_{3\text{pt}}^{D_s \rightarrow \eta^{(i)}}(t, p, k; t_{\text{sep}}) \rightangle \\
= \frac{Z_{\eta^{(i)}}}{2E_{\eta^{(i)}}} \frac{Z_{D_s}}{2E_{D_s}} \langle \eta^{(i)}(k)|S(0)|D_s(p)\rangle \exp \left[ -E_{D_s}(T - t) - E_{\eta^{(i)}}(T - (t_{\text{sep}} - t)) \right] \\
+ \frac{Z_{\eta^{(i)*}}}{2E_{\eta^{(i)*}}} \frac{Z_{D_s}}{2E_{D_s}} \langle \eta^{(i)*}(k)|S(0)|D_s(p)\rangle \exp \left[ -E_{D_s}(T - t) - E_{\eta^{(i)*}}(T - (t_{\text{sep}} - t)) \right] \\
+ \frac{Z_{\eta^{(i)}}}{2E_{\eta^{(i)}}} \frac{Z_{D_s^*}}{2E_{D_s^*}} \langle \eta^{(i)}(k)|S(0)|D_s^*(p)\rangle \exp \left[ -E_{D_s^*}(T - t) - E_{\eta^{(i)}}(T - (t_{\text{sep}} - t)) \right] \\
+ \frac{Z_{\eta^{(i)*}}}{2E_{\eta^{(i)*}}} \frac{Z_{D_s^*}}{2E_{D_s^*}} \langle \eta^{(i)*}(k)|S(0)|D_s^*(p)\rangle \exp \left[ -E_{D_s^*}(T - t) - E_{\eta^{(i)*}}(T - (t_{\text{sep}} - t)) \right] \\
+ \cdots
\]
Fit $\langle C_{2pt}^{D_s}(t, p) \rangle$ to determine ground state $Z_X$, $E_X$ and $\Delta E_X$.

$$R(t) = \frac{\langle C_{3pt}^{D_s \rightarrow \eta'}(t, p, k; t_{sep}) \rangle}{[\text{fitted g.s. } \langle C_{2pt}^{D_s}(t, p) \rangle][\text{fitted g.s. } \langle C_{2pt}^{\eta}(t, k) \rangle]}$$

$$\rightarrow \langle \eta'(k)|S(0)|D_s(p) \rangle + A_1 \exp[-\Delta E_{D_s} t] + A_2 \exp[-\Delta E_{\eta'}(t_{sep} - t)]$$

Asymmetric ensemble.

$D_s$ meson at $t = 16/a$, $\eta/\eta'$ at $t/a = 0, 6, 8$.

Fit $R(t)$ for $t_{sep}/a = 8, 10, 16$ simultaneously to extract the matrix element.
$D_s \rightarrow \eta(\eta')\ell\nu$ scalar form factor

Symmetric ensemble, $M_\pi = M_K = 471$ MeV,

$$f_0(q^2): D_s \rightarrow l\nu\eta^{(i)} \ (m_\pi = 470 \text{ MeV})$$

Asymmetric ensemble, $M_\pi = 370$ MeV, $M_K = 509$ MeV.

$$f_0(q^2): D_s \rightarrow l\nu\eta^{(i)} \ (m_\pi = 370 \text{ MeV})$$

Interpolate to $q^2 = 0$ using a one-pole ansatz: $f_0(q^2) = f_0(0)/(1 - bq^2)$.

$$f_0^{\eta}(0) \quad f_0^{\eta'}(0)$$

<p>| | |</p>
<table>
<thead>
<tr>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Symm.</td>
<td>0.564(11)</td>
</tr>
<tr>
<td>Asymm.</td>
<td>0.542(13)</td>
</tr>
<tr>
<td>LCSR</td>
<td>0.432(33)</td>
</tr>
</tbody>
</table>

Comparison with light cone sum rules (LCSR) Offen et al. [1307.2797]:
Phenomenological relevance

Consider comparison to experiment for the ratio

\[
\frac{\Gamma(D_s^{-} \rightarrow \eta' e^- \bar{\nu}_e)}{\Gamma(D_s^{-} \rightarrow \eta e^- \bar{\nu}_e)} = 0.36(14), \text{ CLEO [0903.0601]}.\]

Calculate

\[
\frac{\Gamma(D_s^{+} \rightarrow \eta' e^+ \nu_e)}{\Gamma(D_s^{+} \rightarrow \eta e^+ \nu_e)} = \frac{\int_0^{(M_{D_s} - M_{\eta'})^2} \lambda_{D_s, \eta'}^3(q^2) |f_{+}^{D_s \rightarrow \eta'}(q^2)|^2 dq^2}{\int_0^{(M_{D_s} - M_{\eta})^2} \lambda_{D_s, \eta}^3(q^2) |f_{+}^{D_s \rightarrow \eta}(q^2)|^2 dq^2},
\]

where \(\lambda_{D_s, \eta'}(q^2) = \frac{1}{4M_{D_s}^2} \left((M_{D_s}^2 + M_{\eta'}^2 - q^2)^2 - 4M_{D_s}^2 M_{\eta'}^2\right)\).

Use Ball-Zwicky parameterisation [hep-ph/0406232] for \(f_+(q^2)\),

\[
f_{+}^{BZ}(q^2) = f_+(0) \left(\frac{1}{1 - q^2/M_{D_s^*}^2} + \frac{r q^2/M_{D_s^*}}{(1 - q^2/M_{D_s^*})(1 - \alpha q^2/M_{D_s^*})}\right)
\]

Use LCSR calculation, Offen et al. [1307.2797], for \(r = 0.284(142)\) and \(\alpha = 0.252(126)\) (with 50% errors).
Some systematics cancel in the ratio.

Extrapolate ratio $f_0^{\eta'}(0)/f_0^{\eta}(0) = 0.705(120)(041)$, mild dependence on $M_{\pi}$.

Vary $r_\eta$, $\alpha_\eta$, $r_{\eta'}$, $\alpha_{\eta'}$ and $f_0^{\eta'}(0)/f_0^{\eta}(0)$ independently within errors.

Final result $1.6\sigma$ below CLEO measurement:

$$\frac{\Gamma(D_s^- \to \eta' e^- \bar{\nu}_e)}{\Gamma(D_s^- \to \eta e^- \bar{\nu}_e)} = 0.128_{-42}^{+51}$$

c.f LCSR Offen et al. [1307.2797] $0.37 \pm 0.09 \pm 0.04$. 
Conclusions and Outlook

- First lattice calculation of $D_s \rightarrow \eta^{(i)}$.
- Technical challenge, requires multiple noise reduction techniques.
- Disconnected contributions are significant.
- Sensitivity to the sampling of topological sectors due to finite volume and limited statistics.
- Obtain scalar form factor at $q^2 = 0$ with 6% statistical errors.
- $\eta$ and $\eta'$ masses and mixing angles consistent with other lattice determinations. Two angles are required to describe the two states.
- Pilot study: smaller quark masses, different volumes, multiple lattice spacings required.
- For comparison with experiment a measurement of $f_+(q^2)$ is needed which is of a similar level of complexity.
- Future: systematic study on CLS ensembles, also $D_s \rightarrow \phi$. 
