A dynamical study of the chirally rotated Schrödinger functional in QCD

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Introduction
Non-perturbative renormalization of lattice QCD with Wilson-fermions

Problem:
Wilson’s regularization explicitly breaks chiral symmetry:

- Mixing among operators with different naive chirality.
- Additional finite renormalization.
- Leading discretization errors are $O(a)$ (and can be large!).

Example: lattice PCAC relation

(Bochicchio et. al. '85)

$$
\langle \partial_\mu (A_R)^a_\mu (x) O_{x \neq 0} \rangle = 2 m_R \langle (P_R)^a(x) O_{x \neq 0} \rangle + O(a),
$$

$$(A_R)^a_\mu = Z_A(g_0) A^a_\mu, \quad m_R = Z_m(m_0 - m_{\text{crit}}(g_0)), \quad (P_R)^a = Z_P P^a,
$$

where,

$$
A^a_\mu(x) = \bar{\psi}(x) \frac{\tau^a}{2} \gamma_\mu \gamma_5 \psi(x), \quad P^a(x) = \bar{\psi}(x) \frac{\tau^a}{2} \gamma_5 \psi(x).
$$
Introduction

Symanzik’s effective description

Cutoff effects in the renormalized lattice theory can be studied using **Symanzik’s effective continuum theory**. This is first specified by the action,

\[ S_{\text{eff}} = S_0 + a S_1 + O(a^2), \quad S_0 \equiv S_{\text{QCD}}, \]

where,

\[ S_1 \propto \int d^4x \overline{\psi}(x) i\sigma_{\mu\nu} F_{\mu\nu}(x) \psi(x) + \text{not important for us}. \]

Analogously for a multiplicatively renormalized lattice field \( O_R \), we have,

\[ O_R = O_0 + a O_1 + O(a^2), \]

where \( O_0 \) is the corresponding continuum field.

Then, the connected **lattice** correlation function \( \langle O_R \rangle \) can be written as,

\[ \langle O_R \rangle = \langle O_0 \rangle^{\text{cont}} - a \langle S_1 O_0 \rangle^{\text{cont}} + a \langle O_1 \rangle^{\text{cont}} + O(a^2), \]

where \( \langle \cdots \rangle^{\text{cont}} \) is taken w.r.t. the **continuum** action \( S_{\text{QCD}} \).
The result:
In a finite volume \((L < \infty)\) w/o boundary and for zero quark-masses \((m_R = 0)\) there is an automatic way to eliminate the \(O(a)\) effects of Wilson-fermions.

The proof:
- Consider the (discrete) chiral symmetry transformation of the continuum massless QCD action,
  \[ \psi \rightarrow \gamma_5 \psi, \quad \bar{\psi} \rightarrow -\bar{\psi} \gamma_5. \]
  \(\Rightarrow\) non-anomalous for \(N_f\) even;
  \(\Rightarrow\) non-spontaneously broken in finite volume.
- Any observable (lattice or not) can be written as \(O = O^+ + O^-\), where,
  \(\gamma_5\)-even: \(O^+ \xrightarrow{\gamma_5} O^+\), \(\gamma_5\)-odd: \(O^- \xrightarrow{\gamma_5} -O^-\).
- In particular,
  \(S_0 \xrightarrow{\gamma_5} S_0, \quad S_1 \xrightarrow{\gamma_5} -S_1, \quad O_1^\pm \xrightarrow{\gamma_5} \mp O_1^\pm.\)
**Introduction**

Automatic $O(a)$ improvement  
(Frezzotti, Rossi ’04; Sharpe, Wu ’05; Shindler ’05; Sint ’05; Aoki, Bär ’05)

**$\gamma_5$-even $O_R^+$:**

\[
\langle O_0^+ \rangle^{\text{cont}} = + \langle O_0^+ \rangle^{\text{cont}} \\
\langle S_1 O_0^+ \rangle^{\text{cont}} = - \langle S_1 O_0^+ \rangle^{\text{cont}} = 0 \\
\langle O_1^+ \rangle^{\text{cont}} = - \langle O_1^+ \rangle^{\text{cont}} = 0
\]

**$\gamma_5$-odd $O_R^-$:**

\[
\langle O_0^- \rangle^{\text{cont}} = - \langle O_0^- \rangle^{\text{cont}} = 0 \\
\langle S_1 O_0^- \rangle^{\text{cont}} = + \langle S_1 O_0^- \rangle^{\text{cont}} \\
\langle O_1^- \rangle^{\text{cont}} = + \langle O_1^- \rangle^{\text{cont}}
\]

**Conclusion:**

\[
\langle O_R^+ \rangle = \langle O_0^+ \rangle^{\text{cont}} + O(a^2), \quad \langle O_R^- \rangle = O(a).
\]

Cutoff effects are located in the $\gamma_5$-odd components, these can be easily identified and projected out for any lattice field!

**Your question:**

Interesting, but why should we care about QCD in a finite volume with massless quarks?
Non-perturbative renormalization of QCD
Connecting low- and high-energy

**General problem:**

\[
\langle f | (\mathcal{O}_R)(\mu) | i \rangle = \lim_{a \to 0} Z_\mathcal{O}(g_0(a), a\mu) \langle f | \mathcal{O}(g_0(a)) | i \rangle,
\]

where \(|i\rangle\) and \(|f\rangle\) are some given (gauge-invariant) states.

- \(\langle f | \mathcal{O}(g_0) | i \rangle\) and \(Z_\mathcal{O}(g_0, a\mu)\) computed non-perturbatively.
- \(\langle f | (\mathcal{O}_R)(\mu) | i \rangle\) needed at \(\mu = \mathcal{O}(10 \text{ GeV})\).

**Examples:**

- Effective weak-transition amplitudes:
  \[
  \mathcal{A} = \langle f | \mathcal{H}_{\text{eff}} | i \rangle = C_W(\mu) \langle f | (\mathcal{O}_R)(\mu) | i \rangle.
  \]

- Fundamental parameters of QCD, i.e. running coupling and quark-masses:
  \[
  \alpha_s(\mu) \text{ or } m_R(\mu) \propto \langle f | (\mathcal{O}_R)(\mu) | i \rangle.
  \]
Non-perturbative renormalization of QCD
Finite-volume renormalization schemes and finite-size scaling

**Problem:** Difficult to fit all relevant energy scales in a single lattice,

\[ L^{-1} \ll \Lambda_{\text{QCD}} \ll \mu \ll a^{-1}. \]

Considering, e.g.,

\[ \Lambda_{\text{QCD}} \sim 0.3 \text{ GeV}, \quad \mu \sim 10 \text{ GeV} \quad \Rightarrow \quad L/a = O(100). \]

**Solution:** Make the renormalization scale run with the volume:

\[ \mu = L^{-1}. \]

*(Wolff '86; Lüscher, Weisz, Wolff '91)*

**Strategy:**

1. Renormalize \( \langle f | O(g_0) | i \rangle \) at some low-energy scale:

\[ \langle f | O_R(\mu_{\text{min}}) | i \rangle, \text{ where } \mu_{\text{min}} \equiv L_{\text{max}}^{-1} \sim \Lambda_{\text{QCD}}. \]

2. Compute the non-perturbative running of \( Z_O(g_0, a\mu, a\lambda_0) \) using **several** small volumes: \( L_{\text{max}} \to L_{\text{max}}/2 \to \ldots \to L_{\text{max}}/2^n. \)

1+2=3. Run to high-energy:

\[ \langle f | O_R(\mu_{\text{min}}) | i \rangle \to \langle f | O_R(\mu_{\text{max}}) | i \rangle, \text{ where } \mu_{\text{max}} \equiv 2^n L_{\text{max}}^{-1} > 10 \text{ GeV}. \]
The Schrödinger functional of QCD
An unfair introduction

\[ Z[C, C'] = \int \mathcal{D}[U] \mathcal{D}[\bar{\psi}, \psi] e^{-S[U, \bar{\psi}, \psi]} \]

Gauge fields:
\[ U_k(x)|_{x_0=0} = e^{aC_k}, \quad U_k(x)|_{x_0=T} = e^{aC'_k}. \]

Quark fields:
\[ P_\pm \equiv \frac{1}{2} (1 \pm \gamma_0) \]
\[ P_+\psi|_{x_0=0} = P_-\psi|_{x_0=T} = 0, \]
\[ \bar{\psi}P_-|_{x_0=0} = \bar{\psi}P_+|_{x_0=T} = 0. \]

Key features:

Simulations in the chiral limit are generally possible, regular perturbative expansion in finite volume, zero-momentum boundary quark-fields, ...
The Schrödinger functional of QCD

An unfair introduction

\( x_0 = T \)

\[ C' \]

\[ x_0 = 0 \]

\[ L \times L \times L \]

\[ T \]

\[ Z[C, C'] = \int \mathcal{D}[U] \mathcal{D}[\bar{\psi}, \psi] \; e^{-S[U, \bar{\psi}, \psi]} \]

Gauge fields:

\[ U_k(x)|_{x_0=0} = e^{aC_k}, \]

\[ U_k(x)|_{x_0=T} = e^{aC'_k}. \]

Quark fields:

\( P_{\pm} \equiv \frac{1}{2}(1 \pm \gamma_0) \)

\[ P_+ \psi|_{x_0=0} = P_- \psi|_{x_0=T} = 0, \]

\[ \bar{\psi} P_-|_{x_0=0} = \bar{\psi} P_+|_{x_0=T} = 0. \]

Key features:

Additional renormalization and cutoff effects (starting at \( O(a) \)) due to boundary counterterms.
The Schrödinger functional of QCD

An unfair introduction

(Symanzik '81; Lüscher et. al. '92; Sint '94, '95)

\[ Z[C, C'] = \int_{\text{SF b.c.}} \mathcal{D}[U] \mathcal{D}[\overline{\psi}, \psi] e^{-S[U, \overline{\psi}, \psi]} \]

**Gauge fields:**
\[ U_k(x)|_{x_0=0} = e^{aC_k}, \]
\[ U_k(x)|_{x_0=T} = e^{aC'_k}. \]

**Quark fields:**
\[ P_{\pm} \equiv \frac{1}{2} (1 \pm \gamma_0) \]
\[ P_+ \psi|_{x_0=0} = P_- \psi|_{x_0=T} = 0, \]
\[ \overline{\psi} P_-|_{x_0=0} = \overline{\psi} P_+|_{x_0=T} = 0. \]

**Key features:**
For Wilson-fermions even if in finite volume and at zero quark-mass, (bulk) $O(a)$ improvement is **NOT** automatic.
The Schrödinger functional of QCD
An unfair introduction

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\[ \bar{\psi} P_+|_{x_0=0} = \bar{\psi} P_-|_{x_0=T} = 0. \]

Key features:
For Wilson-fermions even if in finite volume and at zero quark-mass, (bulk) O(a) improvement is NOT automatic. ✔ for \( N_f \) even!
**The Schrödinger functional of QCD**

**The Schrödinger functional and automatic $O(a)$ improvement**

**Bob:** Why does the argument of automatic $O(a)$ improvement for massless Wilson-fermions in finite volume not hold for the SF?

**Alice:** The SF b.c.'s break chiral symmetry ($\psi \rightarrow \gamma_5 \psi$, $\bar{\psi} \rightarrow -\bar{\psi} \gamma_5$)!

(Frezzotti, Rossi '05; Sint '05)

$$P_+ \psi(x)|_{x_0=0} = 0, \quad \gamma_5 \quad P_- \psi(x)|_{x_0=0} = 0,$$

$$\bar{\psi}(x) P_-|_{x_0=0} = 0, \quad \Gamma_{\bar{\psi}}(x) P_+|_{x_0=0} = 0.$$

**Bob:** OK, but can we not find alternative SF b.c.'s and symmetry transformations of continuum massless QCD so to recover the argument?

**Alice:** Yes, consider for example the symmetry transformation, (Sint '05, '10)

$$\psi \rightarrow \gamma_5 \tau^1 \psi, \quad \bar{\psi} \rightarrow -\bar{\psi} \gamma_5 \tau^1,$$

and the b.c.'s specified by,

$$\tilde{Q}_\pm = \frac{1}{2} (1 \pm i \gamma_0 \gamma_5 \tau^3) \quad \Rightarrow \quad [\gamma_5 \tau^1, \tilde{Q}_\pm] = 0. \quad \checkmark$$
The chirally rotated Schrödinger functional

A chiral rotation to the Schrödinger functional

Given the isospin doublets $\psi$ and $\bar{\psi}$ satisfying standard SF b.c.'s, we consider the chiral rotation ($\alpha = \pi/2$),

$$\psi \rightarrow R\chi = e^{i\alpha\gamma^5 \frac{\tau^3}{2}} \chi, \quad \bar{\psi} \rightarrow \bar{\chi} R = \bar{\chi} e^{i\alpha\gamma^5 \frac{\tau^3}{2}}.$$

The quark fields $\chi$ and $\bar{\chi}$ satisfy the chirally rotated SF ($\chi\text{SF}$) b.c.'s,

$$\tilde{Q}_+ \chi(x)|_{x_0=0} = \tilde{Q}_- \chi(x)|_{x_0=T} = 0,$$

$$\bar{\chi}(x) \tilde{Q}_+|_{x_0=0} = \tilde{Q}_- \chi(x)|_{x_0=T} = 0.$$

Since $R$ is a non-anomalous symmetry transformation of massless QCD, in the continuum we expect the universality relations,

$$\langle \mathcal{O}[\psi, \bar{\psi}] \rangle_{(P \pm)} = \langle \mathcal{O}[R\chi, \bar{\chi} R] \rangle_{(\tilde{Q} \pm)}.$$

On the lattice with Wilson-fermions these relations hold among properly renormalized correlation functions, up to discretization effects!
The chirally rotated Schrödinger functional
Renormalization and \(O(a)\) improvement

(Sint '05, '10)

- For Wilson-fermions the \(\chi_{SF}\) b.c.'s are realized by **fine-tuning** a finite dim. 3 boundary counterterm (e.g. at \(x_0 = 0\))

\[
\overline{\chi} \tilde{Q}_- \chi \xrightarrow{R} -i\overline{\psi} \gamma_5 \tau^3 P_- \psi,
\]

\(\Rightarrow\) **breaks** parity and flavour symmetry: its coefficient, \(z_f(g_0)\), can be fixed by imposing parity/flavour symmetry restoration.

- **Automatic (bulk) \(O(a)\) improvement:**
  - **NO** bulk \(O(a)\) effects for \(\gamma_5 \tau^1\)-even obs.
  - bulk \(O(a)\) effects are located in \(\gamma_5 \tau^1\)-odd obs.

- **Full \(O(a)\)** improvement needs the tuning of a **couple** of \(O(a)\) boundary counterterms. PT seems to be good! (Sint, Vilaseca '14)

- In the unimproved theory \(z_f(g_0)\) and \(m_{\text{crit}}(g_0)\) can be determined only up to \(O(a)\) ambiguities.
  - ✓ These are only \(O(a^2)\) effects in \(\gamma_5 \tau^1\)-even obs.
  - ✗ Their determination can become difficult in practice if these \(O(a)\) effects are large.
The chirally rotated Schrödinger functional
The story so far . . .

**Earlier studies:** (Sint '05, '10; Leder, Sint '10; Gonzalez Lopez et. al. '12; Sint, Vilaseca '12)

The set-up has been investigated in the quenched approximation and in perturbation theory.

- Automatic $O(a)$ improvement, and universality with the SF.
- Determinations of $m_{\text{crit}}$ and $z_f$ are quite independent once the bulk action is improved.

**This work:**
A dynamical study of the $\chi$SF for $N_f = 2$ Wilson-fermions, specifically:

- Is the tuning of $m_{\text{crit}}$ and $z_f$ feasible? Yes! (More if interested)
- Determination of finite renormalization constants through universality relations: $Z_V, Z_A, Z_P/Z_S$.
- Tests of automatic $O(a)$ improvement.
- Checks of universality.
Renormalization in the $\chi$SF

The correlation functions we need . . .

(Leder, Sint '10; Sint, DB '14)

**SF:**

\[
\begin{align*}
    f_X(x_0) &= -\frac{1}{2}\langle \bar{\psi}_{f_1}(x)\Gamma_X \psi_{f_2}(x) \mathcal{O}_5^{f_2f_1} \rangle, \\
    k_Y(x_0) &= -\frac{1}{6}\sum_k \langle \bar{\psi}_{f_1}(x)\Gamma_{Y_k} \psi_{f_2}(x) \mathcal{O}_k^{f_2f_1} \rangle,
\end{align*}
\]

**$\chi$SF:**

\[
\begin{align*}
    g_{X}^{f_1f_2}(x_0) &= -\frac{1}{2}\langle \bar{\chi}_{f_1}(x)\Gamma_X \chi_{f_2}(x) \mathcal{Q}_5^{f_2f_1} \rangle, \\
    l_{Y}^{f_1f_2}(x_0) &= -\frac{1}{6}\sum_k \langle \bar{\chi}_{f_1}(x)\Gamma_{Y_k} \chi_{f_2}(x) \mathcal{Q}_k^{f_2f_1} \rangle,
\end{align*}
\]

where,

- $X = A_0, V_0, P, S$,
- $Y_k = A_k, V_k, T_{0k}, \tilde{T}_{0k}$.

and,

- $f_1, f_2 = u, d, u'd', f_1 \neq f_2$.

**Bilinears of boundary quark-fields**

\[
\mathcal{O}_5^{f_1f_2}, \mathcal{O}_k^{f_1f_2} \xrightarrow{R} \mathcal{Q}_5^{f_1f_2}, \mathcal{Q}_k^{f_1f_2}
\]
Renormalization in the $\chi$SF

Renormalization conditions from universality relations

**Universality relations:**
We consider $\gamma_5\tau^1$-even correlation functions ($\widetilde{V} \equiv$ conserved current),

\[
(f_A)_R = (g^{uu'}_A)_R = (-ig^{ud}_V)_R \quad \Rightarrow \quad Z_A g^{uu'}_A = -ig^{ud}_V + O(a^2),
\]
\[
(k_V)_R = (l^{uu'}_V)_R = (-il^{ud}_A)_R \quad \Rightarrow \quad Z_A l^{ud}_A = il^{uu'}_V + O(a^2).
\]

**Renormalization conditions:**
(Leder, Sint '10; Sint, DB '14)
The universality relations suggest to us simple renormalization conditions for the definition of $Z_A$, e.g.,

\[
Z^g_A \equiv \frac{-ig^{ud}_V(x_0)}{g^{uu'}_A(x_0)} \bigg|_{x_0 = \frac{T}{2}}, \quad Z^l_A \equiv \frac{il^{uu'}_V(x_0)}{l^{ud}_A(x_0)} \bigg|_{x_0 = \frac{T}{2}}.
\]

N.B.: The $Z_A$'s so obtained are fully $O(a)$ improved:
- **NO** need for $O(a)$ operator improvement i.e. $c_A(g_0)$ or $c_{\nabla}(g_0)$.
- $O(a)$ boundary effects **cancel** out in the ratios.
Renormalization in the $\chi$SF

Renormalization conditions from universality relations

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We consider $\gamma_5\tau^1$-even correlation functions ($\tilde{V} \equiv$ conserved current),

\[
(f_A)_R = (g_A^{uu'})_R = (-ig_V^{ud})_R \quad \Rightarrow \quad Z_V g_V^{ud} = g_V^{ud} + O(a^2),
\]

\[
(k_V)_R = (l_V^{uu'})_R = (-il_A^{ud})_R \quad \Rightarrow \quad Z_V l_V^{uu'} = l_V^{uu'} + O(a^2).
\]

**Renormalization conditions:**

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The universality relations suggest to us simple renormalization conditions for the definition of $Z_V$, e.g.,

\[
Z^g_V \equiv \frac{g_V^{ud}(x_0)}{g_V^{ud}(x_0)} \bigg|_{x_0 = \frac{T}{2}}, \quad Z^l_V \equiv \frac{l_V^{uu'}(x_0)}{l_V^{uu'}(x_0)} \bigg|_{x_0 = \frac{T}{2}}.
\]

N.B.: The $Z_V$'s so obtained are fully $O(a)$ improved:

- **NO** need for $O(a)$ operator improvement i.e. $c_V(g_0)$ or $c_{\tilde{V}}(g_0)$.
- $O(a)$ boundary effects **cancel** out in the ratios.
Renormalization in the $\chi$ SF
Renormalization conditions from universality relations

**Universality relations:**
We consider $\gamma_5\tau^1$-even correlation functions,

\[
(f_P)_R = (ig_S^{uu'})_R = (g_P^{ud})_R \quad \Rightarrow \quad Z_S ig_S^{uu'} = Z_P g_P^{ud} + O(a^2),
\]
\[
(k_T)_R = (il_T^{uu'})_R = (l_T^{ud})_R \quad \Rightarrow \quad Z_T il_T^{uu'} = Z_T l_T^{ud} + O(a^2).
\]

**Renormalization conditions:**
The universality relations suggest to us simple renormalization conditions for the definition of $Z_P/Z_S$, or $Z_T/Z_{\tilde{T}}$, e.g.,

\[
\frac{Z_P}{Z_S} \equiv \frac{ig_S^{uu'}(x_0)}{g_P^{ud}(x_0)} \Bigg|_{x_0 = \frac{T}{2}}, \quad \frac{Z_T}{Z_{\tilde{T}}} \equiv \frac{il_T^{uu'}(x_0)}{l_T^{ud}(x_0)} \Bigg|_{x_0 = \frac{T}{2}}.
\]

**N.B.:** The finite ratios so obtained are fully $O(a)$ improved:

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- $O(a)$ boundary effects **cancel** out in the ratios.
Renormalization in the $\chi$SF
Renormalization conditions for the bare parameters and lattice set-up

**Lattice action:**
- In the bulk, $N_f = 2$ non-perturbatively $O(a)$ improved Wilson-fermions, with Wilson gauge action.  
\hfill (Jansen, Sommer '98)
- $O(a)$ boundary improvement at 1-loop order in PT.  
\hfill (Sint, Vilaseca '14)

**Line of constant physics (LCP):**
- Vanish boundary gauge fields: $C = C' = 0$.
- $\chi$SF-geometry: $T = L$, with $L \approx 0.6$ fm.
  $\Rightarrow a \approx [0.076, 0.066, 0.049, 0.038]$ fm, $L/a = 8, 9.2, 12, 16$.
- Tuning conditions:

$$m_{\text{crit}} : m_{\text{PCAC}} = \left. \frac{\partial_0 g_A^{ud}(x_0)}{2 g_P^{ud}(x_0)} \right|_{x_0 = \frac{T}{2}} \neq 0,$$
$$z_f : O_{\gamma_5 T^1\text{-odd}} \neq 0.$$  

**Code:** customized version of the openQCD package.  
\hfill (Lüscher, Schaefer '12)
Renormalization in the $\chi$SF
Renormalization conditions for the bare parameters and lattice set-up

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  \[ m_{\text{crit}} : m_{\text{PCAC}} = \frac{\partial_0 g_A^{ud}(x_0)}{2g_P^{ud}(x_0)} \bigg|_{x_0 = \frac{T}{2}} = 0; \quad z_f : g_A^{ud}(x_0) \bigg|_{x_0 = \frac{T}{2}} = 0. \]

**Code:** customized version of the openQCD package.  
(Lüscher, Schaefer '12)
Renormalization constants
Comparison among different definitions of $Z_V$ as a function of $g_0^2$. (Some points are shifted in $g_0^2$ for your convenience.)

(Della Morte et al. '05)
Renormalization constants
Comparison among different definitions of $Z_V$ as a function of $g_0^2$ after 1-loop order perturbative improvement

(Della Morte et. al. '05)
Renormalization constants

Comparison among different definitions of $Z_A$ as a function of $g_0^2$

(Della Morte et. al. '05; Della Morte, Sommer, Takeda '08; Fritzsch et. al. '12)
Renormalization constants
Comparison among different definitions of $Z_A$ as a function of $g_0^2$ after 1-loop order perturbative improvement

(Della Morte et. al. '05; Della Morte, Sommer, Takeda '08; Fritzsch et. al. '12)

![Graph showing the comparison of renormalization constants $Z_A$ as a function of $g_0^2$. The graph includes different definitions of $Z_A$: $Z_{SF}^A$, $Z_g^A$, and $Z_l^A$. The data points are plotted with error bars, indicating the variation in the values.]
Test of universality
Continuum limit extrapolations for $Z_{V,A}$ differences

\begin{align*}
Z_{g}^{g} - Z_{l}^{l} \\
Z_{g}^{A} - Z_{l}^{A}
\end{align*}
Test of universality
Continuum limit extrapolations for $Z_{V,A}$ differences

\begin{align*}
Z_{gV} - Z_{lV} & \\
Z_{gA} - Z_{lA} & \\
Z_{gA_i} - Z_{gA} &
\end{align*}

\( (a/L)^2 \)
Test of universality
Continuum limit extrapolations for $Z_{V,A}$ differences after 1-loop order perturbative improvement

![Graph showing $Z_{V}^{g} - Z_{V}^{l}$ and $Z_{A}^{g} - Z_{A}^{l}$ vs $(a/L)^{2}$]
Test of automatic $O(a)$ improvement

Continuum limit extrapolations for $\gamma_5\tau^1$-odd correlation functions
More renormalization constants

Determination of $Z^g, l = Z_P / (Z_S Z_A Z^g, l)$ as a function of $g_0^2$ (Fritzsch, Heitger, Tantalo, '10)
Renormalization of the pseudo-scalar density

Renormalization conditions and step-scaling function

**Renormalization conditions:**

\[
Z_{SF}^P (g_0, L/a) = \left. \frac{\sqrt{3} f_1}{f_P(x_0)} \right|_{x_0 = \frac{T}{2}},
\]

\[
Z_{\chi SF}^P (g_0, L/a) = \left. \frac{\sqrt{3} g_{1}^{ud}}{g_P^{ud}(x_0)} \right|_{x_0 = \frac{T}{2}}.
\]

**Step-scaling function:**

\[
\sigma_P(u) = \lim_{a \to 0} \Sigma_P(u, a/L),
\]

\[
\Sigma_P(u, a/L) = \left. \frac{Z_P(g_0, 2L/a)}{Z_P(g_0, L/a)} \right|_{u = \bar{g}^2(L)}.
\]

**Expectations:**

\[
\frac{Z_{\chi SF}^P}{Z_{SF}^P} = 1 + O(a^2), \quad \sigma_P^{SF} = \sigma_P^{\chi SF}.
\]

\[
f_1 = -\frac{1}{2} \langle O_5^{f_1 f_2} O'_5^{f_2 f_1} \rangle
\]

\[
g_{1}^{f_1 f_2} = -\frac{1}{2} \langle Q_5^{f_1 f_2} Q'_5^{f_2 f_1} \rangle
\]
Renormalization of the pseudo-scalar density

Continuum limit extrapolations for \( R = 1 - \frac{Z^\chi_{SF}}{Z^P} \) for different LCPs

(Della Morte et al. '05)
Renormalization of the pseudo-scalar density
Continuum limit extrapolations for the lattice step-scaling functions for $\bar{g}^2(L) = 3.3$

(Della Morte et al. '05)
Outlook & Conclusions

Conclusions:

- Automatic $O(a)$ improvement is at work as expected.
- Very flexible setup to devise renormalization conditions that allows good control over cutoff effects, and good precision.
- The additional tuning of $z_f$ does not present any particular difficulty, and does not add too much work.

Outlook:

- This is a natural setup for renormalization problems in tmLQCD at maximal twist.
- Determination of renormalization constants for $N_f = 2 + 1$ can be obtained from a mixed setup: $2 \chi$SF + 1 SF fermions.
- Renormalization of more complicated operators as for example 4-quark operators relevant for flavour physics.
Determination of $m_{\text{crit}}$ and $z_f$

$m_{\text{PCAC}}$ as a function of $m_q = m_0 - m_{\text{crit}}$, for $a \approx 0.076$ fm, $L/a = 8$, and different $z_f$'s
Determination of $m_{\text{crit}}$ and $z_f$

$\gamma_5 \tau^1$-odd functions $g_A^{ud}$ and $g_P^{uu'}$ at zero quark-mass as a function of $z_f$, $a \approx 0.076$ fm, $L/a = 8$