Walking in a 3-dimensional scalar toy model

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Weinberg, Susskind (1979): a higher strongly coupled gauge theory “Technicolor (TC)” may dynamically explain the EW scale & the origin of masses of gauge bosons

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these still have a tension between suppression of effects caused by flavor-changing neutral currents and a large $m_{\text{top}}$

problem may be overcome in class of asymptotically free “walking TC models” whose characteristic dynamics changes very slowly at LE

Holdom ’85; Yamawaki, Bando, Matumoto; Appelquist, Wijewardhana ’87
candidate models investigated by lattice simulations

SU($N$) gauge theories with large $N_f$ (e.g. Appelquist, Fleming, Neil) or with fermions in higher representations e.g. sextets (e.g. DelDebbio, Patella)

do they produce a $\sim 126\text{GeV}$ Higgs with the right couplings to the gauge bosons?

Early stage: indication that the techni-sigma mass becomes smaller as the number of quarks increases
For QCD with large \( N_f \). PT may give some hints ???:

**Physical running coupling:** \( g^2(L) \)

**\( \beta \)-function:** \( \beta(g^2) = -L \frac{\partial}{\partial L} g^2(L) = -\beta_1 g^4 - \beta_2 g^6 - \ldots \)

For SU(3): \( b_1 = \frac{2}{(4\pi)^2} \left( 11 - \frac{2}{3} N_f \right) \), \( b_2 = \frac{2}{(4\pi)^4} \left( 102 - \frac{38}{3} N_f \right) \)

**2-loop IRFP (Banks & Zaks (1982))**:

\[
\frac{g_*^2}{4\pi^2} = -\frac{b_1}{4\pi^2 b_2} = \frac{132 - 8N_f}{38N_f - 306}
\]

\[
\frac{g_*^2}{4\pi^2} = 0.013 \text{ for } N_f = 16; \quad \frac{g_*^2}{4\pi^2} = 0.24 \text{ for } N_f = 12
\]

no (2-loop) IRFP for \( N_f = 8 \)

**2-loop “conformal window”:** \( 8 < N_f \leq 16 \)
QCD SU(3) probable conformal window: $N_f^{(1)} < N_f < N_f^{(2)} = \frac{33}{2}$

To determine $N_f^{(1)}$ requires non-perturbative methods

Many lattice simulations e.g. Nagai suggests $N_f^{(1)} = 8$, but still much debate (Hasenfratz, Cheng, Petropoulos, Schaich)

Unambiguous identification of walking phenomena in these models is extremely difficult:

- dynamical fermion simulations CPU intensive
- & need control of UV and IR cutoff effects

investigate TOY MODELS EXHIBITING WALKING may be useful
Toy model exhibiting walking

2-d O(3) model with $\theta$-term, proposed by Nogradi

PT results are independent of $\theta \rightarrow$ require non-perturbative methods

Lattice simulations by De Forcrand, Pepe, Wiese (2012)

demonstrated walking phenomena for $\theta$ close to $\pi$. 
LWW coupling; \( d = 2 \) non-linear O(3) model

De Forcrand, Pepe, Wiese (2012)
$\beta$– function; $d = 2$ non-linear O(3) model

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$N$-component scalar field theory in 3-dimensions in $N \to \infty$ limit
Aoki, Balog, P.W (2014)

walking phenomena when
(renormalized mass)/(effective 4 – point coupling) $\ll 1$

can study systematic errors and “robustness of walking phenomena”
1-component $\phi^4_3$ is non-trivial; rigorously constructed
Glimm and Jaffe (1981)

analytic computation of universal amplitude ratios
Gutsfeld, Heitger, Kuster, Münster (1994)
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$\phi^6_3$ model in large $N$ limit has long history (see later)

Klebanov, Polyakov (2002) conjecture: $O(N)$ singlet sector of
critical $(\phi^2)^2_3$ model is, in the large $N$ limit, dual to the minimal
bosonic theory in $AdS_4$ containing infinite number of gauge fields of
even spin (Fradkin, Vasiliev)

Revived interest in phase structure of 3d $U(N)$ symmetric gauged

review on dualities Polchinski (2014)
The model: $N$-component scalar field theory in 3-dimensions

**Fields:** $\Phi^i_n$, $i = 1, \ldots, N$

$n = (n_0, n_1, n_2)$ on a 3-dimensional hyper-cubic lattice (pbc)

**Generating functional:** (Standard action)

$$Z(J) = \int [\mathcal{D}\Phi] \exp \left\{ \sum_n \left[ \sum_{\mu=0}^{2} \Phi_n \cdot \Phi_{n+\hat{\mu}} - N\mathcal{V} \left( \frac{1}{N} \Phi_n^2 \right) + J_n \cdot \Phi_n \right] \right\}$$

$\hat{\mu}$: unit vector in the $\mu$-th direction

**measure:** $[\mathcal{D}\Phi] = \prod_{n,i} d\Phi^i_n$

**Potential:**

$$\mathcal{V}(S) = \frac{1}{2} RS + \frac{1}{4!} U S^2 + \frac{1}{6} E S^3$$

$E \geq 0$, and $U > 0$ if $E = 0$
Large $N$ saddle point expansion

“standard trick”: insert into the path integral

\[ 1 \equiv \int [D\Omega] \prod_n \delta \left( \Omega_n - \frac{1}{N} \Phi_n^2 \right) \]
\[ = \mathcal{N}_1 \int [D\Omega] [D\Lambda] \exp \left\{ -N \sum_n i\Lambda_n \left( \frac{1}{N} \Phi_n^2 - \Omega_n \right) \right\} \]

perform Gaussian $\Phi$ integration to obtain:

\[ Z(J) = \mathcal{N}_2 \int [D\Omega] [D\Lambda] \exp[-NS_{\text{eff}}(\Lambda, \Omega, J)] \]

with effective action:

\[ S_{\text{eff}}(\Lambda, \Omega, J) = \sum_n [\mathcal{V}(\Omega_n) - i\Lambda_n \Omega_n] + \frac{1}{2} \text{Tr} \ln D[\Lambda] \]
\[ -\frac{1}{2N} \sum_{nm,i} J^i_n \left( D^{-1}[\Lambda] \right)_{nm} J^i_m \]

where \((D[\Lambda]F)_n = 2(i\Lambda_n - 3)F_n - \sum_\mu (F_{n+\hat{\mu}} + F_{n-\hat{\mu}} - 2F_n)\)
The large $N$ limit determined by the **saddle point equations**

\[
\frac{\partial S_{\text{eff}}(\Lambda, \Omega, J_0)}{i \partial \Lambda_n} = -\Omega_n + (D^{-1}[\Lambda])_{nn} + H^2 \left( \sum_m (D^{-1}[\Lambda])_{mn} \right)^2 = 0
\]

\[
\frac{\partial S_{\text{eff}}(\Lambda, \Omega, J_0)}{\partial \Omega_n} = \mathcal{V}'(\Omega_n) - i \Lambda_n = 0
\]

in presence of **external field** $J_n = J_0, \ J_0^i = H \delta^{iN}$

**Assuming a translation invariant solution:**

\[
i \Lambda_n = i \Lambda_0 \equiv 3 + \frac{M^2}{2} = \text{const.} \quad (M \geq 0), \quad \Omega_n = \Omega_0 \equiv \text{const.}
\]

the **saddle point equations become**

\[
\frac{H^2}{M^4} + I(M) = \Omega_0
\]

\[
3 + \frac{M^2}{2} = \mathcal{V}'(\Omega_0)
\]

where \( I(M) = \frac{1}{V} \sum_K \left( \hat{K}^2 + M^2 \right)^{-1} \), \( \hat{K}_\nu = 2 \sin \frac{K_\nu}{2} \)
Systematic large $N$ expansion $\sim$ perturbation theory around the saddle point; introduce fluctuations

\[ \Lambda_n = \Lambda_0 + \frac{1}{\sqrt{N}} \tilde{\Lambda}_n, \quad \Omega_n = \Omega_0 + \frac{1}{\sqrt{N}} \tilde{\Omega}_n \]

Then

\[ S_{\text{eff}}(\Lambda, \Omega, J) = S_{\text{eff}}(\Lambda_0, \Omega_0, J) + N^{-1} \sum_{nm} \frac{1}{2} S_{nm}(J) \tilde{\Lambda}_n \tilde{\Lambda}_m \]

\[ + N^{-1} \sum_n \left[ \sqrt{NT_n}(\tilde{J}) i \tilde{\Lambda}_n + \frac{1}{2} \nabla''(\Omega_0) \tilde{\Omega}_n^2 - i \tilde{\Lambda}_n \tilde{\Omega}_n \right] + \ldots \]

where $J_n = J_0 + \tilde{J}_n$

\[ T_n(\tilde{J}) = \frac{1}{N} \sum_i \left[ (G \tilde{J}_i)_n \right]^2 + \frac{2H}{\sqrt{NM^2}} \left( G \tilde{J}^N \right)_n \]

\[ S_{nm}(J) = 2G_{mn}^2 + \frac{4}{N} \sum_i (G J_i)_n G_{nm} (G J_i)_m \]

\[ G_{nm} \equiv (D^{-1}[\Lambda_0])_{nm} = \frac{1}{V} \sum_K G(K)e^{iK(n-m)}, \quad G(K) = \frac{1}{\tilde{K}^2 + M^2}. \]
VEV and Correlation functions

“Pions”: $\Pi^i_n = \Phi^i_n$, $i = 1, \ldots, N - 1$

“Sigma”: $\Sigma_n = \Phi^N_n$

Generator of connected Green functions: $Z(J) = \mathcal{N}_3 \exp[-NW(J)]$

$\textbf{VEV}$: $\Sigma \equiv \frac{1}{\sqrt{N}} \langle \Sigma_n \rangle = \frac{-\sqrt{N} \partial W(J)}{\partial J^N_n} \bigg|_{J \to J_0} = HM^{-2} \left[ 1 + O \left( \frac{1}{N} \right) \right]$

$\Pi$ 2-pt function:

$\langle \Pi^i_n \Pi^j_m \rangle \equiv \frac{-N \partial^2 W(J)}{\partial J^i_n \partial J^j_m} \bigg|_{J \to J_0} = \delta^{ij} \left[ \frac{1}{V} \sum_K G(K) e^{iK(n-m)} + O \left( \frac{1}{N} \right) \right]$

$\textbf{Connected } \Sigma \textbf{ 2-pt function}$:

$\langle \Sigma_n \Sigma_m \rangle_c = \frac{1}{V} \sum_K \left[ G(K) - 4\Sigma^2 G^2(K) K^{-1}(K) \right] e^{iK(n-m)} + O \left( \frac{1}{N} \right)$
Connected $\Pi$ 4-pt function:

$$\langle \Pi_{n_1}^{i_1} \Pi_{n_2}^{i_2} \Pi_{n_3}^{i_3} \Pi_{n_4}^{i_4} \rangle_c =$$

$$\frac{1}{V^3} \sum_{K_1, K_2, K_3, K_4} \exp \left\{ i \sum_{j=1}^{4} K_j n_j \right\} \delta^{(3)} \left( \sum_{l=1}^{4} K_l \right) \times$$

$$\times \frac{1}{N} \left[ \prod_{m=1}^{4} G(K_m) \right] \left[ \delta^{i_1 i_2} \delta^{i_3 i_4} \left\{ \Gamma^{(4)}(K_1 + K_2) + O \left( \frac{1}{N} \right) \right\} + 2 \text{ perms} \right]$$

where $\Gamma^{(4)}_{\Pi}(K) = -4 \mathcal{K}^{-1}(K)$

with $\mathcal{K}(K) = \frac{1}{V''(\Omega_0)} + 2 J(K) + 4 H^2 M^{-4} G(K)$

$$J(K) = \frac{1}{V} \sum_{Q} G(Q) G(K + Q) .$$
Consider the infinite volume limit:

\[
I_\infty(M) = \int_{-\pi}^{\pi} \frac{d^3 K}{(2\pi)^3} \left\{ \hat{K}^2 + M^2 \right\}^{-1}
\]

\[
J_\infty(K) = \int_{-\pi}^{\pi} \frac{d^3 Q}{(2\pi)^3} \left\{ (\hat{Q}^2 + M^2) \left[ \left( \hat{K} + Q \right)^2 + M^2 \right] \right\}^{-1}
\]

For small mass \( M > 0 \) and \( K/M \) fixed:

\[
I_\infty(M) = I_0 - \frac{M}{4\pi} + I_2 M^2 + O(M^3)
\]

\[
J_\infty(K) = \frac{1}{4\pi |K|} \arctan \left( \frac{|K|}{2M} \right) - I_2 + O(M)
\]

constants \( I_0, I_2 \) are regularization dependent
Two phases at $H = 0$

Symmetric phase (SYM): $\lim_{H \to 0} M \neq 0$, $\lim_{H \to 0} \Sigma = 0$

Broken phase (BRO): $\lim_{H \to 0} M = 0$, $\lim_{H \to 0} \Sigma \neq 0$

Phase boundary: $3 = \mathcal{V}'(I_0)$.

Saddle point equations:

SYM case: $I_0 - \frac{M}{4\pi} + I_2 M^2 + O(M^3) = \Omega_0$, $3 + \frac{M^2}{2} = \mathcal{V}'(\Omega_0)$

BRO case: $I_0 + \Sigma^2 = \Omega_0$, $3 = \mathcal{V}'(\Omega_0)$
In the **broken phase** (at least for small $E$)

there is a $\sigma$–**resonance**

Lines of constant physics can be defined

e.g. by keeping $r = \Gamma_\sigma/M_\sigma$ constant
$\phi_3^6$ model in large $N$ limit has long history

Focused mainly on looking for non-Gaussian fixed points

Townsend (1977): Large $N + PT$: 1st non-vanishing order:

$$\beta = \left\{ \frac{1}{N} \right\} \left\{ \frac{3}{(2\pi^2)} \right\} \eta^2 \left( 1 - \frac{\eta}{\eta^*} \right) + O\left( \frac{1}{N^2} \right), \quad \eta^* = 192$$

i.e **UV fixed point at** $E_{\text{cont}} = \eta = \eta^*$ (conjectured also for finite $N$)

Pisarski; Appelquist, Heinz (1982): claimed UV FP at large $N < \infty$
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Bardeen, Moshe, Bander (1984):

1. **PT in \( \eta \) breaks down at** \( N = \infty \) for \( \eta > \eta_c = (4\pi)^2 < \eta^* \)

2. \( \eta_c \) acts as a UV FP with mass generation via dimensional transmutation, and also **spontaneous breaking of scale invariance** signaled by the **appearance of a dilaton**
Excellent papers by David, Kessler & Neuberger (1984)

Study of dominant solutions of gap equations & their stability

1. BMB phenomenon at $N = \infty$ is or is not present depending on details of the regularization!

necessary condition $I_2 > 0$

$(\text{recall } I_\infty(M) = I_0 - M/(4\pi) + I_2M^2 + ..)$

→ BMB limit not obtained for the standard action

consistent with Karsch, Meyer-Ortmanns (1987)
2. **argue BMB phenomenon not present at finite $N$**

here couplings run and there could be a different UV FP behavior

**Bardeen** interested in models with spontaneous breaking of scale symmetries & considers the $N = \infty$ as laboratory

see e.g **Bardeen, Mosche (2014):** Gauged versions:

at leading order $1/N$, $\eta$ and gauge coupling $g$ are unrenormalized and $\eta_c = \eta_c(g)$
Consider \( E = 0 \) in the SYM phase

For the continuum limit restore physical dimensions: \( K = p a \)

**Correlation functions having continuum limit:**

\[
\Gamma^{(2)}(p) = a^2 \Gamma^{(2)}_{\text{latt}}(a p) = \frac{1}{p^2 + (M/a)^2}
\]

\[
\Gamma^{(4)}(p) = a^{-1} \Gamma^{(4)}_{\text{latt}}(a p) = -2/ [6 a/U + aJ_\infty(a p)]^{-1}
\]

**Continuum limits:**

I: \( U = U_0 \neq 0 \)

II. \( U = u_{\text{eff}} a \to 0 \)

A: \( M = m a \to 0 \)

B: \( M = O(a^2) \)

**Interesting Case IIA:**

\[
\Gamma^{(2)} = \frac{1}{p^2 + m^2}
\]

\[
\Gamma^{(4)} = -2 \left[ 6/u_{\text{eff}} + \frac{1}{4\pi |p|} \arctan \left( \frac{|p|}{2m} \right) \right]^{-1}
\]
Lines of constant $\alpha = m/u_{\text{eff}}$ and constant $r = \Gamma_\sigma/M_\sigma$. 
We define a running coupling $g_4$ through the 4-point coupling in units of the energy scale at which it is measured.

$$g_4(\mathcal{E}) \equiv -3\Gamma_\pi^{(4)}(|\mathbf{k}| = \mathcal{E})/\mathcal{E}$$

<table>
<thead>
<tr>
<th>Case</th>
<th>$g_4(\mathcal{E})$</th>
<th>$g_4(0)$</th>
<th>$g_4(\infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>IA</td>
<td>$\frac{24\pi}{\arctan(\mathcal{E}/2m_R)}$</td>
<td>$\infty$</td>
<td>48 UV conformal</td>
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<td>0 UV AF</td>
</tr>
<tr>
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<td>$\frac{48}{1 + 48\mathcal{E}/u_{\text{eff}}}$</td>
<td>48</td>
<td>0 UV AF; IR conformal</td>
</tr>
</tbody>
</table>

**$\beta$–function:** $\beta_4(g_4) = \mathcal{E} \frac{\partial}{\partial \mathcal{E}} g_4(\mathcal{E})$. 
continuum beta-function for various $b = 96m/u_{\text{eff}}$
Lattice $\beta$-function at several values of $am_R$

Diamonds, squares & circles: finite volumes with $m_R L = 3, 4 \& 8$
Scattering amplitude (SYM phase)

\[ T_{i_1i_2,i_1i_3i_4}(k_1, k_2 \mid k_3, k_4) \equiv \lim_{k_{1,2,3,4} \rightarrow \text{on-shell}} \frac{1}{N} \left[ \delta_{i_1i_2} \delta_{i_3i_4} \Gamma^{(4)}_\pi (k_1 + k_2) + 2 \text{ perms} \right] \]

On-shell momenta in the center of mass system:

\[ k_1 = (iE, \vec{p}), \quad k_2 = (iE, -\vec{p}), \quad k_3 = (-iE, \vec{q}), \quad k_4 = (-iE, -\vec{q}) \]

with \( E = \sqrt{\vec{p}^2 + m_R^2}, \quad \vec{p}^2 = \vec{q}^2, \quad \vec{p} = (p_1, p_2), \quad \vec{q} = (q_1, q_2) \)

\[ T_{i_1i_2,i_1i_3i_4}(k_1, k_2 \mid k_3, k_4) = \sum_{I=0}^2 Q_I^{i_1i_2,i_1i_3i_4} T_I(\vec{p}, \vec{q}) \]

"Isospin" Projectors:

\[ Q_0^{i_1i_2,i_3i_4} = \frac{1}{N} \delta_{i_1i_2} \delta_{i_3i_4}, \quad Q_1^{i_1i_2,i_1i_3i_4} = \frac{1}{2} \left( \delta_{i_1i_3} \delta_{i_2i_4} - \delta_{i_1i_4} \delta_{i_2i_3} \right) \]

\[ Q_2^{i_1i_2,i_1i_3i_4} = \frac{1}{2} \left( \delta_{i_1i_3} \delta_{i_2i_4} + \delta_{i_1i_4} \delta_{i_2i_3} \right) - \frac{1}{N} \delta_{i_1i_2} \delta_{i_3i_4} \]
In the large $N$ limit:

$$T_0(\vec{p}, \vec{q}) = \lim_{\varepsilon \to 0} \Gamma^{(4)}_\pi ((i - \varepsilon)W, \vec{0}), \quad T_1(\vec{p}, \vec{q}) = T_2(\vec{p}, \vec{q}) = 0$$

where $W = 2E_p$.

Unitarity:

$$T_0(\vec{p}, \vec{q}) = 16We^{i\delta_0(W)} \sin \delta_0(W)$$

$I = 0$ scattering phase shift $\delta_0(W)$:

$$\cot \delta_0(W) = -48W/u_{\text{eff}} - \frac{2}{\pi} \text{arccoth} \left(\frac{W}{2m_R}\right)$$
$I = 0$ scattering phase shift (SYM phase)
The $\beta$ function of a finite volume coupling illustrates robustness of walking phenomenon in this model.