Loop formulation of supersymmetric models on the lattice
On the relevance of the sign problem

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Supersymmetry is thought to be a crucial ingredient in
the unification of the SM interactions,
the solution of the hierarchy problem.

Low energy physics is, however, not supersymmetric:
SUSY must be broken spontaneously,
this can not be described in perturbation theory.

Lattice provides a non-perturbative regularisation:
discretisation breaks Poincaré symmetry explicitely,
Leibniz’ rule is absent,
fermion doubling,
→ recovered in the continuum limit?

Fermionic sign problem hampers MC simulations.
Accidental symmetries may emerge from a non-symmetric lattice action:

- lattice action enjoys some exact symmetries,
- allows only irrelevant symmetry breaking operators,
- they become unimportant in the IR,

→ full symmetry emerges in the continuum limit

(Euclidean) Poincaré symmetry in lattice QCD.

Supersymmetry in $\mathcal{N} = 1$ SYM:

- only relevant operator is the gaugino mass term $m \bar{\xi}\xi$,
- violates $Z_{2N}$ chiral symmetry,
- chirally symmetric lattice action forbids this term,

→ SUSY automatically recovered in the continuum limit.
For SUSY theories involving scalar fields this is not easily possible:

- scalar mass term $m^2|\phi|^2$ breaks SUSY,
- no other symmetry available to forbid that term.

Some symmetries can be fine tuned with counterterms:

- chiral symmetry for Wilson fermions,
- might be feasible in lower dimensions if theories are superrenormalisable.

Look for subalgebras of the SUSY algebra

[Catterall; Kaplan; Ünsal; etc '01-'09]:

- combine Poincaré and flavour group (twisted SUSY),
- leads to Dirac-Kähler (staggered) fermions,
- consistent with orbifolding approach.
Consider the Lagrangian for supersymmetric quantum mechanics

\[ \mathcal{L} = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + \frac{1}{2} P'(\phi)^2 + \overline{\psi} \left( \frac{d}{dt} + P''(\phi) \right) \psi, \]

- real commuting bosonic 'coordinate' \( \phi \),
- complex anticommuting fermionic 'coordinate' \( \psi \),
- superpotential, e.g. \( P(\phi) = \frac{1}{2} m \phi^2 + \frac{1}{4} g \phi^4 \).

Two supersymmetries in terms of Majorana fields \( \psi_{1,2} \):

\[ \delta_A \phi = \psi_1 \varepsilon_A, \quad \delta_B \phi = \psi_2 \varepsilon_B, \]
\[ \delta_A \psi_1 = \frac{d\phi}{dt} \varepsilon_A, \quad \delta_B \psi_1 = -iP' \varepsilon_B, \]
\[ \delta_A \psi_2 = iP' \varepsilon_A, \quad \delta_B \psi_2 = \frac{d\phi}{dt} \varepsilon_B. \]
Define fields on lattice sites \( x = na, \ n = 0, \ldots, L - 1 \).

To eliminate fermion doubling in 1D use forward or backward derivative

\[
\nabla \phi(x) = \phi(x + a) - \phi(x), \quad \nabla^* \phi(x) = \phi(x) - \phi(x - a).
\]

SUSY variation \( \delta_A \) leads to

\[
\delta_A S_L = i \varepsilon_A \sum_x \psi_2 \left(-\nabla P' + P'' \nabla^* \phi\right),
\]

due to the absence of the Leibniz rule on the lattice, term is \( \mathcal{O}(a) \) and vanishes in the naive continuum limit, vanishes at finite \( a \) if \( g = 0 \).
Standard discretisation at $g = 0$

$m_{\text{phys}} L_{\text{phys}} = 10.0, g_{\text{phys}} = 0$

- Boson
- Fermion

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Loop formulation of SUSY models
Introduce interaction term $P''(\phi) \overline{\psi} \psi \propto g \overline{\psi} \psi \phi^2$:

radiative corrections spoil continuum limit...
Introduce interaction term $P''(\phi)\bar{\psi}\psi \propto g\bar{\psi}\psi\phi^2$:

⇒ can be tuned away by adding counter term $\frac{1}{2}P''(\phi) \propto g\phi^2$
Exact twisted lattice supersymmetry

- Find a combination of supersymmetries which can be transferred to the lattice.
- Recall symmetry breaking of the lattice action:

\[
\delta_A S_L = i \varepsilon_A \sum_x \psi_2 \left( -\nabla P' + P'' \nabla^* \phi \right)
\]

\[
= -i \delta_B \sum_x P' \nabla^* \phi.
\]

- Notice the similar term for \( \delta_B \),

\[
\delta_B S_L = i \delta_A \sum_x P' \nabla^* \phi,
\]

so the linear combination \( \delta = \delta_A + i \delta_B \) gives

\[
\delta S_L = -\delta \sum_x P' \nabla^* \phi.
\]
Correction term $P' \nabla^* \phi$ is a surface term vanishing in the limit $a \to 0$.

Corrected lattice action is invariant under the 'twisted' supersymmetry $\delta$:

$$S_L^{\text{exact}} = \sum_x \frac{1}{2} (\nabla^* \phi)^2 + \frac{1}{2} P'^2 + \overline{\psi} (\nabla^* + P'') \psi + P' \nabla^* \phi$$

Note that the bosonic action can also be written as

$$S_B^{\text{exact}} = \sum_x \frac{1}{2} (\nabla^* \phi + P')^2$$

which exposes the relation to a (local) Nicolai map
- variable transformation $\phi \to \mathcal{N} = \nabla^* \phi + P'(\phi)$,
- action becomes Gaussian,
- Jacobian cancels exactly the fermion determinant.
Now simulate this SUSY-exact (or $Q$-exact) action:
Spontaneous SUSY breaking (SSB) and the Witten index

- Witten index provides a necessary but not sufficient condition for SSB:

\[ W \equiv \lim_{\beta \to \infty} \text{Tr}(\exp(-\beta H)) = 0 \quad \Rightarrow \quad \begin{cases} = 0 & \text{SSB may occur} \\ \neq 0 & \text{no SSB} \end{cases} \]

- Index counts the difference between the number of bosonic and fermionic zero energy states:

\[ W \equiv \lim_{\beta \to \infty} \left[ \text{Tr}_B \exp(-\beta H) - \text{Tr}_F \exp(-\beta H) \right] = n_B - n_F \]

- Index is equivalent to partition function with periodic b.c.:

\[ W = \int_{-\infty}^{\infty} \mathcal{D}\phi \ det[\mathcal{D}(\phi)] \ e^{-S_B[\phi]} = Z_{\text{per}} \]

\[ \Rightarrow \text{Determinant (or Pfaffian) must be indefinite for SSB.} \]
Recall the Lagrangian for supersymmetric quantum mechanics

\[ \mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} P'(\phi)^2 + \overline{\psi} \left( \frac{\partial}{\partial t} + P''(\phi) \right) \psi, \]

The (regulated) fermion determinant can be calculated exactly:

\[ \text{det} \left[ \frac{\partial}{\partial t} + P''(\phi) \right] = \sinh \int_0^T \frac{P''(\phi)}{2} \, dt \quad \implies \quad Z_0 - Z_1 \]

If under some symmetry \( \phi \to \phi' \) of \( S_B(\phi) \) we have

\[ \int_0^T \frac{P''(\phi')}{2} \, dt = \begin{cases} + \int_0^T \frac{P''(\phi)}{2} \, dt & \text{no SSB} \quad \Rightarrow \quad Z_0 \neq Z_1 \\ - \int_0^T \frac{P''(\phi)}{2} \, dt & \text{SSB} \quad \Rightarrow \quad Z_0 = Z_1 \end{cases} \]
On the lattice we find with Wilson type fermions

\[ \det [\nabla^* + P''(\phi)] = \prod_t [1 + P''(\phi_t)] - 1. \]

For even potentials, e.g. \( P(\phi) = \frac{1}{2} m \phi^2 + \frac{1}{4} g \phi^4 \) we have

\[ \det [\nabla^* + P''] = \prod_t [1 + m + 3g\phi_t^2] - 1 \geq 0 \]

for \( m \geq 0, g \geq 0 \).

As a side remark, note that

\[ \lim_{a \to 0} \det [\nabla^* + P''] \sim \exp \int_0^T \frac{P''(\phi)}{2} dt \det [\partial_t + P''(\phi)] , \]

so this term needs 'fine tuning'.
For odd potentials, e.g. $P(\phi) = \frac{m^2}{2\lambda} \phi^2 + \frac{1}{3} \lambda \phi^3$ we have

$$\det [\nabla^* + P''] = \prod_t [1 + 2\lambda \phi_t] - 1$$

no longer positive...

$\Rightarrow$ sign problem!

Every supersymmetric model which allows SSB must have a sign problem:

- SUSY QM with odd potential,
- $\mathcal{N} = 16$ Yang-Mills quantum mechanics [Catterall, Wiseman '07],
- $\mathcal{N} = 1$ Wess-Zumino model in 2D [Catterall '03],
- $\mathcal{N} = (2, 2)$ Super-Yang-Mills in 2D [Giedt '03].
We propose a novel approach circumventing these problems [Wenger '08]:

- based on the exact hopping expansion of the fermion action,
- eliminates critical slowing down,
- allows simulations directly in the massless limit,

⇒ solves the fermion sign problem.

Applicable to the

- Gross-Neveu model in $d = 2$ dimensions,
- Schwinger model in the strong coupling limit in $d = 2$ and 3,
- SUSY QM,
- $\mathcal{N} = 1$ and 2 supersymmetric Wess-Zumino model.
Consider now the $\mathcal{N} = 1$ Wess-Zumino model in 2D:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} P'(\phi)^2 + \frac{1}{2} \bar{\psi} (\phi + P''(\phi)) \psi$$

with $\psi$ a Majorana field,
and $\phi$ real bosonic field,
superpotential, e.g. $P(\phi) = \frac{1}{2} m \phi^2 + \frac{1}{3} g \phi^3$.

Integrating out Majorana fermions yields indefinite Pfaffian.

For Majorana fermions use exact reformulation in terms of loops [old idea]:

$\Rightarrow$ sign of Pfaffian under perfect control

Can also be done for bosonic fields [Prokof'ev, Svistunov '01].
Exact hopping expansion for Majorana Wilson fermions

Using Wilson lattice discretisation for the fermionic part:

\[ \mathcal{L} = \frac{1}{2} \xi^T \mathcal{C} (\gamma_\mu \tilde{\partial}_\mu - \frac{1}{2} \partial^* \partial + P''(f)) \xi, \]

- \( \xi \) is a real, 2-component Grassmann field,
- \( \mathcal{C} = -\mathcal{C}^T \) is the charge conjugation matrix.

Using the nilpotency of Grassmann elements we expand the Boltzmann factor

\[
\int \mathcal{D}\xi \prod_x \left( 1 - \frac{1}{2} M(x) \xi^T(x) \mathcal{C} \xi(x) \right) \prod_{x,\mu} \left( 1 + \xi^T(x) \mathcal{C} \Gamma(\mu) \xi(x + \hat{\mu}) \right)
\]

where \( M(x) = 2 + P''(f) \) and \( \Gamma(\pm \mu) = \frac{1}{2} (1 \mp \gamma_\mu) \).
At each site, the fields $\xi^T C$ and $\xi$ must be exactly paired to give a contribution to the path integral:

$$\int \mathcal{D}\xi \prod_x (M(x)\xi^T(x)C\xi(x))^{m(x)} \prod_{x,\mu} (\xi^T(x)C\Gamma(\mu)\xi(x + \hat{\mu}))^{b_{\mu}(x)}$$

with occupation numbers

- $m(x) = 0, 1$ for monomers,
- $b_{\mu}(x) = 0, 1$ for fermion bonds (or dimers),

satisfying the constraint

$$m(x) + \frac{1}{2} \sum_\mu b_{\mu}(x) = 1.$$

Only closed, non-intersecting paths survive the integration.
Analogous treatment for the bosonic field [Prokof’ev, Svistunov ’01]:

\[ (\partial_\mu \phi)^2 \rightarrow \phi_{x+\hat{\mu}} \phi_x, \]

expand hopping term to all orders:

\[
\int \mathcal{D}\phi \prod x, \mu \sum n_\mu(x) \frac{1}{n_\mu(x)!} (\phi_x \phi_{x+\hat{\mu}})^{n_\mu(x)} \exp (-V(\phi_x)) M(\phi_x)^{m(x)}
\]

with bosonic bond occupation numbers \( n_\mu(x) = 0, 1, 2, \ldots \)

Integrating out \( \phi(x) \) yields bosonic site weights

\[
Q(N) = \int d\phi \, \phi^N \exp (-V(\phi))
\]

where \( N \) includes powers from \( M(\phi) \).
Loop formulation

- Loop representation in terms of fermionic monomers and dimers and bosonic bonds.

- Partition function summing over all non-oriented, self-avoiding fermion loops

\[ Z_L = \sum_{\{\ell\} \in L} \sum_{\{n_\mu\}} |\omega[\ell, n_\mu(x), m(x)]|, \quad L \in \mathcal{L}_{00} \cup \mathcal{L}_{10} \cup \mathcal{L}_{01} \cup \mathcal{L}_{11} \]

represents a system with unspecified fermionic b.c. [Wolff '07].

- Simulate bosons with worm algorithm [Prokof’ev, Svistunov ’01].

- Simulate fermions by enlarging the configuration space by one open fermionic string [Wenger ’08].
The open fermionic string corresponds to the insertion of a Majorana fermion pair \( \{ \xi^T(x) C, \xi(y) \} \) at position \( x \) and \( y \):

- It samples the relative weights between \( Z_{\mathcal{L}00}, Z_{\mathcal{L}10}, Z_{\mathcal{L}01}, Z_{\mathcal{L}11} \).
- Reconstruct the Witten index a posteriori

\[
W \equiv Z^{pp} = Z_{\mathcal{L}00} - Z_{\mathcal{L}10} - Z_{\mathcal{L}01} - Z_{\mathcal{L}11},
\]

or a system at finite temperature

\[
Z^{pa} = Z_{\mathcal{L}00} - Z_{\mathcal{L}10} + Z_{\mathcal{L}01} + Z_{\mathcal{L}11}.
\]
Especially simple for supersymmetric QM:

\[ Z^p = Z_{\mathcal{L}0} - Z_{\mathcal{L}1} \quad \Rightarrow \text{Witten index} \]
\[ Z^a = Z_{\mathcal{L}0} + Z_{\mathcal{L}1} \quad \Rightarrow \text{finite temperature} \]
Especially simple for supersymmetric QM:

\[ Z^p = Z_{L_0} - Z_{L_1} \quad \Rightarrow \text{Witten index} \]
\[ Z^a = Z_{L_0} + Z_{L_1} \quad \Rightarrow \text{finite temperature} \]
Standard discretisation at $g \neq 0$ with counterterm:

![Graph showing unbroken supersymmetry, g/m^2=1.0, improved action]
Standard discretisation at $g \neq 0$ with counterterm:

![Graph showing unbroken supersymmetry, $g/m^2 = 1.0$, improved action](image)
\( Q \)-exact discretisation at \( g \neq 0 \):

![Graph showing unbroken supersymmetry, \( g/m^2 = 1.0 \), \( Q \)-exact action](image-url)
**Q-exact discretisation at** \( g \neq 0 \):
High precision consistency check:

continuum limit, g/m^2=1.0, Q-exact action

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Loop formulation of SUSY models
Broken supersymmetry:
standard, perturbatively improved action, $\lambda^2/m^3 = 1.0$: 

\[ \frac{Z_p}{Z_a} \text{ vs. } mL \]
Broken supersymmetry:
standard, perturbatively improved action, $\lambda^2/m^3 = 1.0$:
Exact calculations can be done via the transfer matrix.

In the loop formulation:
- states are characterised by the bond occupation numbers:
  \[ |b(x), n(x)\rangle \]

- transfer matrix on the dual lattice:
  \[ T |b(x-1), n(x-1)\rangle, \langle b(x), n(x)| \]

- Transfermatrix block diagonalises into bosonic and fermionic parts, \( T^0 \) and \( T^1 \).

Specifically,
\[ T^1 \langle 1, n(x) |, \langle 1, n(x)| = \frac{Q(n(x-1) + n(x))}{\sqrt{n(x-1)!n(x)!}} \]
and \( T^0 \) analogously.
Partition functions on a $L_t$ lattice are given by $Z_i = (T^i)^{L_t}$.

Witten index

$$W \equiv Z_p = Z_0 - Z_1 = (T^1)^{L_t} - (T^0)^{L_t}$$
Mass gaps can be calculated from ratios of eigenvalues of $T^0$ and $T^1$. 

Unbroken SUSY, action with counterterm; $mL=30.0$, $g/m^2 = 1.0$; cutoff $\lambda_1 = 800$
Mass gaps can be calculated from ratios of eigenvalues of $T^0$ and $T^1$. 

Unbroken SUSY, Q-exact action; $m_L=30.0$, $g/m^2 = 1.0$; cutoff$_{11} = 120$; cutoff$_{13} = 30$
Mass gaps can be calculated from ratios of eigenvalues of $T^0$ and $T^1$. 

![Graph showing mass gaps for different states and parameters](image-url)
Construction of $Q$-exact discretisations on the lattice.

The fermionic sign problem and its relevance to the Witten index.

Representation of $\mathcal{N} = 2$ SUSY QM and $\mathcal{N} = 1$ Wess-Zumino model in terms of interacting bosonic and fermionic loops.

Use of topological boundary conditions for the solution of the sign problem.

Results for the Witten index and the spectrum in SUSY QM for both the broken and unbroken case.

Goldstino mode is clearly exposed.

$\mathcal{N} = 1$ $2d$ Wess-Zumino model with SSB under way.