The leading hadronic vacuum polarization on the lattice

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Motivations: The muon $g - 2$, three $\sigma$ between experiment and theory.

Challenges of the lattice approach.

Field theoretical description of connected and disconnected quark diagrams. $PqQCD$ and $Pq\chi PT$.

Improved momentum resolution of the connected contribution by using twisted boundary conditions.

Numerical results (Wilson clover, $N_f = 2$).

Conclusions
Motivations

The anomalous magnetic moment of the muon $a_\mu = \frac{g_\mu-2}{2}$ is defined through the $q^2 \to 0$ limit of the vertex

\[ \gamma(q) \xrightarrow{\quad} \mu(p_2) \]
\[ \mu(p_1) \]

- It is one of the most precisely measured quantity; 0.5 ppm at Brookhaven. Theory is at the same level.
- It mediates helicity flips transition and therefore has a sensitivity $O\left(\frac{m_\mu^2}{M^2}\right)$ to particles of mass $M$ beyond the SM. Best sensitivity to nearby NP.
- 3 sigmas discrepancy between exp and theo [Jegerlehner and Nyffeler, 2009] :

$$a_\mu^{\text{exp}} = 1.16592080(63) \times 10^{-3}$$

$$a_\mu^{\text{the}} = 1.16591790(65) \times 10^{-3}$$
The theory number is obtained by estimating the hadronic contribution to the photon propagator from the hadronic $e^+e^-$ annihilation cross-section. The latter comes from:

- experimental $e^+e^-$ data.
- Hadronic $\tau$ decays. The hadronic matrix elements are related to those in $e^+e^-$ annihilation by isospin (broken symmetry). This would give agreement with the experimental number.

⇒ need for a purely theoretical number.
The Euclidean hadronic vacuum polarisation tensor is defined as

\[ \Pi^{(N_f)}_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle J^{(N_f)}_\mu(x) J^{(N_f)}_\nu(0) \rangle \]

Euclidean invariance and current conservation imply

\[ \Pi^{(N_f)}_{\mu\nu}(q) = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi^{(N_f)}(q^2) \]

The relation between \( \Pi^{(N_f)}_{\mu\nu}(q^2) \) and \( a_{\mu}^{HLO} \) is [T. Blum, 2002]

\[ a_{\mu}^{HLO} = \left( \frac{\alpha}{\pi} \right)^2 \int_0^\infty dq^2 f(q^2) \hat{\Pi}(q^2) \]

with

\[ f(q^2) = \frac{m_\mu^2 q^2 Z^3 (1 - q^2 Z)}{1 + m_\mu^2 q^2 Z^2} , \quad Z = -\frac{q^2 - \sqrt{q^4 + 4m_\mu^2 q^2}}{2m_\mu^2 q^2} \]

and \( \hat{\Pi}(q^2) = 4\pi^2 [\Pi(q^2) - \Pi(0)] \).
Main challenges

- The integral is dominated by the \( q^2 \approx m^2_\mu \) region

\[
 f(q^2) \cdot q^2
\]

The lowest Fourier mode \( 2\pi/L \) is typically much larger than \( m_\mu \), and the VP has to be extrapolated to the relevant region [FSE].

- Correlator of flavor-diagonal currents \( \rightarrow \) disconnected diagrams.

- The vector resonances \((\rho, \omega, \phi)\) give a large contribution. Those have to be properly included in the lattice computation.

- Previous lattice efforts
  \[
  a^{HLO}_\mu = 748(21) \times 10^{-10} \quad [\text{Aubin and Blum, 2007}] \quad N_f = 2 + 1 (\sqrt{\text{staggered}}).
  \]
  \[
  a^{HLO}_\mu = 446(23) \times 10^{-10} \quad [\text{QCDSF, 2003}] \quad \text{quenched}.
  \]
  computation in progress by ETMC \[\text{[Renner, 2009,2010]} \quad N_f = 2.\]
Let's consider the 2 flavor case and the contribution form the $u$ quark only

$$C_{uu,QCD}^{\mu\nu}(q) = \frac{4}{9} \int d^4 x e^{i q x} \langle J^{uu}_\mu(0) J^{uu}_\nu(x) \rangle$$

After Wick contractions the correlator $\langle J^{uu}_\mu(0) J^{uu}_\nu(x) \rangle$ is written in terms of connected and disconnected quark diagrams

We add a quark $r$ degenerate with $u$ and quenched it away (valence quark)

$$L_{QCD}^{(2)} \rightarrow L_{QCD}^{(2)} + \bar{r}(\not{D} + m_u) r + \bar{r}^\dagger (\not{D} + m_u) \bar{r} = L_{PqQCD}$$

$\bar{r}$ is a commuting spin 1/2 filed, a ghost. $PqQCD$ is an unphysical theory. However the partition function is the same as in QCD and in this theory the (dis)connected diagrams above can be written as correlation functions.
\[ \mathcal{C}_{\mu \nu, QCD}(q) = \frac{4}{9} \int d^4 x e^{i q x} \langle J^{ur}_{\mu}(0) J^{ru}_{\nu}(x) \rangle_{PqQCD} + \frac{4}{9} \int d^4 x e^{i q x} \langle J^{uu}_{\mu}(0) J^{rr}_{\nu}(x) \rangle_{PqQCD} \]

= \mathcal{C}^{\text{conn}}_{\mu \nu, PqQCD}(q) + \mathcal{C}^{\text{disc}}_{\mu \nu, PqQCD}(q)

---

- A low energy effective theory (Pq\(\chi\)PT) exists, which should describe correlation functions in PqQCD. This theory is based on an extended (graded) flavor-symmetry group.

- The connected part is not flavor-diagonal anymore in PqQCD. Twisting \cite{Sachrajda and Villadoro, 2005} can be used there to improve the momentum resolution.
For the case of 2 degenerate flavors there’s no need for the quark $r$, making the charge factors explicit and using iso-spin invariance

$$\Pi^{(2)}_{\mu\nu}(q^2) = \frac{5}{9} \int d^4x e^{iqx} \langle J^u_d \mu(0) J^d_u \nu(x) \rangle + \frac{1}{9} \int d^4x e^{iqx} \langle J^{uu} \mu(0) J^{dd} \nu(x) \rangle$$

$$= \Pi^{(2),conn}(q^2) + \Pi^{(2),disc}(q^2)$$

For a computation in SU(2) $\chi$PT it is convenient to rewrite the currents using the generators

$$J^{uu}_\mu = \frac{1}{2} \overline{\psi} (\sqrt{2}\sigma_0 + \sigma_3) \gamma_\mu \psi, \quad J^{dd}_\mu = \frac{1}{2} \overline{\psi} (\sqrt{2}\sigma_0 - \sigma_3) \gamma_\mu \psi,$$

$$J^{ud}_\mu = \frac{1}{2} \overline{\psi} (\sigma_1 + i\sigma_2) \gamma_\mu \psi, \quad J^{du}_\mu = \frac{1}{2} \overline{\psi} (\sigma_1 - i\sigma_2) \gamma_\mu \psi,$$

where $\psi^T = (u, d)$, and $\sigma_0 = \sqrt{1/2} I_{2\times2}$. 
The vector current is described by external fields $v_\mu = \frac{\lambda_a}{2} \cdot v_a^\mu$ coupled through the covariant derivative

$$ D_\mu U = \partial_\mu U + iv_\mu U - iUv_\mu $$

Note: Green functions of singlet current do not receive contributions at LO. The expansion produces the vertices

$$ L_{\text{int}} = -f_{abc}\phi_c\partial_\mu \phi_a v^b_\mu + \frac{1}{2} f_{abg} f_{cdg} \phi_b \phi_c v^a_\mu v^d_\mu $$
These give the 1-loop diagrams contributing to the vector-vector correlator

they both diverge. The divergences are removed by tree level insertions of the $O(p^4)$ terms prop. to $l_5$ and $h_1$, which get scale dependent. In addition a coupling $h_s$ is introduced to parameterise the dynamics of flavor-singlet contributions [Kaiser, 2001]. At this order it is scale-independent.

$$\Pi^{(2)}(q^2) = - \left( \Lambda^{(2)}(M_\pi) + \frac{2}{9} h_s + i4\bar{B}_{21}(q^2, M_\pi^2) \right),$$

$$\Pi^{(2)}_{\text{conn}}(q^2) = - \frac{10}{9} \Lambda^{(2)}(M_\pi) + i4\bar{B}_{21}(q^2, M_\pi^2),$$

$$\Pi^{(2)}_{\text{disc}}(q^2) = \frac{1}{9} \left( \Lambda^{(2)}(M_\pi) - 2h_s + i4\bar{B}_{21}(q^2, M_\pi^2) \right),$$

with $\Lambda^{(2)}(M_\pi) = 2(l_5(M_\pi) + h_1(M_\pi))$
This leads to the simple result

\[ \frac{\hat{\Pi}_{\text{disc}}^{(2)}(q^2)}{\hat{\Pi}_{\text{conn}}^{(2)}(q^2)} = - \frac{1}{10} \]

at NLO in mesonic $\chi$PT.

For the 2+1 or 2+ quenched strange cases one has to introduce one and two valence quarks at least (there’s no degenerate partner of the strange quark) and therefore consider $SU(4|1)$ and $SU(4|2) \, Pq\chi$PT.
The flavor-symmetry group of PqQCD is $SU(N_s + N_v | N_v)_L \times SU(N_s + N_v | N_v)_R$. An element $U$ of $SU(N_s + N_v | N_v)$ can be written as

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
$$

$A$ is $(N_s + N_v) \times (N_s + N_v)$ commuting numbers, $D$ is $N_v \times N_v$ commuting numbers, $B$ and $C$ are made of anti-commuting numbers. One introduces

$$
\text{Str}(U) = \text{Tr}(A) - \text{Tr}(D) \Rightarrow \text{Sdet}(U) = e^{\text{Str \, log}(U)} = \frac{\det(A - BD^{-1}C)}{\det(D)}
$$

s.t. the Str has the cyclic property, if in addition $(ab)^* = b^* a^*$ then the $U$ matrices are unitary and with $\text{Sdet}(U) = 1$. The Lie algebra is defined through commutation and anti-commutation relations

$$
t_a t_b - (-1)^{\eta_a \eta_b} t_b t_a = i \sum_c C_{ab}^c t_c
$$

$\eta_a = 1$ if $t_a$ mixes valence or sea with ghosts.
For our purposes $Pq\chi PT$ is obtained by $Tr \rightarrow Str$ in the $\chi PT$ Lagrangian

(a) \[ N_f = 2 \] with a quenched strange quark,

(b) \[ N_f = 2 + 1 \]. From top to bottom: $M_\pi = M_K$, then fixed $M_K = 495\text{MeV}$ and $M_\pi = 400, 300, 200, 139 \text{ MeV}$. The bottom most-line at -1/10 is the result for $N_f = 2$. 

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Partially Twisted boundary conditions

Valence quarks periodic in space up to a phase $\rightarrow$ spatial momentum to hadrons up to exponentially small finite volume effects [Sachrajda and Villadoro, 2005]

$$E_\pi = \sqrt{m_\pi^2 + (\vec{n}^{2\pi}_L)^2}$$

$$E_\pi = \sqrt{m_\pi^2 + (\vec{n}^{2\pi}_L + \frac{\vec{\partial}_u - \vec{\partial}_d}{L})^2}$$

[Flynn, Jüttner and Sachrajda, 2006]
Numerical results

Ensembles of gauge configurations for $N_f = 2$ NP $O(a)$ improved Wilson fermions generated within the CLS community effort by using the DD-HMC [Lüscher, 2005] algorithm.

<table>
<thead>
<tr>
<th>run</th>
<th>$\kappa_{\text{sea}}$</th>
<th>$\beta$</th>
<th>$L$ [fm]</th>
<th>$N_{\text{cfg}}$</th>
<th>$m_\pi$ [MeV]</th>
</tr>
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<tbody>
<tr>
<td>$A2$</td>
<td>0.13590</td>
<td>5.2</td>
<td>2.6</td>
<td>125</td>
<td>$\sim 580$</td>
</tr>
<tr>
<td>$A3$</td>
<td>0.13605</td>
<td>5.2</td>
<td>2.6</td>
<td>133</td>
<td>$\sim 460$</td>
</tr>
<tr>
<td>$A4$</td>
<td>0.13610</td>
<td>5.2</td>
<td>2.6</td>
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<tr>
<td>$A5$</td>
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<td>2.6</td>
<td>108</td>
<td>$\sim 300$</td>
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<tr>
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<td>5.3</td>
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<td>158</td>
<td>696.5(0.9)</td>
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<tr>
<td>$E3$</td>
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<td>2.2</td>
<td>156</td>
<td>593.4(1.1)</td>
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<tr>
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<td>5.3</td>
<td>2.2</td>
<td>162</td>
<td>554.2(1.1)</td>
</tr>
<tr>
<td>$E5$</td>
<td>0.13625</td>
<td>5.3</td>
<td>2.2</td>
<td>168</td>
<td>414.4(1.4)</td>
</tr>
<tr>
<td>$F6$</td>
<td>0.13635</td>
<td>5.3</td>
<td>3.3</td>
<td>200</td>
<td>297.9(0.9)</td>
</tr>
<tr>
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<td>3.3</td>
<td>205</td>
<td>$\sim 250$</td>
</tr>
<tr>
<td>$N5$</td>
<td>0.13660</td>
<td>5.5</td>
<td>2.5</td>
<td>161</td>
<td>$\sim 410$</td>
</tr>
</tbody>
</table>

$0.05 \leq a \leq 0.08$ fm.
We twist the $d$ quark in $C_{\mu\nu}^{\text{conn}}(q^2) \propto \int d^4 x e^{i q x} \langle J_\mu^u(0) J_\nu^d(x) \rangle$. We use the conserved 1-point split vector current.

- Dispersion relation fits from a model for $\sigma_{e^+e^- \rightarrow \text{hadrons}}(s)$:
  $$\Pi(q^2) = B \log(a^2 q^2 + a^2 s_0) - \frac{A}{(q^2 + m_V^2)} + K$$ [QCDSF, '03]
  - Polynomial
  - Padé
$4\alpha^2 f(q^2)\hat{\Pi}(q^2)/10^{-7}$

Convolution integral for $a_{\mu}^{\text{had}}$

kernel of integral
- twisted bc F7
- periodic bc F7

$q^2 \ [\text{GeV}^2]$
Fits to $\beta = 5.3$ data. Chiral fit: $a_\mu(m_\pi^2) = a + b m_\pi^2(1 + c \log(m_\pi^2))$
Reduction of systematic errors from twisting ($\beta = 5.3$)

Simulation results for $N_f = 2$ using partially twisted bcs

Simulation results for $N_f = 2$ using Fourier modes only

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Cutoff effects

We expect $O(a)$ cutoff effects as the VP receives off-shell contributions.

[The value at $\beta = 5.3$ is interpolated]
Adding a quenched strange quark ($\beta = 5.3$)
Conclusions and outlook

- The anomalous magnetic moment of the muon is a phenomenologically very interesting quantity. Lattice determinations of the leading hadronic vacuum polarization can play an important role in producing a solid theoretical number.
- The appearance of disconnected diagrams and the poor momentum resolution represent the main difficulties in a lattice computation.
- $Pq\chi$PT can be used to estimate the size of the disconnected contribution. Interesting way to look also at other quantities where disconnected diagrams appear.
- We use the $\chi$PT result for VP only to estimate the size of the disconnected piece, we will include vectors to see how stable the result is and we will compute disconnected contribution numerically.
- However this estimate tells us we should control the disconnected contribution to a 10-20% level to have a overall result which is as accurate as the present 'theoretical' one. That is challenging.
• Numerically these disconnected contributions seem to be even smaller than predicted by $Pq\chi PT$ at NLO. That is good.

![Graph showing disconnected and connected contributions](image_url)

[D. Renner, LAT10]

• The target precision appear to be reachable for the connected contribution also thanks to the use of twisted boundary conditions, which help reducing the systematic errors from the fits to the momentum dependence of the VP.

• We want to be able to control these effects to the desired level with 2 dynamical fermions, before considering $N_f > 2$. We see that the inclusion of a quenched strange quark is an effect $O(10\%)$. 

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