Closed $SU(N)$ Flux Tubes as Bosonic Strings

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* With Dr. Barak Bringoltz and Dr. Mike Teper
based on:
and mostly on unpublished results
Overview

1. Introduction.

2. Theoretical Expectations.
   • Nambu Goto.
   • Effective String Theory.

3. Lattice Calculation.

4. $D = 2 + 1$.
   • Quantum Numbers/Operators.
   • Results.

5. $D = 3 + 1$.
   • Quantum Numbers/Operators.
   • Results.

6. Conclusions.

7. Future Work.
1. Introduction: General

General question:
→ What effective string theory describes flux tubes in $SU(N)$ gauge theories?

Two cases:
→ Open string
→ Closed string (torelon)

During the last decade:
→ $3D, 4D$ with $Z_2, Z_4, SU(N \leq 8)$ (Caselle and collaborators, Gliozzi and collaborators, Kuti and collaborators, Lüscher & Weisz, Majumdar and collaborators, Teper and collaborators)

Recently in $D = 2 + 1$: Nambu-Goto works well for:

First attempt to extract the $D = 3 + 1$ torelon spectrum in $SU(3)$:
K. J. Juge, J. Kuti, F. Maresca, C. Morningstar and M. Peardon in [arXiv:hep-lat/0309180]
1. Introduction: General

**General question:**

→ What effective string theory describes flux tubes in $SU(N)$ gauge theories?

**Two cases:**

→ Open string

→ **Closed string** (torelon) ← in $D = 2 + 1$ & $D = 3 + 1$

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1. Introduction: Open-Closed (torelons) flux tubes

Open flux tube

$$\Phi(l, t) = \psi^\dagger(0, t) U(0, l; t) \psi(l, t)$$

Closed flux tube

$$\Phi(l, t) = \text{Tr} U(l; t)$$
2. Theoretical Expectations: Nambu-Goto String

→ **Spectrum:**

\[ E_{N_L, N_R, q, w}^2 = (l w)^2 + 8 \pi \sigma \left( \frac{N_L + N_R}{2} - \frac{D-2}{24} \right) + \left( \frac{2\pi q}{l} \right)^2. \]

→ **Described by:**

1. The winding number \( w \) \((w=1)\),
2. The winding momentum \( p_\parallel = 2\pi q/l \) with \( q = 0, \pm 1, \pm 2, \ldots \).
3. The transverse momentum \( p_\perp \) \((p_\perp = 0)\),
4. \( N_L \) and \( N_R \) connected through the relation: \( N_L - N_R = q w \).

\[ N_L = \sum_{k>0} \sum_{n_L(k)>0} n_L(k) k \quad \text{and} \quad N_R = \sum_{k'>0} \sum_{n_R(k')>0} n_R(k') k' \]

→ **Construction of states:**

\[ (\alpha_{-k_1}^{i_1})^{n_L(k_1)} \ldots (\alpha_{-k_{m_L}}^{i_{m_L}})^{n_L(k_{m_L})} (\bar{\alpha}_{-k'_1}^{i'_1})^{n_R(k'_1)} \ldots (\bar{\alpha}_{-k'_{m_R}}^{i'_{m_R}})^{n_R(k'_{m_R})} |0\rangle \]

\((i = 1, \ldots, D - 2)\)

Example: \( \alpha_{-2}\alpha_{-1}\bar{\alpha}_{-1} |0\rangle \)
2. Theoretical Expectations: Nambu-Goto String

→ **Spectrum:**

\[ E_{N_L,N_R,q,w}^2 = (\sigma lw)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{D-2}{24} \right) + \left( \frac{2\pi q}{l} \right)^2. \]

→ **Described by:**

1. The winding number \( w \) (\( w=1 \)),
2. The winding momentum \( p_{\parallel} = 2\pi q/l \) with \( q = 0, \pm 1, \pm 2, \ldots \),
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→ **Construction of states:**

\[(\alpha_{-k_1}^{i_1})\cdots(\alpha_{-k_mL}^{i_mL})n_R(k_mL)(\tilde{\alpha}_{-k'_1}^{i'_1})\cdots(\tilde{\alpha}_{-k'_mR}^{i'_mR})n_R(k'_mR) | 0\rangle\]

\((i = 1, \ldots, D - 2)\)

Example: \( \alpha_{-2}\alpha_{-1}\tilde{\alpha}_{-1} | 0\rangle \rightarrow N_L = 3, N_R = 1, q = 2 \ (w = 1) \)
2. Theoretical Expectations: Effective String Approaches


\[
E_n = \sigma l + \frac{4\pi}{l} \left( n - \frac{D-2}{24} \right) + \mathcal{O}\left(1/l^2 \right).
\]

→ **Lüscher & Weisz at** \(D = 2 + 1\) [JHEP 0407 (2004) 014] (N.L.O):

\[
E_n = \sigma l + \frac{4\pi}{l} \left( n - \frac{1}{24} \right) - \frac{8\pi^2}{\sigma l^3} \left( n - \frac{1}{24} \right)^2 + \mathcal{O}\left(1/l^4\right)
\]

→ **Drummond and Dass&Matlock** [arXiv:hep-th/0411017/0606265] (N.L.O):

\[
E_n = \sigma l + \frac{4\pi}{l} \left( n - \frac{D-2}{24} \right) - \frac{8\pi^2}{\sigma l^3} \left( n - \frac{D-2}{24} \right)^2 + \mathcal{O}\left(1/l^4\right)
\]

→ **Aharony&Karzbrun at any** \(D\) [arXiv:0903.1927] (N.N.L.O):

\[
E_n = \sigma l + \frac{4\pi}{l} \left( n - \frac{D-2}{24} \right) - \frac{8\pi^2}{\sigma l^3} \left( n - \frac{D-2}{24} \right)^2 + \frac{32\pi^3}{\sigma^2 l^5} \left( n - \frac{D-2}{24} \right)^3 + \mathcal{O}\left(1/l^7\right)
\]

→ **Recently Dass&collaborators at any** \(D\) [arXiv:0911.3236] (All.O):

\[
E_n = \sigma l + \frac{4\pi}{l} \left( n - \frac{D-2}{24} \right) - \frac{8\pi^2}{\sigma l^3} \left( n - \frac{D-2}{24} \right)^2 + \cdots + \mathcal{O}\left(1/l^\infty\right) = E_{n\,N.G}
\]
3. Lattice Calculation: General Strategy

- Use the standard Wilson action
- Cabbibo-Marinary & Kennedy-Pendelton heat bath
- Construct a basis of operators $\Phi_i : i = 1, 2, \ldots$ described by the right quantum numbers
- Calculate the correlation function (Matrix) $C_{ij}(t) = \langle \Phi_i^\dagger(t)\Phi_j(0) \rangle$
- Apply the variational technique
- Extract the correlators for a number of excited states $C_n(t) = \langle \Psi_n^\dagger(t)\Psi_n(0) \rangle$
- Energies can be calculated using the correlation functions of specific operators:

\[
C(t) = \langle \Psi^\dagger(t)\Psi(0) \rangle = \langle \Psi^\dagger(0)e^{-Ht}\Psi(0) \rangle \\
= |\langle 0|\Psi(0)|vac \rangle|^2 e^{-E_0t} + \sum_{n=1} |\langle n|\Psi(0)|vac \rangle|^2 e^{-E_nt} \xrightarrow{t \to \infty} |\langle 0|\Psi(0)|vac \rangle|^2 e^{-E_0t}
\]

- By fitting the results, we extract the mass (energy) for each state.
3. Lattice Calculation: Correlation Function

Pictorialisation of the Correlation Function
4. $D = 2 + 1$
4. D=2+1: Lattice Calculation

→ **We define our theory on a 3D Euclidean lattice with volume:**

\[ L_\parallel \times L_{\perp 1} \times L_T. \]

→ **We use the standard Wilson Action:**

\[ S_L = \beta \sum_p \left\{ 1 - \frac{1}{N} \text{ReTr} U_p \right\} \quad \text{with} \quad \beta = \frac{2N}{ag^2} \]

→ **Monte Carlo simulations:**

- \( SU(3) \) at \( \beta = 21, \ a\sqrt{\sigma} \simeq 0.174 \)
- \( SU(3) \) at \( \beta = 40, \ a\sqrt{\sigma} \simeq 0.087 \)
- \( SU(4) \) at \( \beta = 50, \ a\sqrt{\sigma} \simeq 0.131 \)
- \( SU(5) \) at \( \beta = 80, \ a\sqrt{\sigma} \simeq 0.130 \)
- \( SU(6) \) at \( \beta = 80, \ a\sqrt{\sigma} \simeq 0.172 \)
- \( SU(6) \) at \( \beta = 171, \ a\sqrt{\sigma} \simeq 0.086 \)
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- **\( SU(6) \) at \( \beta = 171 \), \( a \sqrt{\sigma} \simeq 0.086 \)** Today:
  * \( q = 0 \)
  * \( q = 1 \)
  * \( q = 2 \)
4. D=2+1: Symmetries

→ Flux tube:
  - Lattice Symmetry of Rotations in $D = 2 + 1$ → Reflections on transverse line
  - States are described by the quantum number of Parity $P = \pm$
  - Need to construct operators transforming irreducibly under $P$

→ Bosonic String:
  - String states can be characterized as irreducible representations of $SO(D - 2)$.
  - In $D = 2 + 1$ this becomes the transverse parity ($P = \pm$).
  - Under Parity: $\alpha_{-k} \xrightarrow{P} - \alpha_{-k}$ and $\bar{\alpha}_{-k} \xrightarrow{P} - \bar{\alpha}_{-k}$
  - Even number of phonons $\rightarrow P = + \ (\alpha_{-2}\bar{\alpha}_{-1} | 0\rangle)$
  - Odd number of phonons $\rightarrow P = - \ (\alpha_{-1} | 0\rangle)$
4. D=2+1: Symmetries: Bosonic String

The seven lowest \((q = 0, 1, 2)\) NG energy levels for the \(w = 1\) closed string

\[
\begin{array}{c|c|c|c|c|c}
\text{level} & N_L & N_R & q & P = + & P = - \\
\hline
0 & 0 & 0 & 0 & |0\rangle & \alpha_{-1}|0\rangle \\
1 & 1 & 1 & 0 & \alpha_{-1}\alpha_{-1}|0\rangle & \alpha_{-1}|0\rangle \\
2 & 1 & 1 & 0 & \alpha_{-1}\alpha_{-1}|0\rangle & \alpha_{-2}|0\rangle \\
3 & 2 & 0 & 0 & \alpha_{-2}\alpha_{-1}|0\rangle & \alpha_{-1}\alpha_{-1}\alpha_{-1}|0\rangle \\
4 & 2 & 1 & 1 & \alpha_{-2}\alpha_{-1}|0\rangle & \alpha_{-1}\alpha_{-1}\alpha_{-1}|0\rangle \\
5 & 2 & 2 & 0 & \alpha_{-2}\alpha_{-2}|0\rangle, \alpha_{-1}\alpha_{-1}\alpha_{-1}\alpha_{-1}|0\rangle & \alpha_{-2}\alpha_{-1}|0\rangle, \alpha_{-1}\alpha_{-1}\alpha_{-1}|0\rangle \\
6 & 3 & 1 & 2 & \alpha_{-3}\alpha_{-1}|0\rangle, \alpha_{-1}\alpha_{-1}\alpha_{-1}\alpha_{-1}|0\rangle & \alpha_{-2}\alpha_{-1}\alpha_{-1}|0\rangle \\
\end{array}
\]
4. D=2+1: Quantum Numbers: Parity

Parity reflection plane

Example:

We can project onto $P = \pm$ using operators such as:

$$\Phi(q_\parallel, P) = \frac{1}{L_\parallel L_\perp} \sum_{x_\parallel, x_\perp} \{ \text{Tr} \{ \quad \} \pm \text{Tr} \{ \quad \} \} \ e^{i2\pi q_\parallel x_\parallel / L_\parallel}$$
4. $D=2+1$: Transverse deformations

$\rightarrow \sim 200$ operators

Transverse deformations in 2 spatial dimensions:
4. D=2+1 Results: $N_L = N_R = 0, q = 0, P = +$ (Trivial)

- $N_L = N_R = 0$ corresponds to the Ground State (No movers)
- String theoretically: $|0\rangle$
- Even number of phonons
- Expected to appear as $\{P = +, q = 0\}$ ground state
- Apply variational technique to the correlation matrix $P = +, q = 0$.
- Extraction of the ground state.
  - Fit data with Nambu-Goto or L.O/N.L.O/N.N.L.O $\rightarrow a^2\sigma_f$ (string tension)
  - This provides parameter free predictions for the energies in N.G:

$$E_{N_L,N_R,q,w}^2 = (\sigma_f L w)^2 + 8\pi\sigma_f \left(\frac{N_L + N_R}{2} - \frac{1}{24}\right) + \left(\frac{2\pi q}{l}\right)^2.$$

In lattice units:

$$a^2E_{N_L,N_R,q,w}^2 = (a^2\sigma_f L w)^2 + 8\pi a^2\sigma_f \left(\frac{N_L + N_R}{2} - \frac{1}{24}\right) + \left(\frac{2\pi q}{L}\right)^2.$$
4. D=2+1 Results: $N_L = N_R = 0, \ q = 0, \ P = +$

\[ E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{2\pi} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
4. $D=2+1$ Results: $N_L = N_R = 0$, $q = 0$, $P = +$
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4. D=2+1 Results: $N_L = N_R = 0$, $q = 0$, $P = +$

$$E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left( \frac{2\pi q}{l} \right)^2$$
4. D=2+1 Results: $N_L = N_R = 1$, $q = 0$, $P = +$

- $N_L = N_R = 1$ corresponds to the $q = 0$ first excited state.
- String theoretically: $\alpha_{-1} \bar{\alpha}_{-1} | 0\rangle$
- Even number of phonons
- Expected to appear as $\{P = +, q = 0\}$ first excited state.
- Apply variational technique to the correlation matrix $\{P = +, q = 0\}$.
- Extraction of the first excited state.
4. D=2+1 Results: \( N_L = N_R = 1, \; q = 0, \; P = + \)

\[
E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left( \frac{2\pi q}{l} \right)^2
\]

\[
\frac{E}{\sqrt{\sigma f}}
\]

Graph showing the relationship between \( E \) and \( l\sqrt{\sigma f} \) for different values of \( N_L = N_R \).
4. D=2+1 Results: \( N_L = N_R = 1, \ q = 0, \ P = + \)
4. D=2+1 Results: $N_L = N_R = 1$, $q = 0$, $P = +$

\[ E = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
4. D=2+1 Results: $N_L = N_R = 1$, $q = 0$, $P = +$
4. D=2+1 Results: $N_L = N_R = 1$, $q = 0$, $P = +$

$$E^2 = (\sigma l)^2 + 8\pi\sigma \left(\frac{N_L + N_R}{2} - \frac{1}{24}\right) + \left(\frac{2\pi q}{l}\right)^2$$
4. D=2+1 Results: $N_L = N_R = 2$, $q = 0$, $P = \pm$

- $N_L = N_R = 2$ corresponds to the $q = 0$ second excited state.

- String theoretically, four states can be constructed:
  - $\alpha_{-2} \bar{\alpha}_{-2} \vert 0 \rangle$
  - $\alpha_{-1} \alpha_{-1} \bar{\alpha}_{-1} \bar{\alpha}_{-1} \vert 0 \rangle$
  - $\alpha_{-2} \bar{\alpha}_{-1} \bar{\alpha}_{-1} \vert 0 \rangle$
  - $\alpha_{-1} \alpha_{-1} \bar{\alpha}_{-2} \vert 0 \rangle$

- Two states with even number of phonons, expected to appear as 
  $\{P = +, q = 0\}$ second and third excited states.

- Two states with odd number of phonons, expected to appear as $\{P = -, q = 0\}$ ground and first excited states.

- Apply variational technique to the correlation matrices $\{P = \pm, q = 0\}$. 
4. \(D=2+1\) Results: \(N_L = N_R = 2,\ q = 0,\ P = +\)

\[
E^2 = (\sigma l)^2 + 8\pi(\frac{N_L + N_R}{2} - \frac{1}{24}) + \left(\frac{2\pi q}{l}\right)^2
\]
4. \( D=2+1 \) Results: \( N_L = N_R = 2, \ q = 0, \ P = + \)

\[
E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left( \frac{2\pi q}{l} \right)^2
\]
4. $D=2+1$ Results: $N_L = N_R = 2$, $q = 0$, $P = +$
4. D=2+1 Results: $N_L = N_R = 2, \ q = 0, \ P = -$
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4. D=2+1 Results: $N_L = N_R = 2, \ q = 0, \ P = -$ 

\[ E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
4. D=2+1 Results: $N_L = N_R = 2, q = 0$

\[ E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
\[ 0 = b \quad \Rightarrow \quad \gamma_N = \tau_N \quad \text{Results:} \quad D=2+1 \]
4. **D=2+1 Results:** \( N_L = 1, N_R = 0, q = 1, P = - \)

- \( N_L = 1, N_R = 0 \) corresponds to the \( q = 1 \) ground state.
- String theoretically: \( \alpha_{-1} | 0 \rangle \)
- Odd number of phonons
- Expected to appear as \( \{ P = - , q = 1 \} \) ground state
- Apply variational technique to the correlation matrix \( \{ P = - , q = 1 \} \).
- Extraction of the ground state.
4. $D=2+1$ Results: $N_L = 1, N_R = 0, q = 1, P = -$
4. $D=2+1$ Results: $N_L = 1$, $N_R = 0$, $q = 1$, $P = -$
4. D=2+1 Results: $N_L = 2, N_R = 1, q = 1, P = \pm$

- $N_L = 2, N_R = 1$ corresponds to the $q = 1$ first excited state.

- String theoretically, two states can be constructed:
  
  $-\alpha_{-2}\bar{\alpha}_{-1} | 0\rangle$
  
  $-\alpha_{-1}\alpha_{-1}\bar{\alpha}_{-1} | 0\rangle$

- One state with odd number of phonons, expected to appear as \{P = -, q = 1\} first excited state.

- One state with even number of phonons, expected to appear as \{P = +, q = 1\} ground state.

- Apply variational technique to the correlation matrices \{P = \pm, q = 1\}.
4. D=2+1 Results: $N_L = 2, N_R = 1, q = 1, P = -$
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4. D=2+1 Results: $N_L = 2, N_R = 1, q = 1, P = -$
4. $D=2+1$ Results: $N_L = 2, N_R = 1, q = 1, P = +$

$$E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left( \frac{2\pi q}{l} \right)^2$$
4. $D=2+1$ Results: $N_L = 2, N_R = 1, q = 1, P = +$

\[ E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
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$$E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{2}\right) + \left(\frac{2\pi q}{l}\right)^2$$
4. D=2+1 Results: $N_L = 2, N_R = 0, q = 2, P = \pm$

- $N_L = 2, N_R = 0$ corresponds to the $q = 2$ ground state.
- String theoretically, two states can be constructed:
  - $\alpha_{-2} | 0\rangle$
  - $\alpha_{-1}\alpha_{-1} | 0\rangle$
- One state with odd number of phonons, expected to appear as $\{P = -, q = 2\}$ ground state.
- One state with even number of phonons, expected to appear as $\{P = +, q = 2\}$ ground state.
- Apply variational technique to the correlation matrices $\{P = \pm, q = 2\}$. 
4. D=2+1 Results: \( N_L = 2, N_R = 0, q = 2, P = + \)

\[ E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
4. D=2+1 Results: \( N_L = 2, N_R = 0, q = 2, P = + \)

\[
E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left( \frac{2\pi q}{l} \right)^2
\]
4. **D=2+1 Results:** $N_L = 2, N_R = 0, q = 2, P = -$

\[ E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
4. D=2+1 Results: \( N_L = 3, N_R = 1, q = 2, P = \pm \)

- \( N_L = 3, N_R = 1 \) corresponds to the \( q = 2 \) first excited state.

- String theoretically, three states can be constructed:
  - \( \alpha_{-3} \bar{\alpha}_{-1} \left| 0 \right> \)
  - \( \alpha_{-2} \alpha_{-1} \bar{\alpha}_{-1} \left| 0 \right> \)
  - \( \alpha_{-1} \alpha_{-1} \alpha_{-1} \bar{\alpha}_{-1} \left| 0 \right> \)

- One state with odd number of phonons, expected to appear as \( \{ P = -, q = 2 \} \) first excited state.

- Two states with even number of phonons, expected to appear as \( \{ P = +, q = 2 \} \) first and second excited states.

- Apply variational technique to the correlation matrices \( \{ P = \pm, q = 2 \} \).
4. D=2+1 Results: \( N_L = 3, N_R = 1, q = 2, P = + \)

\[
E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left( \frac{2\pi q}{l} \right)^2
\]
4. D=2+1 Results: $N_L = 3, N_R = 1, q = 2, P = +$

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4. D=2+1 Results: \( N_L = 3, N_R = 1, q = 2, P = - \)

\[
E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{2l} \right) + \left( \frac{2\pi q}{l} \right)^2
\]
4. D=2+1 Results: $N_L = 3, N_R = 1, q = 2, P = -$
5. $D=3+1$
5. D=3+1: Lattice Calculation

→ **We define our theory on a 4D Euclidean lattice with volume:**

\[
L_{\parallel} \times L_{\perp 1} \times L_{\perp 2} \times L_T.
\]

→ **We use the standard Wilson Action:**

\[
S_L = \beta \sum_p \left\{ 1 - \frac{1}{N} \text{Re} \text{Tr} U_p \right\}
\]

with \( \beta = \frac{2N}{g^2} \)

→ **Monte Carlo simulations:**

- First: \( SU(3) \), \( \beta = 6.0625 \) with \( a\sqrt{\sigma_f} \simeq 0.195 \) (\( a \simeq 0.09 \text{fm} \)).
- \( a \to 0 \): \( SU(3) \), \( \beta = 6.3380 \) with \( a\sqrt{\sigma_f} \simeq 0.129 \) (\( a \simeq 0.06 \text{fm} \)).
- \( N \to \infty \): \( SU(5) \), \( \beta = 17.630 \) with \( a\sqrt{\sigma_f} \simeq 0.197 \) (\( a \simeq 0.09 \text{fm} \)).

→ **Our Approach:**

- Create a large basis of operators \( \Phi_{i,q,J,P_R,P_P} : i = 1, 2..., N_O, (\sim 700) \).
- Calculate the correlation matrix

\[
C_{ij,q,J,P_R,P_P}(t) = \langle \Phi_{i,q,J,P_R,P_P}(t) \Phi_{j,q,J,P_R,P_P}(0) \rangle
\]

- Use the variational technique to extract correlators of different states.
5. D=3+1: Lattice Calculation

→ **We define our theory on a 4D Euclidean lattice with volume:**

\[ L_{\parallel} \times L_{\perp 1} \times L_{\perp 2} \times L_T. \]

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\[
S_L = \beta \sum_p \left\{ 1 - \frac{1}{N} \text{ReTr} U_p \right\} \quad \text{with} \quad \beta = \frac{2N}{g^2}
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→ **Our Approach:**

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\]

- Use the variational technique to extract correlators of different states.
5. D=3+1: Symmetries: Flux-Tubes

More complicated structure than $D = 2 + 1$: 

→ Lattice Symmetry of Rotations about the string axis.

→ $C_{4\nu} \otimes Z(\mathcal{R})$ for zero longitudinal momentum.
  - Rotations of $\pi/2 \rightarrow$ angular momentum $J$
  - Reflections in orthogonal plane ($\mathcal{P}$-Parity)
  - Reflections about the mid-point on the principal axis ($\mathcal{R}$-Parity)
→ 10 irreducible representations $\equiv$ 10 correlation matrices

→ $C_{4\nu}$ for non-zero longitudinal momentum.
  - Rotations of $\pi/2 \rightarrow$ angular momentum $J$
  - Reflections in orthogonal plane ($\mathcal{P}$-Parity)
→ 5 irreducible representations $\equiv$ 5 correlation matrices

→ $A_1 \equiv (|J| = 0, 4, ..., 4N, P_\mathcal{P} = +)$, $A_2 \equiv (|J| = 0, 4, ..., 4N, P_\mathcal{P} = -)$
→ $E \equiv (|J| = 1, 3, ..., \pm 2N + 1, P_\mathcal{P} = \pm)$
→ $B_1 \equiv (|J| = 2, 6, ..., 4N + 2, P_\mathcal{P} = +)$, $B_2 \equiv (|J| = 2, 6, ..., 4N + 2, P_\mathcal{P} = -)$
Two transverse directions:

→ Define $\alpha_{-k}^+$ and $\alpha_{-k}^-$ as ($x$, $y$ are the transverse directions):
  - $\alpha_{-k}^+ = \alpha_{-k}^x + i\alpha_{-k}^y$
  - $\alpha_{-k}^- = \alpha_{-k}^x - i\alpha_{-k}^y$

→ Spin $J$.
  - $J = |\#(+) - \#(-)|$

→ $\mathcal{P}$-Parity
  - Under $\mathcal{P}$-Parity: $\alpha_{-k}^+ \xrightarrow{\mathcal{P}} \alpha_{-k}^-$ and $\bar{\alpha}_{-k}^+ \xrightarrow{\mathcal{P}} \bar{\alpha}_{-k}^-$

→ $\mathcal{R}$-Parity
  - Under $\mathcal{R}$-Parity: $\alpha_{-k}^\pm \xrightarrow{\mathcal{R}} \bar{\alpha}_{-k}^\pm$

• Example: $(\alpha_{-1}^+ \bar{\alpha}_{-1}^+ \pm \alpha_{-1}^- \bar{\alpha}_{-1}^-) |0\rangle$
  - $J = 2$
  - $P_{\mathcal{P}} = \pm$
  - $P_{\mathcal{R}} = +$
5. D=3+1: Quantum Numbers: Parity

\( \mathcal{P} \)-Parity reflection plane

\( \mathcal{R} \)-Parity reflection plane

Example:
5. D=3+1: Quantum Numbers: Spin

→ Operator $\phi$ is given by the trace of path ordered product of blocked links

$\phi(\theta) = \text{Tr}\{\text{Path Ordered Product}\}$

→ We can then form an operator of spin $J$:

Continuum: $\phi(J) = \int d\theta e^{iJ\theta} \phi_\theta$

Lattice: $\phi_L(J) = \sum_n e^{iJn\frac{\pi}{2}} \phi_{n\frac{\pi}{2}}$

→ Example $J = 1$:

$\phi_L(J = 1) = i\phi_{\frac{\pi}{2}} - \phi_{\pi} - i\phi_{\frac{3\pi}{2}} + \phi_{2\pi}$

→ If $\phi_{\theta=0} = \text{Tr}\{\text{Path Ordered Product}\}$

$\phi_L(J = 1) = \text{Tr}\{i\phi_{\frac{\pi}{2}} - \phi_{\pi} - i\phi_{\frac{3\pi}{2}} + \phi_{2\pi} \}$
5. D=3+1: Operators: Example $J = 0$, $P_P = +$, $P_R = +$

Irreducible representation: $A_1$, $P_R = +$.

Operator before being traced:

\[\begin{align*}
&+ \left[ \text{Diagram 1} \right] \\
&+ \left[ \text{Diagram 2} \right] \\
&+ \left[ \text{Diagram 3} \right] \\
&+ \left[ \text{Diagram 4} \right] \\
&+ \left[ \text{Diagram 5} \right]
\end{align*}\]
5. D=3+1: Operators: Example $J = 0$, $P_P = -$, $P_R = +$

Irreducible representation: $A_2$, $P_R = +$.

Operator before being traced:

\[
\begin{align*}
\text{Diagram} + \left[ \text{Diagram} \right] + \left[ \text{Diagram} \right] + \left[ \text{Diagram} \right] \\
- \left[ \text{Diagram} \right] + \left[ \text{Diagram} \right] + \left[ \text{Diagram} \right] + \left[ \text{Diagram} \right] \\
- \left[ \text{Diagram} \right] + \left[ \text{Diagram} \right] + \left[ \text{Diagram} \right] + \left[ \text{Diagram} \right]
\end{align*}
\]
5. D=3+1: Operators: Example $J = 0$, $P_P = +$, $P_R = -$ 

Irreducible representation: $A_1$, $P_R = -$.

Operator before being traced:
5. $D=3+1$: Operators: Example $J = 0$, $P_P = -$, $P_R = -$

Irreducible representation: $A_2$, $P_R = -$.

Operator before being traced:

\[
\begin{align*}
&\quad + \quad + \quad + \quad + \\
&- \left[ \quad + \quad + \quad + \quad + \right] \\
&- \left[ \quad + \quad + \quad + \quad + \right] \\
&+ \left[ \quad + \quad + \quad + \quad + \right]
\end{align*}
\]
5. D=3+1: Operators: Transverse Deformations

→ ~ 700 operators

Transverse deformations in 3 spatial dimensions:

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5. D=3+1 Results: $N_L = N_R = 0$

- $N_L = N_R = 0$ corresponds to the ground state (No movers)
- String theoretically: $|0⟩$
- No phonons
- Expected to appear as $\{J = 0, P_P = +, P_R = +, q = 0\}$ ground state
- Apply variational technique to the correlation matrix $\{A_1, P_R = +, q = 0\}$.
- Extraction of the ground state.
  - Fit data with Nambu-Goto or L.O/N.L.O/N.N.L.O $→ a^2 \sigma_f$ (string tension)
  - This provides parameter free predictions for the energies in N.G:

$$E_{N_L,N_R,q,w}^2 = (\sigma_f lw)^2 + 8\pi \sigma_f \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2.$$  

In lattice units:

$$a^2 E_{N_L,N_R,q,w}^2 = (a^2 \sigma_f Lw)^2 + 8\pi a^2 \sigma_f \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{L} \right)^2.$$
5. D=3+1 Results: $N_L = N_R = 0$

\[
E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2
\]
5. $D=3+1$ Results: $N_L = N_R = 0$

\[
E^2 = (\sigma l)^2 + 8 \pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2
\]

$J = 0$, $P_P = +$, $P_R = +$, $q = 0$
5. D=3+1 Results: \( N_L = 1, N_R = 0 \)

- \( N_L = 1, N_R = 0 \) corresponds to the \( q = 1 \) ground state
- String theoretically: \((\alpha_{-1}^+ + \alpha_{-1}^- , \alpha_{-1}^+ - \alpha_{-1}^- ) | 0 \rangle\)
- One phonon \( \rightarrow J = 1 \)
- Expected to appear as \{\( J = 1, q = 1 \)\} ground state
- Apply variational technique to the correlation matrix \{\( E, q = 1 \)\}.
- Extraction of the \{\( E, q = 1 \)\} ground state.
5. **D=3+1 Results: \(N_L = 1, N_R = 0\)**

\[
E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2
\]

\(J = 0, P_P = +, P_R = +, q = 0\)

\(N_L = 1, N_R = 0\)

\(N_L = N_R = 0\)
5. D=3+1 Results: $N_L = 1, N_R = 0$

\[ E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
5. **D=3+1 Results:** $N_L = 2, N_R = 0$

- $N_L = 2, N_R = 0$ corresponds to the $q = 2$ ground state

- **String theoretically:**
  - $\alpha_{-1}^+ \alpha_{-1}^- |0\rangle \sim \text{g.s } J = 0, P_P = +, q = 2$
  - $(\alpha_{-2}^+ + \alpha_{-2}^-, \alpha_{-2}^+ - \alpha_{-2}^-) |0\rangle \sim \text{g.s } J = 1, q = 2$
  - $(\alpha_{-1}^+ \alpha_{-1}^+ + \alpha_{-1}^- \alpha_{-1}^-) |0\rangle \sim \text{g.s } J = 2, P_P = +, q = 2$
  - $(\alpha_{-1}^+ \alpha_{-1}^+ - \alpha_{-1}^- \alpha_{-1}^-) |0\rangle \sim \text{g.s } J = 2, P_P = -, q = 2$

- One phonon $\rightarrow J = 1$, Two phonons $\rightarrow J = 0, 2$.

- One state expected to appear as $\{A_1, q = 2\}$ ground state

- One state expected to appear as $\{E, q = 2\}$ ground state

- One state expected to appear as $\{B_1, q = 2\}$ ground state

- One state expected to appear as $\{B_2, q = 2\}$ ground state

- Apply variational technique to the correlation matrices.
5. D=3+1 Results: $N_L = 2, N_R = 0$

\[ E = \frac{\sqrt{\sigma}}{\sqrt{\sigma_f}} \]

\[ E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
5. $D=3+1$ Results: $N_L = 2, N_R = 0$
5. \( D=3+1 \) Results: \( N_L = 2, N_R = 0 \)

\[
E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2
\]
5. D=3+1 Results: $N_L = 2, N_R = 0$

\[ E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
5. D=3+1 Results: $N_L = 2, N_R = 0$

\[ E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
5. D=3+1 Results: $N_L = N_R = 1$

- $N_L = N_R = 1$ corresponds to the $q = 0$ first excited state.
- String theoretically:
  
  $\begin{align*}
  &- (\alpha_{-1}^+ \bar{\alpha}_{-1}^- + \alpha_{-1}^- \bar{\alpha}_{-1}^+) |0\rangle \sim \text{f.e.s } J = 0, P_P = +, P_R = +, q = 0 \\
  &- (\alpha_{-1}^+ \bar{\alpha}_{-1}^- - \alpha_{-1}^- \bar{\alpha}_{-1}^+) |0\rangle \sim \text{g.s } J = 0, P_P = -, P_R = -, q = 0 \\
  &- (\alpha_{-1}^+ \bar{\alpha}_{-1}^+ + \alpha_{-1}^- \bar{\alpha}_{-1}^-) |0\rangle \sim \text{g.s } J = 2, P_P = +, P_R = +, q = 0 \\
  &- (\alpha_{-1}^+ \bar{\alpha}_{-1}^+ - \alpha_{-1}^- \bar{\alpha}_{-1}^-) |0\rangle \sim \text{g.s } J = 2, P_P = -, P_R = +, q = 0
  \end{align*}$

- Two phonons $\rightarrow J = 0, 2$.
- One state expected to appear as $\{A_1, P_R = +, q = 0\}$ first excited state
- One state expected to appear as $\{A_2, P_R = -, q = 0\}$ ground state
- One state expected to appear as $\{B_1, P_R = +, q = 0\}$ ground state
- One state expected to appear as $\{B_2, P_R = +, q = 0\}$ ground state
- Apply variational technique to the correlation matrices.
5. **D=3+1 Results:** $N_L = N_R = 1$

\[ E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L+N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2 \]

- $g.s$ $J = 0$, $P_P = +$, $P_R = +$, $q = 0$

- $N_L = N_R = 1$

- $N_L = N_R = 0$
5. D=3+1 Results: \( N_L = N_R = 1 \)

\[
E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2
\]

\[
f.e.s \ J = 0, \ P_P = +, \ P_R = +, \ q = 0
\]

\[
g.s \ J = 0, \ P_P = +, \ P_R = +, \ q = 0
\]

\( N_L = N_R = 1 \)

\( N_L = N_R = 0 \)
5. D=3+1 Results: $N_L = N_R = 1$

\[ E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
5. D=3+1 Results: $N_L = N_R = 1$

$$E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2$$
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\[ E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
5. D=3+1 Results: $N_L = 2, N_R = 1$

- $N_L = 2, N_R = 1$ corresponds to the $q = 1$ first excited state.

- String theoretically:
  
  - $\left(\alpha_{-2}^+ \bar{\alpha}_{-1}^- + \alpha_{-2}^- \bar{\alpha}_{-1}^+\right) |0\rangle \to \text{g.s.} \{A_1, q = 1\}$
  - $\left(\alpha_{-2}^+ \bar{\alpha}_{-1}^- - \alpha_{-2}^- \bar{\alpha}_{-1}^+\right) |0\rangle \to \text{g.s.} \{A_2, q = 1\}$
  - $\left(\alpha_{-2}^+ \bar{\alpha}_{-1}^+ + \alpha_{-2}^- \bar{\alpha}_{-1}^+\right) |0\rangle \to \text{g.s.} \{B_1, q = 1\}$
  - $\left(\alpha_{-2}^+ \bar{\alpha}_{-1}^- - \alpha_{-2}^- \bar{\alpha}_{-1}^+\right) |0\rangle \to \text{g.s.} \{B_2, q = 1\}$
  - $\left(\alpha_{-1}^+ \alpha_{-1}^- \bar{\alpha}_{-1}^- - \alpha_{-1}^- \alpha_{-1}^- \bar{\alpha}_{-1}^+\right) |0\rangle \to \text{f.e.s, s.e.s and t.e.s} \{E, q = 1\}$
5. D=3+1 Results: $N_L = 2, N_R = 1$

$$E^2 = (\sigma l)^2 + 8\pi \sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + (\frac{2\pi q}{l})^2$$
5. \( D=3+1 \) Results: \( N_L = 2, N_R = 1 \)

\[ E^2 = (\sigma l)^2 + 8\pi\sigma \left( \frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left( \frac{2\pi q}{l} \right)^2 \]
6. Conclusions

- $D = 2 + 1$
  - Closed flux tube can be well described by Nambu-Goto even to small $l$!
  - Nambu-Goto is the best approach.

- $D = 3 + 1$
  - The spectrum is mostly closed to Nambu-Goto down to very small $l$!
  - Nambu-Goto is better than any other effective string theory model!
  - However, some states ($A_2$) are far from Nambu-Goto.
  - Non-Stringy dynamics of some kind?
7. Future Work

- Extract $k = 2$-flux tube excitation spectrum for $SU(6)$, $\beta = 171$
- Investigation of states with large deviations in $D = 3 + 1$
- Extract more excited states in $D = 3 + 1$
- Increase projection onto massive states in $D = 3 + 1$
  1. Tuning Blocking/Smearing???
  2. Include more operators???
  3. More statistics???
  4. Any Suggestions???
- Investigation of $k = 2$-flux tube excitation spectrum in $D = 3 + 1$. 