All-to-all Propagators in Lattice Hadron Spectrum Calculations

John Bulava

DESY
Zeuthen, Germany

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DESY, Zeuthen
Germany
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2 Distillation - An Exact All-to-all Method

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Background

Distillation - An Exact All-to-all Method

Variance-Reduced Stochastic LapH (VRSL)
Dedicated to mapping out the low-lying excited hadron spectra.

Members:


**Carnegie Mellon**: J. Foley, C. Morningstar, D. Lenkner, C. H. Wong

**DESY, Zeuthen**: JB

**U. of Maryland**: E. Engelson, S. Wallace

**U. of the Pacific**: K. J. Juge

**Tata Inst., Mumbai**: N. Mathur

**Trinity College, Dublin**: M. J. Peardon, S. M. Ryan
Correlation functions between hadron operators are an ensemble average over gauge configurations.

\[
\langle 0 | \mathcal{O}_i \left[ \psi, \bar{\psi}, U \right](t) \mathcal{O}_j \left[ \psi, \bar{\psi}, U \right](t_0) | 0 \rangle = \langle F_{ij} [M^{-1}(U), U] \rangle_U
\]

\(M^{-1}(U)\), is a \((V \times L_t \times N_{\text{spin}} \times N_{\text{color}})\)-dimensional matrix. Can only solve equations like

\[
M(x, y) \phi(y) = \eta(x) \rightarrow \phi(x) = M^{-1}(x, y) \eta(y)
\]

‘Point-to-all’ \(\Rightarrow\) \(\eta(x) \propto \delta(x, x_0)\)

‘All-to-all’ \(\Rightarrow\) Use of \(M^{-1}(x, y) \forall x, y\)
Spectral decomposition of two-point correlation functions:

\[ \langle 0 | \mathcal{O}(t + t_0) \mathcal{O}(t_0) | 0 \rangle = \sum_n \langle 0 | \mathcal{O}(t_0) | n \rangle^2 \exp[-E_n t] \]

Difficult (or impossible) to fit sub-leading exponentials, instead form a matrix of two-point correlators:

\[ C_{ij}(t) = \langle 0 | \mathcal{O}_i(t + t_0) \mathcal{O}_j(t_0) | 0 \rangle \]

Define new operators \( \Omega_n(t) \) such that:

\[ \langle 0 | \Omega_a(t + t_0) \Omega_b(t_0) | 0 \rangle = \delta_{ab} \lambda_a(t) \]

Fit \( \lambda_n(t) \) with a single exponential to obtain \( E_n \)
THE NEED FOR ALL-TO-ALL PROPAGATORS

- Finite-volume stationary states are comprised of resonance states as well as scattering states.
- Some resonance states may have multi-particle content.

\[
\langle 0 | B(p = 0, t) \bar{B}(p = 0, t_0) | 0 \rangle = \frac{1}{\sqrt{2}} \sum_{x, y} \langle 0 | \varphi_B(x, t) \overline{\varphi}_B(y, t_0) | 0 \rangle
\]

\[
B(p, t) M(-p, t) = \frac{1}{\sqrt{2}} \sum_{x, y} \varphi_B(x, t) \varphi_M(y, t) e^{ip \cdot (x - y)}
\]

- Sum over \( y \) can be eliminated in Eq. 1, but not in correlation functions containing the operator from Eq. 2. Solving \( M \phi = \eta \) for all \( y \) is not feasible.
Simulation Details

- **Anisotropic Wilson Gauge** action with spatial rectangle. (C. Morningstar, M. Peardon ‘99)

- **Anisotropic Clover-Wilson** quark action, with tadpole improvement

- Spatial links are stout smeared (Morningstar, Peardon ’04), parameters are tuned non-perturbatively (Edwards, Lin, Joo ’08).

- $N_f = 2 + 1$, $a_s = 0.12\text{fm}$, $a_t = a_s/3.5$. Range of pion masses: $\approx 700\text{MeV} - 230\text{MeV}$, and lattice sizes $12^3 \times 96$ to $32^2 \times 256$. 
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**LapH (Laplacian Heaviside) Smearing**

- Quark smearing damps out unwanted excited states by applying \( S = \exp[\frac{\sigma^2}{4}\Delta] \).

\[
S_{ab}[U; t](x|y) = \sum_{i=1}^{N} e^{\frac{\sigma^2}{4} \lambda_i[U; t]} v^{(i)}_a[U; t](x) v^{*(i)}_b[U; t](y)
\]

- **Truncate** at a finite number \( (n_{\text{max}} < N) \) of eigenmodes (M. Peardon, JB, et al. (2009))

\[
\tilde{M}^{-1}_{(a\alpha|b\beta)}(x, t|x_0, t_0) = \sum_{i=1}^{n_{\text{max}}} \sum_{\bar{i}=1}^{n_{\text{max}}} v^{(i)}_a[t](x) K^{(i)(\bar{i})}_{\alpha\beta}(t|t_0) \times v^{*(\bar{i})}_b[t_0](x_0)
\]

with

\[
K^{(i)(\bar{i})}_{\alpha\beta}(t|t_0) = v^{*(i)}_c[t](y) M^{-1}_{(c\alpha|d\beta)}(y, t|z, t_0) v^{(\bar{i})}_d[t_0](z)
\]
**Volume Dependence**

- $N_f = 2 + 1$ ensembles: $12^3 \times 96$ and $16^3 \times 128$ with $a_s = 0.12\text{fm}$, $a_t = a_s/3.5$, $m_\pi \approx 700\text{MeV}$
- Examine the number of eigenvalues in $[0.3, 0.4]$: $n_{12}/n_{16} = (12/16)^3$
- For fixed $\lambda_{\text{max}}$, $n_{\text{max}} \propto V$
LapH Conclusions

- *Exact* (smeared) all-to-all propagator for a finite number of inversions!

- $n_{max}$ (and thus required number of inversions) grows linearly with the volume.

- Cannot completely separate initial and final degrees of freedom.

- Still useful for smaller volumes ($L_s < 2 - 2.5\text{fm}$). Results!
Spatially extended operators have overlap with orbital and radial excitations.

Operators must transform irreducibly under lattice symmetries.

Generate a large set of operators from simple elemental building blocks C. Morningstar, et al. (2005)

Select $O(10)$ with good low-lying state overlap which form a well-conditioned correlator matrix.
100 configurations from $N_f = 2 + 1$, $16^3 \times 128$ ensemble, with $L_s \approx 2\text{fm}$, $a_s \approx 0.12\text{fm}$, $a_t = a_s/3.5$, and $m_\pi \approx 380\text{MeV}$. Also, $n_{max} = 32$.

Operators transform under lattice irreps: ‘g’→ +ve parity, ‘u’ → -ve parity. $J = 1/2 \rightarrow G_1$, $J = 3/2 \rightarrow H$, $J = 5/2 \rightarrow G_2$ and $H$.

About 12 (single-hadron) operators chosen in each channel.
**Nucleon (JB) and Delta (E. Engelson) Spectra**

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**Σ (D. Lenkner) and Ξ (C. H. Wong)**

**Baryon Spectra**

Simulated Sigma Spectrum, \( V=16^3 \)

Simulated Cascade Spectrum, \( V=16^3 \)
$N_f = 2 + 1 \textbf{ Results: } I^G = 1^- \text{ and } I^G = 1^+$

**Meson Spectrum (C. H. Wong)**
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**Stochastic All-to-all**

- Generate $N_r$ independent stochastic sources
  \[ \eta_{c\alpha}^{(r)}(x, t) \in \mathbb{Z}_4 \]
  and solve $M\phi^{(r)} = \eta^{(r)}$ for $\phi^{(r)}$.
- Can estimate the quark propagator:
  \[
  M^{-1} \approx \frac{1}{N_r} \sum_{r=1}^{N_r} \phi^{(r)} \eta^{(r)*}
  \]
- Variance Reduction: **Dilute** each noise source
  $\eta^{(r)[d]} = P^{[d]} \eta^{(r)}$, with $\sum_{d=1}^{D} P^{[d]} = 1$ and $P^{[d]} P^{[d']} = \delta_{dd'} P^{[d]}$. 
Noise in the Subspace Only

- A new way to introduce noise:

\[ \eta_{c\alpha}^{(r)}(x, t) = \sum_{n=1}^{N_{ev}} a_{\alpha n}^{(r)}[t] \nu_c^{(n)}[U; t](x) \]

- Dilute in the subspace only

\[ \eta_{c\alpha}^{(r)[d]}(x, t) = \sum_{n=1}^{N_{ev}} a_{\alpha n}^{(r)[d]}[t] \nu_c^{(n)}[U; t](x) \]

\[ a_{\alpha n}^{(r)[d]}[t] = P_{(\alpha n|\alpha' n')}(t|t') a_{\alpha' n'}^{(r)}[t'] \]

- Is VRSL more efficient than conventional dilution? What about the volume dependence?
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VRSL vs. Conventional Dilution

\[ \frac{\sigma(N_{inv})}{\sigma_g} \]

\[ N_{inv}^{-1/2} \]

- Time LAPH
- Time+Every Other
- Time+2 Groups
- Time+Spin
- Time+Every 4th
- Time+4 Groups
- Time+Every 8th
- Time+8 Groups
- Time+Every 12th
- Time+Every 16th
- Time
- Time+Space\_eo
- Time+Color
- Time+Spin
- Time+Color\_eo\_space\_eo
- Time+Spin\_space\_eo
- Time+Spin\_Color
- Time+Spin\_Color\_space\_eo

Max Dil. Limit
VRSL Volume Dependence

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The graph shows the dependence of the variance $\sigma(N) / \sigma_g$ on $N_{inv}^{-1/2}$ for different volume sizes and time-slice configurations. The configurations include:

- Time+Spin $16^3$
- Time+Every 4th $16^3$
- Time+4 Groups $16^3$
- Time+Every 8th $16^3$
- Time+8 Groups $16^3$
- Time+Every 12th $16^3$
- Time+Every 16th $16^3$
- Time+Spin $20^3$
- Time+Every 4th $20^3$
- Time+4 Groups $20^3$
- Time+Every 8th $20^3$
- Time+8 Groups $20^3$
- Time+Every 12th $20^3$
- Time+Every 16th $20^3$
- Max Dil. Limit
Same-Time ('Disconnected') Diagrams

- Required for flavor-singlet quantities and single hadron/multi-hadron correlators.

- For ‘Connected’ correlators, only need $\eta(x_0, t_0)$ for a single $t_0$ but here we need many $t_0$’s.

- Exact distillation result on all $t_0$’s for a small lattice: 100 cфgs., $N_f = 2 + 1$, $12^3 \times 96$, $m_\pi \approx 700\text{MeV}$, $n_{max} = 12$, $N_{inv} = 4608$.

- Try different time, spin, and eigenvector dilution schemes. (C. H. Wong, M. Peardon)
Disc. Term from $\eta'$ Meson Correlator - 192 Inversions

Eta', V=12^3 \times 96, in the Chosen Dilution Scheme

No. of Interlace Projectors in (time, vector, spin) in the labels

$$C(\tau)$$

- 96,12,4(Dist)
- 12,4,4
Disc. Term from $\sigma$ Meson Correlator - 192 Inversions

Sigma, $V=12^3 \times 96$, for the Chosen Dilution Scheme

No. of interlace projectors in (time, vector, spin) in the labels

\[ C(\tau) \]

- 96,12,4 (Dist)
- 12,4,4
Single Term from Scalar Meson Decay - 192 Inversions

Scalar Decay, $V=12^3 \times 96$, for the Chosen Dilution Scheme

No. of Interlace Projectors in $[t,v,s][t,v,s]$ for the labels (1st [] is $t_0-t_1$, 2nd [] is $t_0-t_0$)

Re($\mathcal{C}(\tau)$) vs $\tau$

- $[96,12,4][96,12,4]$(dist)
- $[96,6,4][12,4,4]$
VRSL Conclusions

- VRSL is more efficient than conventional dilution. For a fixed number of inversions, VRSL gives a considerably lower error.

- The smeared subspace is low-dimensional, so the gauge-noise ($\sigma_g$) limit can be obtained with a MUCH smaller number of projectors.

- Unlike exact method, volume dependence seems mild.

- A moderate level of dilution projectors works for disconnected terms.
Future Plans

- Employ the VRSL method to study multi-hadron (specifically nucleon-pion) operators.
  J. Foley: Group theory for moving hadrons
  J. Juge, C. H. Wong, J. Bulava: First Multi-pion results

- Repeat the spectrum calculations for a variety of larger volumes (up to $32^3 \times 256$, $L_s \approx 3.8\text{fm}$) with lighter quarks (pion mass down to $\approx 230\text{MeV}$) including multi-hadron operators.

- Differentiate resonances from multi-hadron states and identify resonance quantum numbers.