

Algorithmic advances in NSPT

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Motivation

Why perturbation theory?

Renormalization

$$\text{Ex.: } O(q) \stackrel{q \rightarrow M_Z}{\approx} \alpha_{\overline{\text{MS}}}(q) + k_1 \alpha_{\overline{\text{MS}}}(q)^2 + k_2 \alpha_{\overline{\text{MS}}}(q)^3 + \dots \quad [\alpha_{\overline{\text{MS}}}(M_Z) \approx 0.12]$$

$\approx 10\% \qquad \qquad \approx 1\%$

but also

- ▶ quark-masses
- ▶ effective electro-weak Hamiltonians
- ▶ small-flow time expansions
- ▶ quasi-PDF
- ▶ ...

Improvement

$$\text{Ex.: } \tilde{g}_0^2 = g_0^2 \left[1 + \frac{1}{3} b_g(g_0) \text{tr}\{aM_q\} \right], \quad b_g = O(g_0^2)$$

or in general

$$\frac{\Sigma(g^2, a/L) - \sigma(g^2)}{\sigma(g^2)} \stackrel{g^2 \rightarrow 0}{\approx} g^2 \delta_1(a/L) + g^4 \delta_2(a/L) + \dots$$

Motivation

Need for automation

Perturbative lattice calculations can too easily become difficult, tedious, and (human) time consuming ...

Why?

- ▶ Feynman rules can be very complicated
- ▶ In gauge-theories new vertices appear at each order
⇒ the number of diagrams grows very rapidly
- ▶ Requires numerical evaluation even for simple diagrams
⇒ naive computational cost $\propto V^N$ @ N loops

A possible **solution** ...

Numerical stochastic perturbation theory (NSPT)

- ✓ Fully automated
- ✓ Easy to set-up and very flexible tool
- ✓ Allows for high-order computations
- ✗ Systematic and statistical errors

Introduction

The Parisi and Wu way

(Parisi, Wu '81)

Ex.: Lattice φ^4 -theory

$$S(\varphi) = \sum_x \left\{ \frac{1}{2} \partial_\mu \varphi(x) \partial_\mu \varphi(x) + \frac{1}{2} (m^2 + \delta m^2) \varphi(x)^2 + \frac{g_0}{4!} \varphi(x)^4 \right\}$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int d\varphi \mathcal{O}(\varphi) e^{-S(\varphi)}$$

Langevin equation

$$\partial_t \phi(t, x) = - \frac{\delta S(\phi)}{\delta \phi(t, x)} + \eta(t, x)$$

$$\langle \eta(t, x) \eta(s, y) \rangle_\eta = 2 \delta_{xy} \delta(t - s)$$

Stochastic quantization

$$\lim_{t \rightarrow \infty} \langle \phi(t, x_1) \dots \phi(t, x_n) \rangle_\eta = \langle \varphi(x_1) \dots \varphi(x_n) \rangle$$

$$\lim_{t \rightarrow \infty} P(t, \phi) = P_{\text{eq}}(\phi) \propto e^{-S(\phi)}$$

Introduction

Stochastic perturbation theory

(Parisi, Wu '81)

Perturbative field

$$\phi = \phi_0 + g_0 \phi_1 + g_0^2 \phi_2 + \dots$$

Forces & c.t.

$$\frac{\delta S(\phi)}{\delta \phi} = F_0(\phi_0) + g_0 F_1(\phi_0, \phi_1) + \dots$$

$$\delta m^2 = g_0 \delta m_1^2 + g_0^2 \delta m_2^2 + \dots$$

Dynamics

$$\partial_t \phi_0 = -\Delta \phi_0 + \eta \quad [\Delta = -\partial_\mu^* \partial_\mu + m^2]$$

$$\partial_t \phi_1 = -\Delta \phi_1 - \delta m_1^2 \phi_0 - \frac{1}{3!} \phi_0^3$$

...

$$\partial_t \phi_r = -F_r(\phi_0, \dots, \phi_r) + \delta_{r0} \eta \quad [F_r(\phi) = \Delta \phi_r + V_r(\phi_0, \dots, \phi_{r-1})]$$

Observables

$$\mathcal{O}(\phi) = \mathcal{O}_0(\phi_0) + g_0 \mathcal{O}_1(\phi_0, \phi_1) + \dots \quad \text{Ex.: } \phi^2 = \phi_0^2 + g_0 2\phi_0 \phi_1 + \dots$$

$$\lim_{t \rightarrow \infty} \langle \mathcal{O}_r(\phi_0, \dots, \phi_r) \rangle_\eta = k_r^\mathcal{O} \quad \text{where} \quad \langle \mathcal{O} \rangle = k_0^\mathcal{O} + g_0 k_1^\mathcal{O} + \dots$$

Numerical stochastic perturbation theory

Go numerical!

(Di Renzo et. al. '94)

Computerize

$$t \rightarrow t_n = n\varepsilon, \quad n \in \mathbb{N} \quad \text{and} \quad r = 0, 1, \dots, N$$

Discrete dynamics

$$\text{Ex.: } \phi_0(t_{n+1}) = -\varepsilon \Delta \phi_0(t_n) + \sqrt{\varepsilon} \eta(t_n)$$

$$\phi_1(t_{n+1}) = -\varepsilon \Delta \phi_1(t_n) - \varepsilon \delta m_1^2 \phi_0(t_n) - \frac{\varepsilon}{3!} \phi_0(t_n)^3$$

...

$$\phi_r(t_{n+1}) = -\varepsilon F_r(\phi_0(t_n), \dots, \phi_r(t_n)) + \delta_{r0} \sqrt{\varepsilon} \eta(t_n)$$

Order-by-order ops.

$$\phi = \{\phi_0, \dots, \phi_N\} \quad \text{Ex.: } \phi + \chi \rightarrow \phi_r + \chi_r, \quad \phi^2 \rightarrow \phi_r^2 = \sum_{s=0}^r \phi_{r-s} \cdot \phi_s$$

Stochastic estimates

$$\bar{\mathcal{O}}_r = \frac{1}{N_{\text{cnfg}}} \sum_{n=0}^{N_{\text{cnfg}}} \mathcal{O}_r(\phi_0(t_n), \dots, \phi_r(t_n)) \quad \text{Ex.: } \phi^2 = \{\phi_0^2, 2\phi_0\phi_1, \dots\}$$

Langevin based NSPT

Some known (and proven) facts

[Statistical errors]

$$\sigma(\overline{\mathcal{O}}_r)^2 = N_{\text{cnfg}}^{-1} \times \tau_{\text{int}}(\mathcal{O}_r) \times \text{var}(\mathcal{O}_r)$$

[$\mathcal{O} \equiv$ multiplicatively ren.]

► **Autocorrelations**

(Zinn-Justin '86; Zinn-Justin, Zwanziger '88)

$$\tau_{\text{int}}(\mathcal{O}_r) \stackrel{a \rightarrow 0}{\propto} 1/a^2$$

► **Variances**

(Lüscher '15)

$$\text{var}(\mathcal{O}_r) = \overline{\mathcal{O}_r^2} - (\overline{\mathcal{O}_r})^2 \stackrel{a \rightarrow 0}{\propto} \ln(a)$$

IMPORTANT: $\text{var}(\mathcal{O}_r) \neq 2r$ -order of $\langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2$ [$r \neq 0$]

\Rightarrow they are **NOT** given by the theory **ALONE!**

[Systematic errors]

(Parisi '81; Batrouni et. al. '85)

► $\lim_{N_{\text{cnfg}} \rightarrow \infty} \overline{\mathcal{O}}_r = k_r^{\mathcal{O}} + \mathcal{O}(\varepsilon^p)$

[$p \equiv$ order integration scheme]

IMPORTANT: NO accept-reject step: **NOT** smooth in $g_0!$

\Rightarrow inexact algorithm and step-size errors

Stochastic molecular dynamics algorithm

The Horowitz way

(Horowitz '85, '87, '91; Jansen, Liu '95)

Stochastic molecular dynamics (SMD)

$$\partial_t \phi(t, x) = \pi(t, x)$$

$$\partial_t \pi(t, x) = -\frac{\delta S(\phi)}{\delta \phi(t, x)} - \gamma \pi(t, x) + \eta(t, x)$$

$$\langle \eta(t, x) \eta(s, y) \rangle_\eta = 2\gamma \delta_{xy} \delta(t - s)$$

- ▶ Adjustable “friction” parameter $\gamma > 0$
- ▶ Coincides with Langevin equation for $\gamma \rightarrow \infty$
- ▶ On the lattice this limit is also reached if $\hat{\gamma} \equiv a\gamma = \text{const.}$

(Lüscher, Schaefer '11)

In this case:

(MDB, Garofalo, Kennedy '17; MDB, Lüscher '17)

- ▶ **Autocorrelations:** $\tau_{\text{int}}(\mathcal{O}_r) \stackrel{a \rightarrow 0}{\approx} c_{\mathcal{O}_r}(\hat{\gamma})/a^2$

- ▶ **Variances:** $\text{var}(\mathcal{O}_r) \stackrel{a \rightarrow 0}{\approx} d_{\mathcal{O}_r}(\hat{\gamma}) \ln(a)$

Phase-space stochastic quantization

$$\lim_{t \rightarrow \infty} \langle \phi(t, x_1) \dots \phi(t, x_n) \rangle_\eta = \langle \varphi(x_1) \dots \varphi(x_n) \rangle$$

$$P_{\text{eq}}(\pi, \phi) \propto e^{-H(\pi, \phi)}, \quad H(\pi, \phi) = \frac{1}{2}(\pi, \pi) + S(\phi)$$

SMD based NSPT

(MDB, Garofalo, Kennedy '17; MDB, Lüscher '17)

The algorithm in a nutshell

Start:

$$t \rightarrow t_n = n\epsilon, \quad n \in \mathbb{N}, \quad \text{and} \quad \phi = \{\phi_0, \dots, \phi_N\}, \quad \pi = \{\pi_0, \dots, \pi_N\}$$

Step 1: Momentum rotation

$$\pi_r \rightarrow c_1 \pi_r + c_2 \delta_{r0} v, \quad c_1 = e^{-\gamma\epsilon}, \quad c_1^2 + c_2^2 = 1$$

$$\langle v(x)v(y) \rangle_v = \delta_{xy},$$

Step 2: Molecular dynamics step

$$\left. \begin{aligned} I_{\pi_r, h} : \pi_r &\rightarrow \pi_r - hF_r(\phi) \\ I_{\phi_r, h} : \phi_r &\rightarrow \phi_r + h\pi_r \end{aligned} \right\} \begin{array}{l} \text{Symplectic reversible int.} \\ t_n \rightarrow t_{n+1} \end{array}$$

$$\text{Ex. LPF: } I_\epsilon = I_{\pi, \epsilon/2} I_{\phi, \epsilon} I_{\pi, \epsilon/2}$$

Step 1 \rightarrow Step 2 \rightarrow Step 1 $\rightarrow \dots \rightarrow$ Step 2

Step 3: Do not forget to measure

$$\overline{\mathcal{O}}_r = \frac{1}{N_{\text{cnfg}}} \sum_{n=0}^{N_{\text{cnfg}}} \mathcal{O}_r(\phi_0(t_n), \dots, \phi_r(t_n))$$

SMD based NSPT

(MDB, Lüscher '17)

Convergence to a unique and stationary distribution

The discrete stochastic processes $\{\phi_r(t_n), \pi_r(t_n)\}_{r=0,1,\dots,N}$, all converge for $t_n \rightarrow \infty$ to a **unique** and **stationary** distribution iff:

1. $\Delta > 0$
2. $\epsilon^2 \|\Delta\| < \kappa$

$$S_0(\phi_0) = \frac{1}{2}(\phi_0, \Delta\phi_0)$$

$$\|\Delta\| \equiv \lambda_{\max} \approx 16$$

Leading-order distros [$r = 0$ - free theory]

$$\hat{P}_{\text{eq}}(\pi_0, \phi_0) \propto e^{-\hat{H}_0(\pi_0, \phi_0)}$$

$$\hat{H}_0(\pi_0, \phi_0) = \frac{1}{2}(\pi_0, \pi_0) + \hat{S}_0(\phi_0), \quad \hat{S}_0(\phi_0) = \frac{1}{2}(\phi_0, \hat{\Delta}\phi_0)$$

Examples

(Omelyan, Mryglod, Folk '03)

- ▶ $\hat{\Delta}_{\text{L呢}} = \Delta(1 + a_0 \epsilon^2 \Delta)$ $a_0 = -2.5 \times 10^{-1}$, $\kappa = 4$
- ▶ $\hat{\Delta}_{\text{OMF2}} = \Delta(1 + a_1 \epsilon^2 \Delta + \dots)$ $a_1 \approx -2.5 \times 10^{-3}$, $\kappa = 6.51$
- ▶ $\hat{\Delta}_{\text{OMF4}} = \Delta(1 + a_2 \epsilon^4 \Delta^2 + \dots)$ $a_2 \approx -2.6 \times 10^{-5}$, $\kappa = 9.87$

Effectiveness

$$\|1 - \hat{\Delta}_{\text{L呢}}/\Delta\| \approx 25 \|1 - \hat{\Delta}_{\text{OMF2}}/\Delta\| \approx 300 \|1 - \hat{\Delta}_{\text{OMF4}}/\Delta\| \quad [@ \text{ fixed cost per MDU }]$$

NSPT in lattice QCD

SU(3) Yang-Mills theory

(MDB, Lüscher '17)

SMD equations

$$\partial_t U_t(x, \mu) = g_0 \pi_t(x, \mu) U_t(x, \mu)$$

$$\partial_t \pi_t(x, \mu) = -g_0 (\partial_{x,\mu}^a S_G)(U_t) T^a - \gamma \pi_t(x, \mu) + \eta_t(x, \mu)$$

$$\langle \eta_t^a(x, \mu) \eta_s^b(y, \nu) \rangle = 2\gamma \delta^{ab} \delta_{\mu\nu} \delta_{xy} \delta(t-s)$$

Perturbative fields

$$U = e^{g_0 A} = 1 + g_0 U_0 + g_0^2 U_1 + \dots, \quad \pi = \pi_0 + g_0 \pi_1 + \dots$$

Step 1. Momentum rotation

$$\pi \rightarrow c_1 \pi + c_2 v$$

Step 2. Molecular dynamics step

$$\left. \begin{aligned} I_{\pi,h} : \pi &\rightarrow \pi - hg_0 \partial S_G(U) \\ I_{U,h} : U &\rightarrow e^{hg_0 \pi} U \end{aligned} \right\} \text{Symplectic reversible int.} \quad t_n \rightarrow t_{n+1}$$

Step 3. Gauge damping ...

NOTE: Convergence to a unique and stationary distribution requires care due to the **gauge modes**, but it can **rigorously** be guaranteed!

The gradient flow coupling

A non-trivial case study

Gradient flow (Lüscher '10)

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu\mu}(t, x) \quad [t \equiv \text{flow time}]$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], \quad B_\mu(0, x) = A_\mu(x)$$

Schrödinger functional (Lüscher et. al. '92)

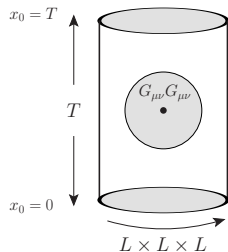
$$A_k(x)|_{x_0=0} = A_k(x)|_{x_0=T} = 0$$

$$A_\mu(x + \hat{k}L) = A_\mu(x)$$

Basic quantity (Lüscher '10; Fritsch, Ramos '13)

$$t^2 \langle E(t, x) \rangle, \quad E(t, x) = \frac{1}{4} G_{\mu\nu}^a(t, x) G_{\mu\nu}^a(t, x)$$

$$T = L, \quad x_0 = T/2, \quad \sqrt{8t} = 0.3 \times L$$



NSPT observables

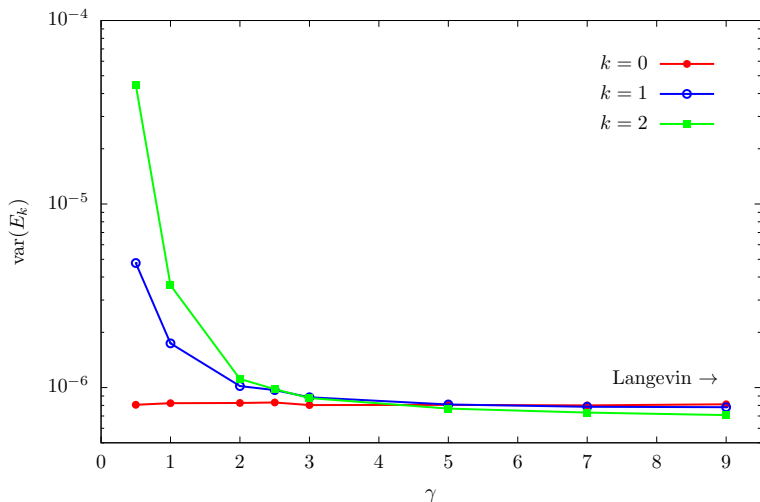
[using $\alpha_{\overline{\text{MS}}} \leftrightarrow g_0$ (Lüscher, Weisz '95)]

$$t^2 \langle E(t, x) \rangle = g_0^2 E_0(L/a) + g_0^4 E_1(L/a) + g_0^6 E_2(L/a) + \dots$$

$$\stackrel{a/L \rightarrow 0}{=} k_0 \{ \alpha_{\overline{\text{MS}}}(q) + k_1 \alpha_{\overline{\text{MS}}}(q)^2 + k_2 \alpha_{\overline{\text{MS}}}(q)^3 + \dots \}, \quad q = 1/\sqrt{8t}$$

Algorithm-dependence of variances

Variances $\text{var}(E_k)$ as a function of γ for $L/a = 16$

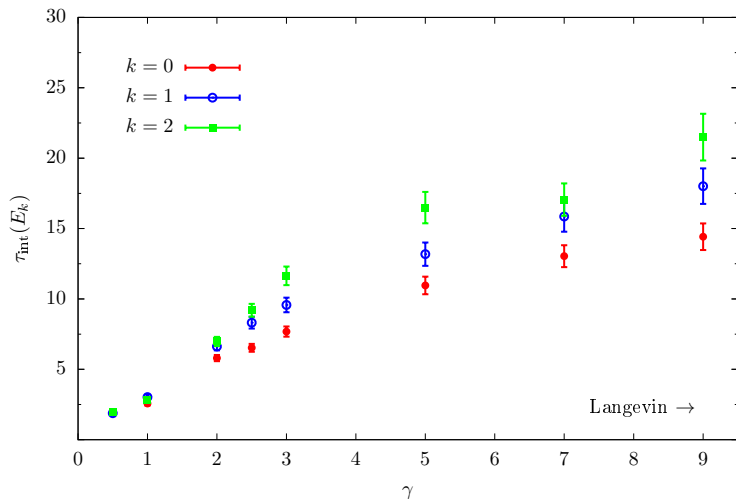


NOTE:

$\text{var}(E_{k>0}) \neq t^4 [\langle E^2 \rangle - \langle E \rangle^2] \Big|_{g_0^{4(k+1)}} \Rightarrow \text{var}(E_{1,2})$ are **algorithm-dependent!**

Autocorrelations

Integrated autocorrelations $\tau_{\text{int}}(E_k)$ (in MDUs) as a function of γ for $L/a = 16$

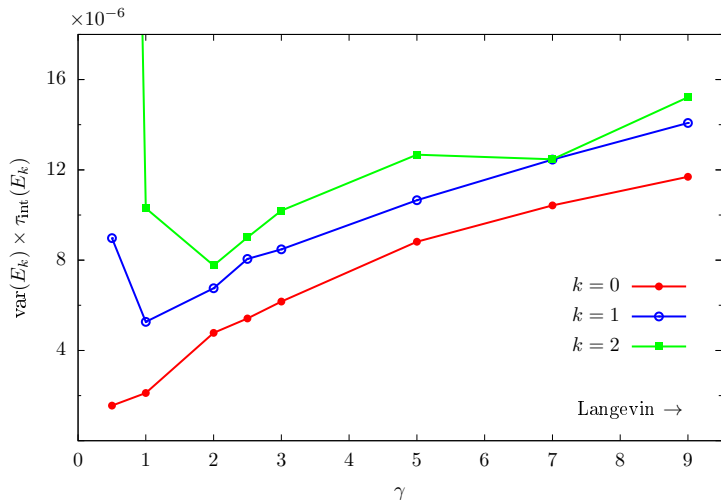


NOTE:

The autocorrelations of the different orders appear to have similar γ -dependence

Statistical errors

Products $\text{var}(E_k) \times \tau_{\text{int}}(E_k)$ as a function of γ for $L/a = 16$

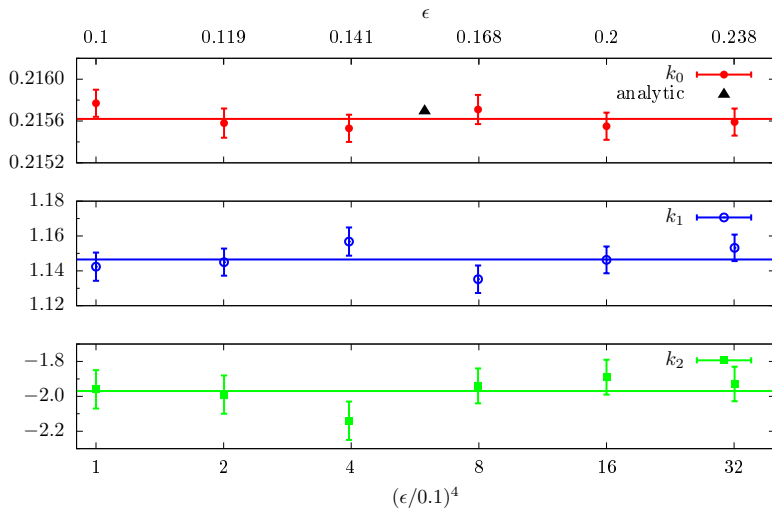


NOTE:

Algorithms tuned for **small autocorrelations** tend to have **large variances** and vice-versa

Systematic errors

Coefficients k_0, k_1, k_2 for $L/a = 24$, OMF4 integrator, 10k meas. per ϵ



Simulation parameters

... and other info

Lattice sizes

$$10 \leq L/a \leq 40$$

Step sizes

- ▶ $L/a \leq 20$: $\epsilon = 0.168$
- ▶ $L/a \geq 24$: $\epsilon = 0.168 \times 24 \times (a/L)$
⇒ Step size errors $\propto (a/L)^4$ w/ OMF4

Gamma

- ▶ $L/a \leq 20$: $\gamma = 4 - 5$
- ▶ $L/a \geq 24$: $\gamma = 3$
⇒ Langevin scaling (eventually)

Autocorrelations

$L/a = 24 \rightarrow 40$: $\tau_{\text{int}}(E_k)$ increased 33% less than $1/a^2$

Variances

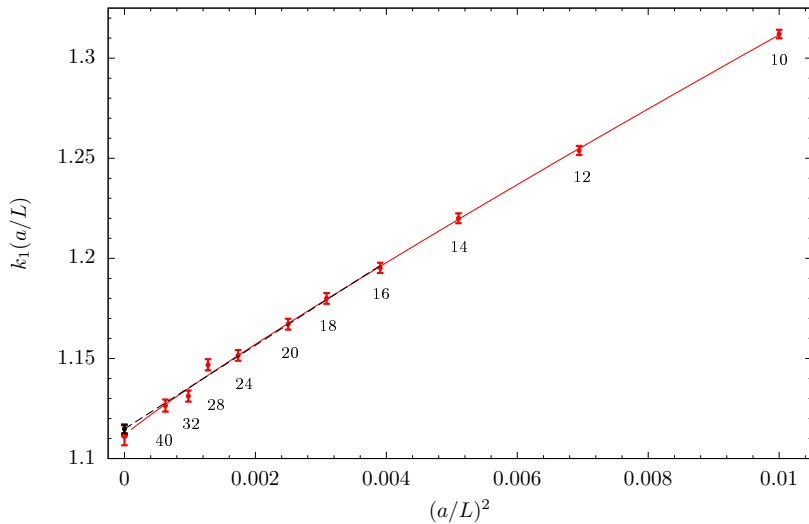
$L/a = 24 \rightarrow 40$: $\text{var}(E_1) : +19\%$, $\text{var}(E_2) : +30\%$

Measurements

60k – 80k

Continuum limit of k_1

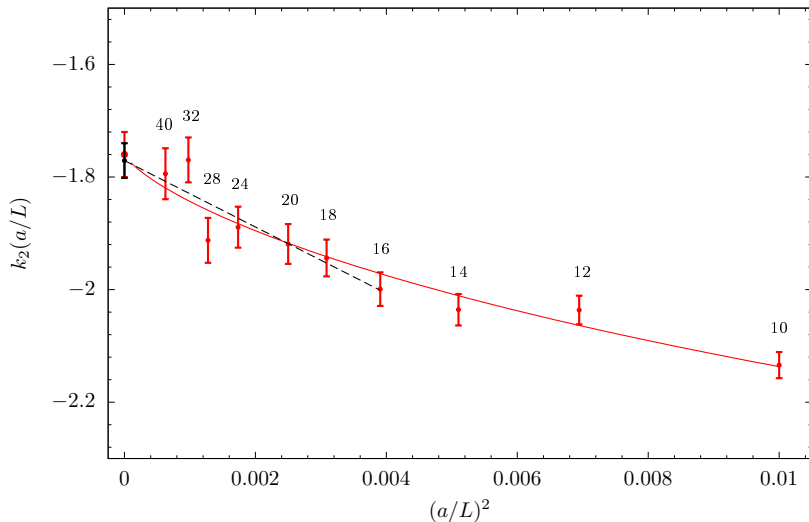
$O(a)$ -improved data, Wilson action, Wilson flow, E_{plaq}



$$k_1 \stackrel{a/L \rightarrow 0}{=} a_0 + \{a_1 + b_1 \ln(L/a)\} (a/L)^2 + \dots$$

Continuum limit of k_2

$O(a)$ -improved data, Wilson action, Wilson flow, E_{plaq}



$$k_2 \stackrel{a/L \rightarrow 0}{=} a_0 + \{a_1 + b_1 \ln(L/a) + c_1 (\ln(L/a))^2\} (a/L)^2 + \dots$$

The gradient flow coupling and its family

...are they all nice?

More schemes

$$\alpha(\mu) \propto \begin{cases} t^2 \langle G_{\mu\nu} G_{\mu\nu} \rangle \\ t^2 \langle G_{kl} G_{kl} \rangle & s \equiv \text{spatial} \\ t^2 \langle G_{0k} G_{0k} \rangle & t \equiv \text{temporal} \end{cases} \quad \mu^{-1} = \sqrt{8t} = cL$$

Parametric uncertainty

$$\begin{aligned} \delta k_1 &\approx 5 \times 10^{-3} \\ \delta k_2 &\approx 5 \times 10^{-2} \end{aligned} \quad \Rightarrow \quad \frac{\delta \alpha_{\overline{\text{MS}}}(M_Z)}{\alpha_{\overline{\text{MS}}}(M_Z)} \approx 0.1\%$$

Who runs faster?

$$L \frac{\partial \alpha}{\partial L} = \beta_0 \alpha^2 + \beta_1 \alpha^3 + \beta_2 \alpha^4 + \dots$$

$$\beta_0 = \frac{11}{2\pi}, \quad \beta_1 = \frac{51}{(2\pi)^2}$$

c	β_2/β_0	$\beta_{2,s}/\beta_0$	$\beta_{2,t}/\beta_0$
0.4	-4.88(6)	-8.04(7)	-2.21(8)
0.3	-2.99(4)	-3.74(5)	-2.29(6)
0.2	-2.38(6)	-2.51(6)	-2.22(7)
0.0	-2.1751(3)	-2.1751(3)	-2.1751(3)

Conclusions & outlook

Conclusions

- ▶ NSPT is a **powerful** tool for automatizing LPT calculations
- ▶ Complicated lattice set-ups and observables can **easily** be considered
- ▶ The recent algorithmic advances opened the way for **precise** and **accurate** results
- ▶ A full implementation for SU(3) gauge theory can be downloaded at:

luscher.web.cern.ch/luscher/NSPT

Outlook

- ▶ The inclusion of fermions is in principle straightforward and does not pose any technical difficulty
- ▶ Fermions are not expected to slow down the simulations by a big factor

(MDB, Lüscher '17)

(Di Renzo, Scorzato '04)



BACKUP

The gradient flow coupling

Lattice formulation

Wilson flow

(Lüscher '10)

$$\partial_t V_t(x, \mu) V_t(x, \mu)^{-1} = -g_0^2 (\partial_{x,\mu}^a S_G)(V_t) T^a, \quad V_{t=0}(x, \mu) = U(x, \mu)$$

Schrödinger functional

$$U(x, k)|_{x_0=0} = 1 = U(x, k)|_{x_0=T} \quad \Rightarrow \quad U_r(x, k)|_{x_0=0, T} = 0$$

Perturbative solution

(MDB, Hesse '13)

$$V_t = e^{g_0 B_\mu(t)} = 1 + g_0 V_{t,0} + g_0^2 V_{t,1} + \dots, \quad V_{t,r} = U_r$$

$\mathcal{O}(a)$ action counterterm

(Lüscher et. al. '92; Bode, Wolff, Weisz '98)

$$\propto c_t \sum_x F_{0k}^a(x) F_{0k}^a(x)|_{x_0=0, T}, \quad c_t = 1 - 0.08900 g_0^2 - 0.0294 g_0^2 + \dots$$

Basic observable

$$\langle E(t, x) \rangle = g_0^2 E_0(L/a) + g_0^4 E_1(L/a) + g_0^6 E_2(L/a) + \dots$$

Gradient flow coupling

(using $\alpha_{\overline{\text{MS}}} \leftrightarrow g_0$ (Lüscher, Weisz '95))

$$t^2 \langle E(t, x) \rangle = k_0(a/L) \{ \alpha_{\overline{\text{MS}}}(q) + k_1(a/L) \alpha_{\overline{\text{MS}}}(q)^2 + k_2(a/L) \alpha_{\overline{\text{MS}}}(q)^3 + \dots \}$$

$$\lim_{a/L \rightarrow 0} k_i(a/L) = k_i$$

NSPT in lattice QCD

Convergence to a unique and stationary distribution

Leading order dynamics

$$S_G = S_0 + g_0 S_1 + g_0^2 S_2 + \dots$$

$$S_0 = \frac{1}{2} \sum_{x,y} A_\mu(x) \Delta_{\mu\nu}(x,y) A_\nu(y), \quad A_\mu(x) = U_0(x, \mu)$$

PROBLEM: $\Delta \geq 0$

1. Gauge modes $A_\mu(x) \sim 0$
2. In a finite volume w/ periodic bc. $\exists A_\mu(x) \not\sim 0$ s.t. $\Delta A = 0$

(Gonzalez-Arroyo, Jurkiewicz, Korthals-Altes '83)

N.B.: Issues of the perturbative expansion not (N)SPT!

SOLUTION:

1. Gauge damping aka stochastic gauge fixing
2. Choose proper boundary conditions for the fields
e.g. $A_k(x)|_{x_0=0} = A_k(x)|_{x_0=T} = 0$

(Zwanziger '81)

RESULT: Separate convergence criteria for the gauge modes

(MDB, Lüscher '17)

Gauge damped SMD equations

The Zwanziger way

Time-dependent gauge transf.

$$\pi_t(x, \mu) \rightarrow \Lambda_t(x) \pi_t(x, \mu) \Lambda_t(x)^{-1}$$

$$U_t(x, \mu) \rightarrow \Lambda_t(x) U_t(x, \mu) \Lambda_t(x + \hat{\mu})^{-1}, \quad \Lambda_t(x) \in \text{SU}(3)$$

Modified SMD equations

$$\partial_t U_t(x, \mu) = g_0 \{ \pi_t(x, \mu) - \nabla_\mu \omega_t(x) \} U_t(x, \mu)$$

$$\partial_t \pi_t(x, \mu) = -g_0 (\partial_{x, \mu}^a S_G)(U_t) T^a - \gamma \pi_t(x, \mu) + \eta_t(x, \mu) + g_0 [\omega_t(x), \pi_t(x, \mu)]$$

$$\langle \eta_t^a(x, \mu) \eta_s^b(y, \nu) \rangle = 2\gamma \delta^{ab} \delta_{\mu\nu} \delta_{xy} \delta(t - s)$$

with

$$\omega_t(x) = g_0^{-1} \partial_t \Lambda_t(x) \Lambda_t(x)^{-1}$$

$$\nabla_\mu \omega_t(x) = U_t(x, \mu) \omega_t(x + \hat{\mu}) U_t(x, \mu)^{-1} - \omega_t(x)$$

Gauge damping

$$(\partial_t + \gamma) \omega_t(x) = \lambda (d^* C_t)(x), \quad \lambda > 0$$

$$C_t(x, \mu) = \frac{1}{2g_0} \left\{ U_t(x, \mu) - U_t(x, \mu)^{-1} - \frac{1}{3} \text{tr} \left[U_t(x, \mu) - U_t(x, \mu)^{-1} \right] \right\}$$

NSPT in lattice QCD

Gauge damping

Gauge transformation

$$\pi(x, \mu) \rightarrow \Lambda(x)\pi(x, \mu)\Lambda(x)^{-1}, \quad U(x, \mu) \rightarrow \Lambda(x)U(x, \mu)\Lambda(x + \hat{\mu})^{-1}$$

Gauge damping field

$$\Lambda(x) = e^{\epsilon g_0 \omega(x)} \in \text{SU}(3), \quad \omega(x) \in \mathfrak{su}(3), \quad \omega(x) = \omega_0 + g_0 \omega_1(x) + \dots$$

Gauge-damping dynamics @ leading-order [$L = T = \infty$]

1. $\omega_0(x) \rightarrow c_1 \omega_0(x)$ $c_1 = e^{-\epsilon \gamma}$
2. $\omega_0(x) \rightarrow \omega_0(x) + \epsilon \lambda (d^* A)(x)$ $(d^* A)(x) = -\partial_\mu^* A_\mu(x)$
3. $A_\mu(x) \rightarrow A_\mu(x) - \epsilon (d\omega_0)_\mu(x)$ $(d\omega_0)_\mu(x) = \partial_\mu \omega_0(x)$

$\Rightarrow \omega(x)$ assures a **restoring force** for the (longitudinal) **gauge modes!**

Convergence criteria

1. $\Delta|_{AT} > 0$
2. $\epsilon^2 \|\Delta\| < \kappa$
3. $\epsilon^2 \lambda \|dd^*\| < 2(1 + c_1)$ **(N.B.:** $\|dd^*\| \leq 16$)