Lattice string sigma model

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Anti-de Sitter / Conformal Field Theory correspondence

- **conjectured** equivalence of two models
- complementary perturbative expansions
- prospect: strong coupling results from dual weak coupling computation

$\mathcal{N}=4$ super Yang-Mills in 4d with coupling g_{YM} and gauge group $SU(N)$	Type IIB superstring theory in $AdS_5 \times S^5$ with coupling g_s and N units of Ramond-Ramond five-form flux
't Hooft limit	no joining and splitting of strings
$g_{\mathrm{YM}} \to 0,\ N \to \infty$	$g_s \to 0,\ N \to \infty$
$\lambda = g_{\mathrm{YM}}^2 N = \text{constant}$	$\lambda = 4\pi N g_s = \text{constant}$
expansion for *small* λ	expansion for *large* λ

$\mathcal{N} = 4$ Super Yang-Mills (SYM)

- believed to be exactly integrable (all-loop/quantum integrability)
- non-perturbative (lattice) definition possible (Catterall et al.)
- problems?

“strings”

- objects spatially extended in one dimension
- embedded in higher-dimensional target space ($AdS_5 \times S^5$)
- time evolution: sweep out two-dimensional surface (worldsheet)
- $N \to \infty$ means neither holes nor handles

Field content

- fields defined on 2d worldsheet (coordinates t, s)
- $X(t,s)$: 8 bosonic embedding coordinates of the worldsheet into target space ($10 - 2 = 8$)
- $\Psi(t,s)$: 16 anti-commuting degrees of freedom (fermionic)

Quantization

- fluctuations around a classical solution $X_{\mathrm{cl}}, \Psi = 0$
- large symmetry needs to be partly fixed (κ-symmetry, diffeomorphism invariance)

Correspondence

Wilson loop expectation value equals string partition function

$$\langle W[\mathcal{C}]\rangle = Z_{\mathrm{string}} \equiv \int DX\ D\Psi\ \mathrm{e}^{-S[X_{\mathrm{cl}}+X,\Psi]}$$

where the string embedding ends on contour $\mathcal{C}$

Outline

1. **Continuum**
   - Action, Symmetries
   - Perturbation Theory Results
   - All Order Results
2. **Lattice Discretization**
   - Four-Fermion Interactions $\rightarrow$ Auxiliary Fields
   - Naive Discretization
   - Lattice Perturbation Theory / Wilson Term
   - Spectrum of the Fermionic Operator
3. **Monte-Carlo Simulations**
   - Observables
   - Sign Problem
   - Symmetry Breaking

$$S_{\text{cusp}} = g \int \mathrm{d}t \int \mathrm{d}s \, \mathcal{L}_{\text{cusp}} \,, \quad g = \frac{\sqrt{\lambda}}{4\pi}$$

$$\mathcal{L}_{\text{cusp}} = |\partial_t x + \tfrac{1}{2} m x|^2 + \tfrac{1}{z^4} |\partial_s x - \tfrac{1}{2} m x|^2 + \left(\partial_t z^M + \tfrac{1}{2} m z^M + \tfrac{i}{z^2} z_N \eta_i \left(\rho^{MN}\right)^i{}_j \eta^j \right)^2$$
$$+ \frac{1}{z^4} \left(\partial_s z^M - \tfrac{1}{2} m z^M \right)^2 + i \left(\theta^i \partial_t \theta_i + \eta^i \partial_t \eta_i + \theta_i \partial_t \theta^i + \eta_i \partial_t \eta^i \right) - \tfrac{1}{z^2} \left(\eta^i \eta_i \right)^2$$
$$+ \tfrac{2i}{z^3} z^M \eta^i \left(\rho^M \right)_{ij} \left(\partial_s \theta^j - \tfrac{1}{2} m \theta^j - \tfrac{i}{z} \eta^j \left(\partial_s x - \tfrac{1}{2} m x \right) \right)$$
$$+ \tfrac{2i}{z^3} z^M \eta_i (\rho_M^\dagger)^{ij} \left(\partial_s \theta_j - \tfrac{1}{2} m \theta_j + \tfrac{i}{z} \eta_j \left(\partial_s x - \tfrac{1}{2} m x \right)^* \right)$$

- bosons: x, x^*, z^M with $M = 1, \dots, 6$ and $z = \sqrt{z_M z^M}$
- fermions: θ_i, η_i with $i = 1, 2, 3, 4$ are complex Grassmann ($\theta^i = \theta_i^\dagger$)
- ρ^M are off-diagonal blocks of $SO(6)$ Dirac matrices γ^M
- $\left(\rho^{MN}\right)^i{}_j$ are the $SO(6)$ generators

Continuum Symmetries

Two global symmetries:

1. $SU(4) \sim SO(6)$ symmetry
   - the fields z^M transform in the vector representation
   - the fermions transform in the (anti) fundamental representation
   - x, x^* are neutral
2. $U(1) \sim SO(2)$ symmetry
   - x, x^* transform (charge 1 and -1)
   - fermions η_i, θ^i have charge $\frac{1}{2}$, η^i, θ_i have charge $-\frac{1}{2}$
   - z^M are neutral

Continuum PT

- Classical solution is homogenous: all fields zero, except $z = 1$
- Perturbation theory is done by picking a direction in z^M-space, e.g. $u^M = (0, 0, 0, 0, 0, 1)$
- Define new fluctuation fields that vanish on the vacuum
- No infinite renormalization, i.e., no logarithms
- β-functions zero

Effective Action / Scaling function

$$\langle W[\mathcal{C}]\rangle = Z_{\mathrm{string}} \equiv \mathrm{e}^{-\Gamma} = \mathrm{e}^{-\frac{Vm^2}{8}f(g)}$$

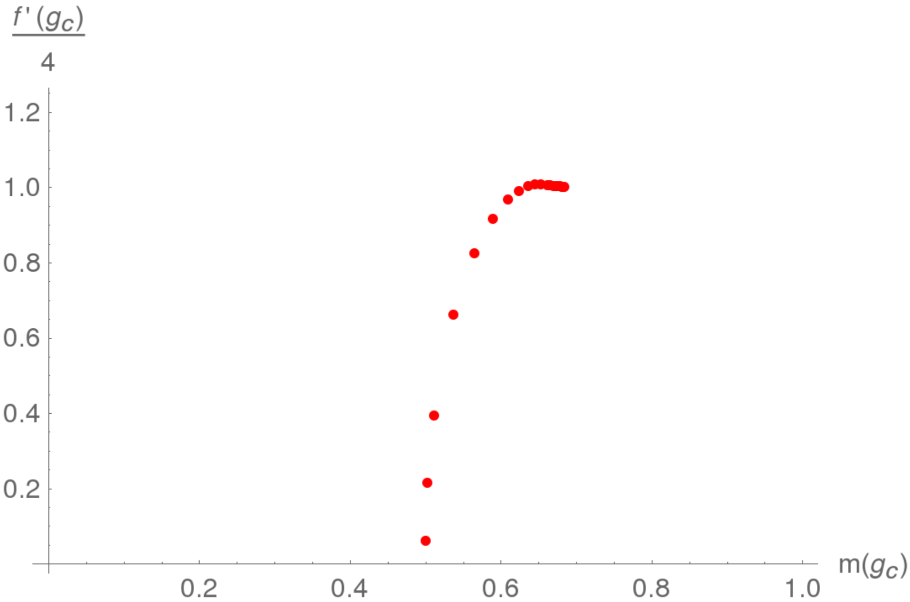
$$f(g) = 4g - \frac{3\log(2)}{\pi} - \frac{K}{4\pi g} + O(g^{-2})$$

Mass spectrum

- Free bosons: one field with m^2, two with $\frac{1}{2}m^2$ and five massless
- Fermions: all $\frac{1}{2}m$
- Corrections to the mass of x, x^*:

$$m_x(g) = \frac{1}{2}m^2\left(1 - \frac{1}{8g} + O(g^{-2}\right)$$

- Bethe ansatz techniques based on assumed quantum integrability of $\mathcal{N} = 4$ SYM
- Lead to Beisert-Eden-Staudacher (BES) integral equation
- Numerical computation of $f(g)$ and $m_x(g)$ at finite g



Lattice Discretization

1. Discretize Euclidean spacetime
2. Integrate out Grassmann fields (Pfaffian)
3. Exponentiate using pseudo-fermions
4. Use Hybrid Monte-Carlo (HMC) to estimate expectation values

- Bosons and fermions defined on $N = T/a \times L/a$ sites
- Discretization of purely bosonic part straightforward
- Four-fermion Interactions: in a minute
- Quadratic fermion action: $\mathcal{L}_2 = \Psi^T O_F \Psi$ mit $\Psi \equiv (\theta^i, \theta_i, \eta^i, \eta_i)$

$$O_F = \begin{pmatrix} 0 & -i\Delta_0 & (i\Delta_1 - i\frac{m}{2})\rho^M \frac{z^M}{z^3} & 0 \\ -i\Delta_0 & 0 & 0 & \rho^\dagger_M (i\Delta_1 - i\frac{m}{2})\frac{z^M}{z^3} \\ -(i\Delta_1 + i\frac{m}{2})\rho^M \frac{z^M}{z^3} & 0 & 2\frac{z^M}{z^4}\rho^M \left(\partial_s x - m\frac{x}{2}\right) & -i\Delta_0 \\ 0 & -\rho^\dagger_M (i\Delta_1 + i\frac{m}{2})\frac{z^M}{z^3} & -i\Delta_0 & -2\frac{z^M}{z^4}\rho^\dagger_M \left(\partial_s x^* - m\frac{x}{2}^*\right) \end{pmatrix}$$

4×4 matrix of $(4N) \times (4N)$ blocks

$$\mathcal{L}_4 = \frac{1}{z^2}\left[-(\eta^2)^2 + \left(i\,\eta_i(\rho^{MN})^i{}_j \frac{z^N}{z}\eta^j\right)^2\right]$$

Hubbard-Stratonovich:

$$\exp\left\{-g\int \mathrm{dtds}\,\mathcal{L}_4\right\} \sim \int D\phi D\phi^M \exp\left\{-g\int \mathrm{dtds}\,\mathcal{L}_{aux}\right\}$$

$$\mathcal{L}_{aux} = \tfrac{1}{2}\phi^2 + \tfrac{\sqrt{2}}{z}\phi\,\eta^2 + \tfrac{1}{2}(\phi_M)^2 - i\,\tfrac{\sqrt{2}}{z^2}\phi^M z_N\left(i\,\eta_i {\rho^{MN}}^i{}_j\eta^j\right)$$

Note:

- Last term is non-hermitian, resulting in a complex Pfaffian
- 7 auxiliary fields: ϕ and ϕ^M
- Additional terms in the bosonic action and the fermion opertor O_F

$$O_F = \begin{pmatrix} 0 & -i\Delta_0 & (i\Delta_1 - i\frac{m}{2})\rho^M \frac{z^M}{z^3} & 0 \\ -i\Delta_0 & 0 & 0 & \rho_M^\dagger(i\Delta_1 - i\frac{m}{2})\frac{z^M}{z^3} \\ -(i\Delta_1 + i\frac{m}{2})\rho^M \frac{z^M}{z^3} & 0 & 2\frac{z^M}{z^4}\rho^M\left(\partial_s x - m\frac{x}{2}\right) & -i\Delta_0 - A^T \\ 0 & -\rho_M^\dagger(i\Delta_1 + i\frac{m}{2})\frac{z^M}{z^3} & -i\Delta_0 + A & -2\frac{z^M}{z^4}\rho_M^\dagger\left(\partial_s x^* - m\frac{x}{2}^*\right) \end{pmatrix}$$

Grassmann integral yields Pfaffian:

$$\int D\Psi \; e^{-\int \mathrm{d}t\mathrm{d}s\, \Psi^T O_F \Psi} = \mathrm{Pf}\, O_F \longrightarrow (\det O_F\, O_F^\dagger)^{\frac{1}{4}}$$

Last step ignores possible complex phase

- skew-symmetric matrix, i.e., eigenvalue pairs λ, $-\lambda$
- if $A^\dagger = A$ then O_F satisfies

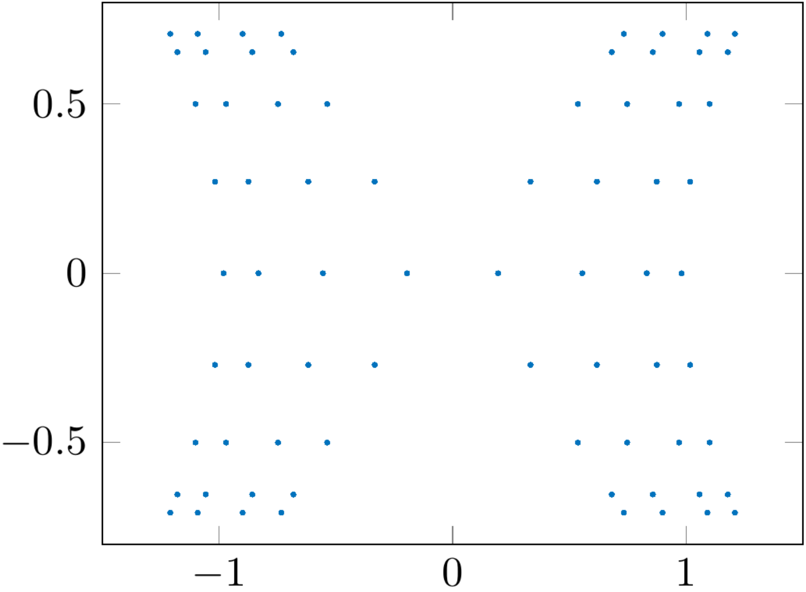
$$O_F^\dagger = \Gamma_5 O_F \Gamma_5$$

with

$$\Gamma_5 = \begin{pmatrix} 0 & \mathbb{1} & 0 & 0 \\ -\mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} \\ 0 & 0 & -\mathbb{1} & 0 \end{pmatrix} \,, \quad \Gamma_5^\dagger \Gamma_5 = \mathbb{1} \,, \quad \Gamma_5^\dagger = -\Gamma_5$$

- therefore eigenvalue pairs λ, $-\lambda^*$
- general complex eigenvalue quartets $(\lambda, -\lambda^*, -\lambda, \lambda^*)$
- Pfaffian is real, but sign flips may occur
- during HMC: only if eigenvalue goes through zero

Spectrum of free fermion operator, $L/a = T/a = 16$, $am = 0$, $u^M = (1,1,1,1,1,1)/\sqrt{6}$



- A in O_F is **not** hermitian and therefore Pfaffian complex
- Rewrite four-fermion Lagrangian: Given ${\Sigma_i}^j \equiv \eta_i \eta^j$ and $n^M = \frac{z^M}{z}$ define dual matrix ${\tilde{\Sigma}_j}^{\ i} = n_N n_L (\rho^N)^{ik} (\rho^L)_{jl} {\Sigma_k}^l$
- Then $\mathcal{L}_4 = \frac{1}{z^2}\left(-6\,(\eta^2)^2 - {\Sigma_+}_i^j {\Sigma_+}_j^i\right)$ with $\Sigma_\pm = \Sigma \pm \tilde{\Sigma}$
- And

$$\mathcal{L}_{aux} = \frac{12}{z}\eta^2\phi + 6\phi^2 + \frac{2}{z}{\Sigma_+}_j^i \phi_i^j + \phi_j^i \phi_i^j$$

- ϕ_i^j are 4×4 complex hermitian matrices
- in total 17 auxiliary (real) fields
- yields $A^\dagger = A$

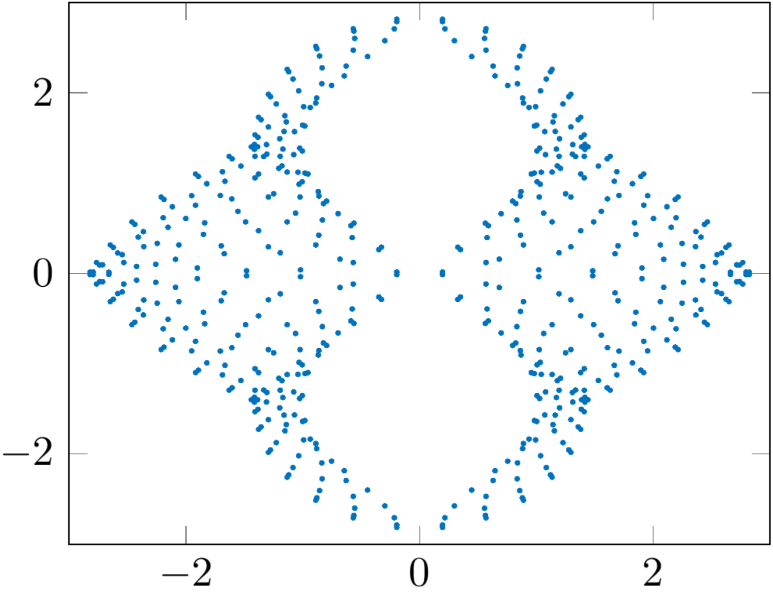
Wilson Term

- naive discretization of fermions has doublers as can be seen in the free propagator and in the 1-loop computation of the effective action
- need to add a Wilson term to action: $O_F + W$
  - mass dimension 3 operator (formally vanishes in continuum limit)
  - respect as much of the continuum symmetries as possible ($SO(6)$ and $U(1)$)
  - respect Γ_5-hermiticty (no complex phase)

$$W = \begin{pmatrix} W_+ & & & \\ & -W_+^\dagger & & \\ & & W_- & \\ & & & -W_-^\dagger \end{pmatrix} , \quad W_\pm = \frac{ra}{4}\left[(\Box_0 \pm i\Box_1)\, \rho^M \frac{z^M}{z^2} + \rho^M \frac{z^M}{z^2} (\Box_0 \pm i\Box_1) \right]$$

... breaks $U(1)$

Spectrum of free fermion operator, $L/a = T/a = 16$, $am = 0$, $u^M = (1, 1, 1, 1, 1, 1)/\sqrt{6}$



Monte-Carlo Simulations

1. Line of constant physics
2. Observables
3. Algorithms

- two bare parameters: dimensionless coupling g, mass scale m
- in the continuum: no infinite renormalization, β-functions vanish
- therefore continuum limit at **fixed** bare g
- also masses can be fixed by

$$L^2 m_x = \mathrm{const} \quad \rightarrow \quad L^2 m^2 = \mathrm{const}$$

- observables are then expected as

$$F_{LAT}(g, L/a, am) = F(g) + O(a/L) + O(\mathrm{e}^{-Lm})$$

- choose $Lm \leq 4$ to control finite volume effects

Effective Action / Scaling Function

$$Z \equiv \mathrm{e}^{-\Gamma} = \mathrm{e}^{-\frac{Vm^2}{8} f(g)}$$

- can be accessed via a derivative of the partition function Z

$$-g\frac{\mathrm{d}\ln Z}{\mathrm{d}g} = g\frac{Vm^2}{8} f'(g) \quad \text{or} \quad -m\frac{\mathrm{d}\ln Z}{\mathrm{d}m} = \frac{Vm^2}{4} f(g)$$

- first choice leads to the expectation value of the bosonic action
- second choice needs evaluation of trace of O_F^{-1}

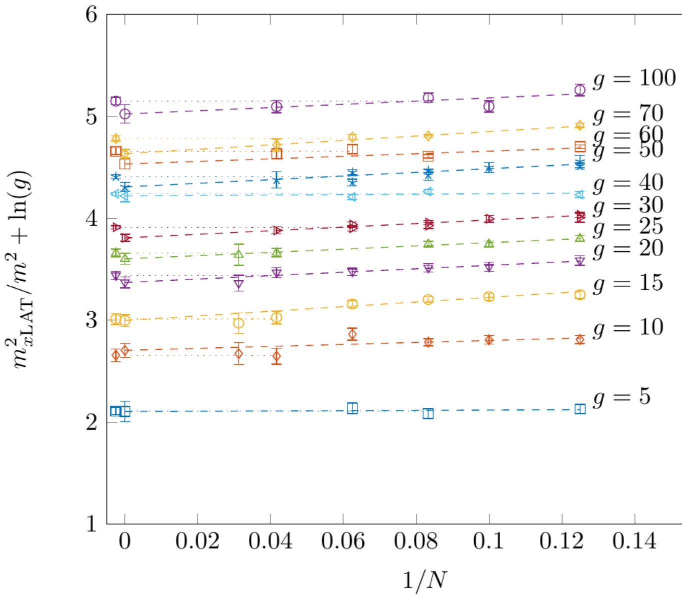
Masses

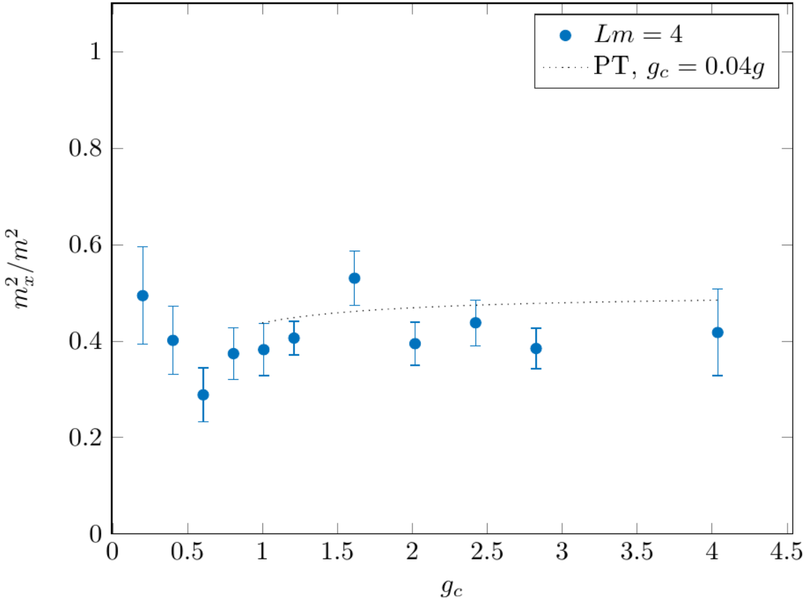
- from large time behavior of connected two-point functions
- done for mass of the x, x^*
- also possible for fermions (no gauge invariance, in progress)

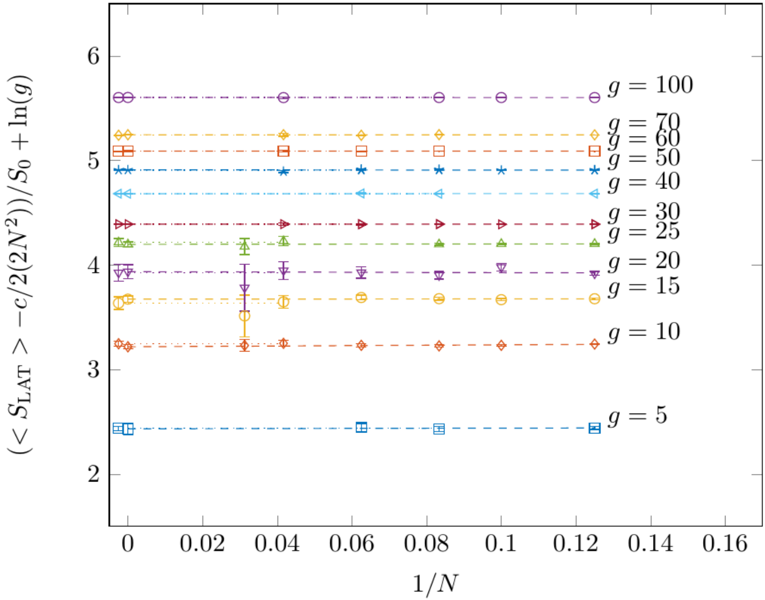
Algorithms

- Rational Hybrid Monte-Carlo
- Conjugate Gradient multi-shift solver
- Fortran and Matlab code
- not parallelized yet

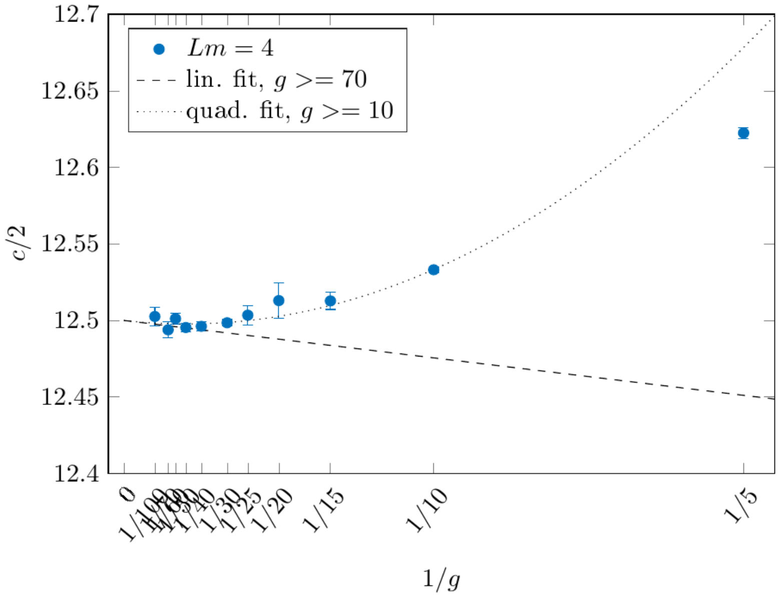
Bosonic Correlator, m_x

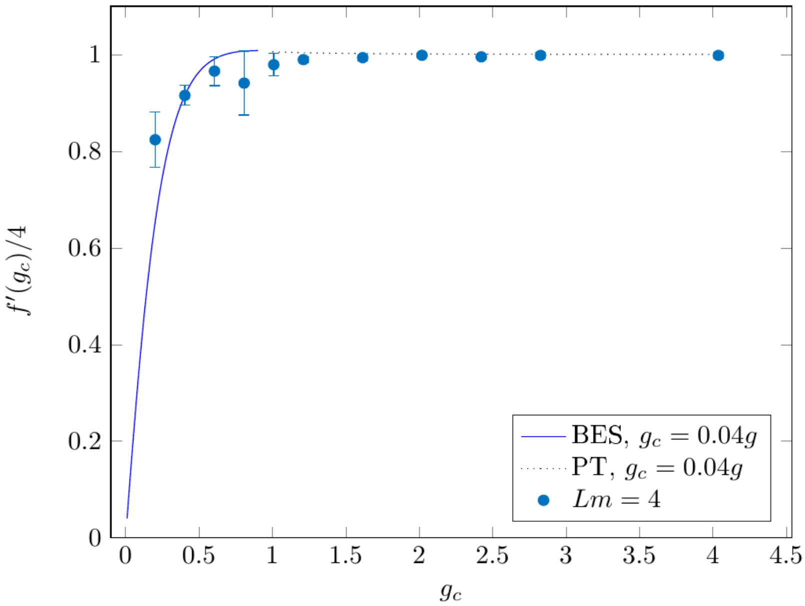



$Lm = 4$
PT, $g_c = 0.04g$
m_x^2/m^2
g_c
0
0.2
0.4
0.6
0.8
1
0.5
1.5
2
2.5
3
3.5
4
4.5



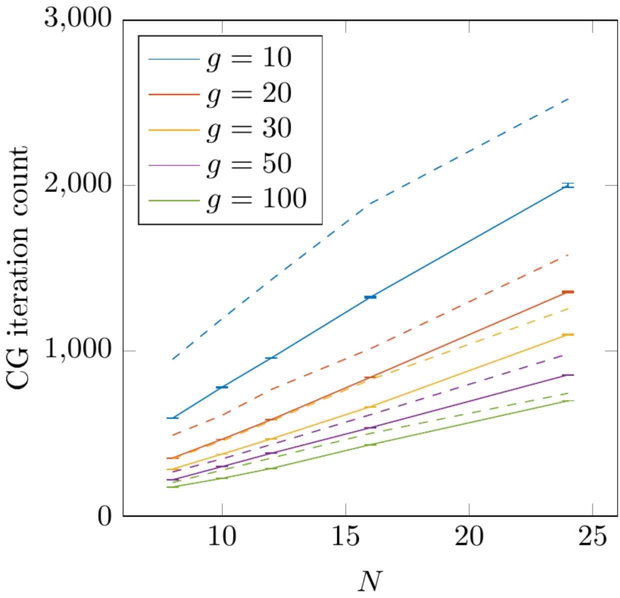
$(< S_{\mathrm{LAT}} > -c/2(2N^2))/S_0 + \ln(g)$
6
5
4
3
2
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
$1/N$
$g = 100$
$g = 70$
$g = 60$
$g = 50$
$g = 40$
$g = 30$
$g = 25$
$g = 20$
$g = 15$
$g = 10$
$g = 5$


12.7
12.65
12.6
12.55
12.5
12.45
12.4
c/2
Lm = 4
lin. fit, g >= 70
quad. fit, g >= 10
0
1/15
1/10
1/5
1/g



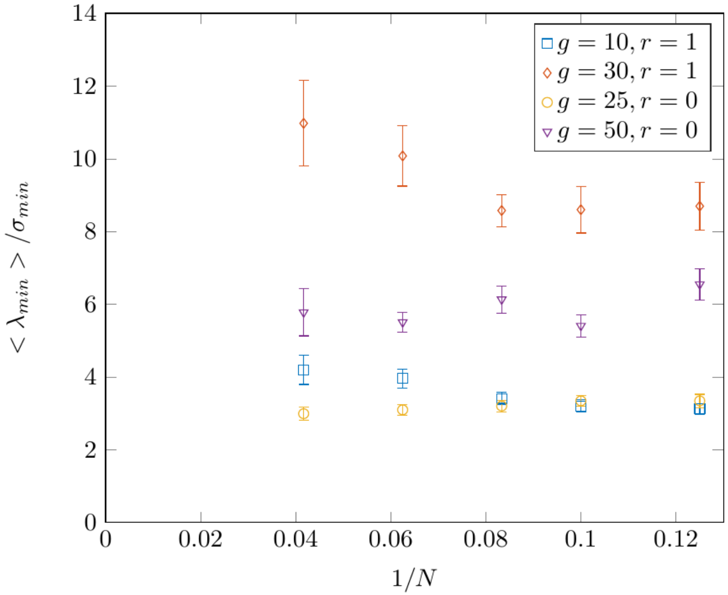
$f'(g_c)/4$
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
g_c
BES, $g_c = 0.04g$
PT, $g_c = 0.04g$
$Lm = 4$

CG iteration count

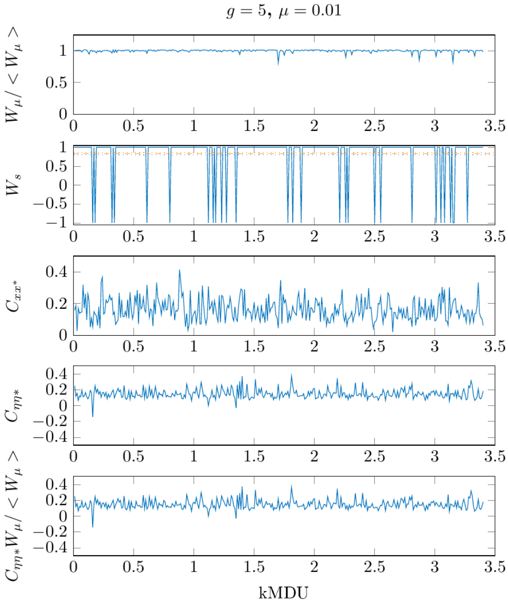


signals small eigenvalues for small g

Separation of the smallest eigenvalue of $O_F^\dagger O_F$ from zero



Simulation employing μ-reweighting ($O_F^\dagger O_F \to O_F^\dagger O_F + \mu^2$)



Wilson term breaks $U(1)$

- $\langle x \rangle = 0$ not guaranteed, indeed we find $\langle x \rangle \sim 1/a$
- also the mass of the fermions appear to receive additive renormalization
- tuning is possible employing $U(1)$ Ward-Takahashi identities (in progress)

Summary

- First principle proof of predictions from AdS/CFT and integrability is appealing
- Lattice string sigma model promising:
  - two-dimensional, no supersymmetry (compared to $\mathcal{N} = 4$ SYM)
  - non-perturbative effects just before sign problem sets in
- ToDo:
  - lattice observable for $f(g)$ (use $d\ln Z/dm$)
  - recover $U(1)$ symmetry in the continuum limit (employ WT identities)
  - algorithmic improvements (precondition CG)
  - parallelized code (GPU/SMP/MPI)
- In the future: extend study to other backgrounds, string settings