Condensation thresholds and scattering data - a study in the relativistic Bose gas at finite density

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Introduction

- Sign problem $\Rightarrow e^{-S[\psi]} \not\geq 0$
- Possible solution: Dual formulation $\Rightarrow e^{-S'[\psi']} \geq 0$
- Replacing old fields ψ with new fields ψ'
- Different physical interpretation of the new fields, from gauge fields to fluxes with particles worldline variables
- Is it possible to use a dual formulation for something more?
  Dual formulation $\Rightarrow$ Charge condensation method $\Rightarrow$ Spectroscopy results
- Seminal paper for 2-D $O(3)$: F. Bruckmann, C. Gattringer, Christof, T. Kloiber and T. Sulejmanpasic **PhysRevLett.115.231601**

Dual formulation for ϕ^4

- The Lagrangian of the system is:

$$S = \sum_x \left(\eta |\phi_x|^2 + \lambda |\phi_x|^4 - \sum_{\nu=1}^{4} \left[e^{\mu \delta_{\nu,4}} \phi_x^* \phi_{x+\hat{\nu}} + e^{-\mu \delta_{\nu,4}} \phi_x^* \phi_{x-\hat{\nu}} \right] \right)$$

  Where $\eta = 2D + m^2 \Rightarrow 8 + m^2$

- The partition sum is:

$$Z = \int \mathcal{D}[\phi] e^{-S} \Rightarrow \mathcal{D}[\phi] = \prod_x \int_{\mathbb{C}} \frac{d\phi_x}{2\pi}$$

Dual formulation for ϕ^4

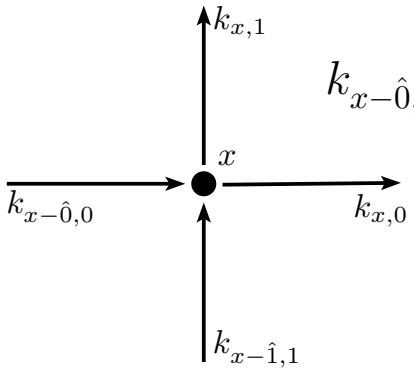
- The partition function can be rewritten with the following dual variables:

$$Z = \sum_{\{k,l\}} \left(\prod_{x,\nu} \frac{1}{(|k_{x,\nu}| + l_{x,\nu})! l_{x,\nu}!} \right) \left(\prod_x \delta \left(\sum_\nu \left[k_{x,\nu} - k_{x-\hat{\nu},\nu} \right] \right) \right)$$
$$\times \left(\prod_x e^{\mu k_{x,4}} \; W \left(\sum_\nu \left[|k_{x,\nu}| + |k_{x-\hat{\nu},\nu}| + 2(l_{x,\nu} + l_{x-\hat{\nu},\nu}) \right] \right) \right)$$

- The new variables are not gauge fields
- $l_{x,\nu} \in \mathbb{N}$ is a background variable
- $l_{x,\nu}$ can be updated with a standard Monte Carlo simulation

Dual formulation for ϕ^4

- $k_{x,\nu} \in \mathbb{Z}$ is the flux variable subjected to constraints
- The constraints $\prod_x \delta\left(\sum_\nu \left[k_{x,\nu} - k_{x-\hat{\nu},\nu}\right]\right)$ represents the flux conservation for every site x
- In two dimensions:



$$k_{x-\hat{0},0} + k_{x,0} + k_{x-\hat{1},1} + k_{x,1} = 0$$

- This means that the fluxes k_x, ν appear only in closed loops
- $k_{x,\nu}$ is usually updated using a Worm algorithm to satisfy the constraints

Dual formulation for ϕ^4

- The chemical potential is acting through the term

$$\prod_x e^{\mu k_{x,4}}$$

- Its meaning is manifest looking at the particle number observable:

$$N = T\frac{\partial \log Z}{\partial \mu} = \frac{1}{L_T}\left\langle \sum_x k_{x,4} \right\rangle$$

- A particle is represented by line of fluxes winding around the temporal side
- The chemical potential helps at locale level:
  fluxes in the particle direction for chemical potential $\mu > 0$
  fluxes in the antiparticle direction for chemical potential $\mu < 0$
- Causing the creation of particles or antiparticles

Dual formulation for ϕ^4

- The weights

$$W\left(\sum_{\nu}\left[|k_{x,\nu}| + |k_{x-\hat{\nu},\nu}| + 2(l_{x,\nu} + l_{x-\hat{\nu},\nu})\right]\right)$$

  are represented by the integral

$$W(n) = \int_0^\infty dr\ r^{n+1}\ e^{-\eta r^2 - \lambda r^4}$$

- These weights can be computed and stored in advance in a lookup table
- Usually for a simulation $\mathcal{O}(\#W) = 100$ is more than enough
- The weights represent the parameters η and λ in the dual theory

Dual formulation for ϕ^4

- Simulation strategy:
  1. Update the background $l_{x,\nu}$ with conventional local Monte Carlo steps
  2. Update the variable $k_{x,\nu}$ with a worm step
  3. Go back to 1

- Strictly speaking a fixed order of the updates breaks Detailed Balance....

  .... usually this is harmless because the balance condition is still satisfied

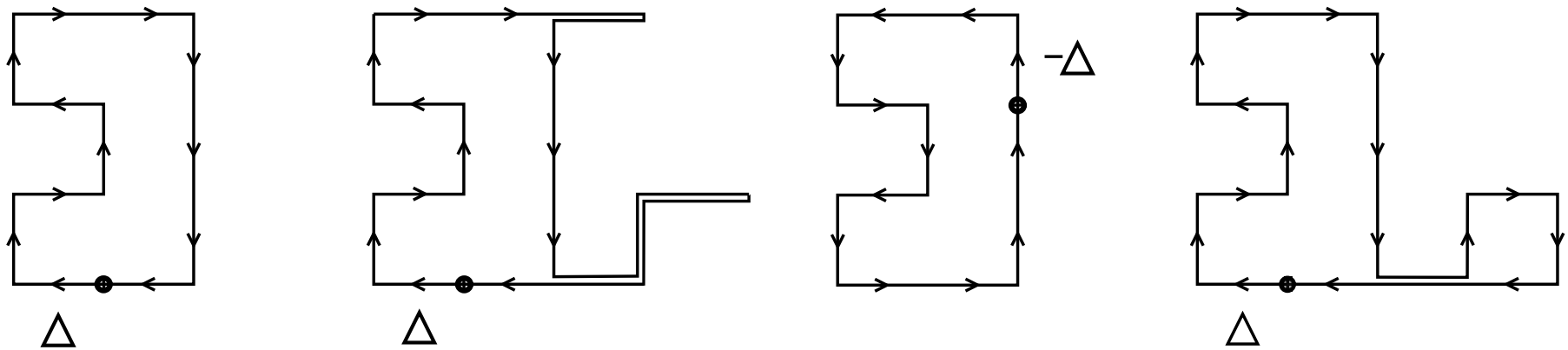
- This approach can be harmful because the propagation of the worm in a fixed background can produce a large autocorrelation length

- A more complex strategy with the update of the background $l_{x,\nu}$ during the worm propagation is advisable

Dual formulation for ϕ^4

- The worm is a Random Walk on the lattice where the directions are weighted by the partition function

- A closing worm automatically satisfies the constraints



Dual formulation for ϕ^4

- To describe the algorithm it is useful to separate the local part and the unlocal part of the partition function:

$$L_{x,\nu}(k_{x,\nu}) = \frac{e^{\mu\,\delta_{\nu,\mathbf{4}}\,k_{x,\nu}}}{(|k_{x,\nu}| + l_{x,\nu})!\,l_{x,\nu}!}\ ,$$

$$S_x(\{k_{x,\bullet}\}) = W\left(\sum_{\nu}\left[|k_{x,\nu}| + |k_{x-\widehat{\nu},\nu}| + 2(l_{x,\nu} + l_{x-\widehat{\nu},\nu})\right]\right)$$

- So the rules are:

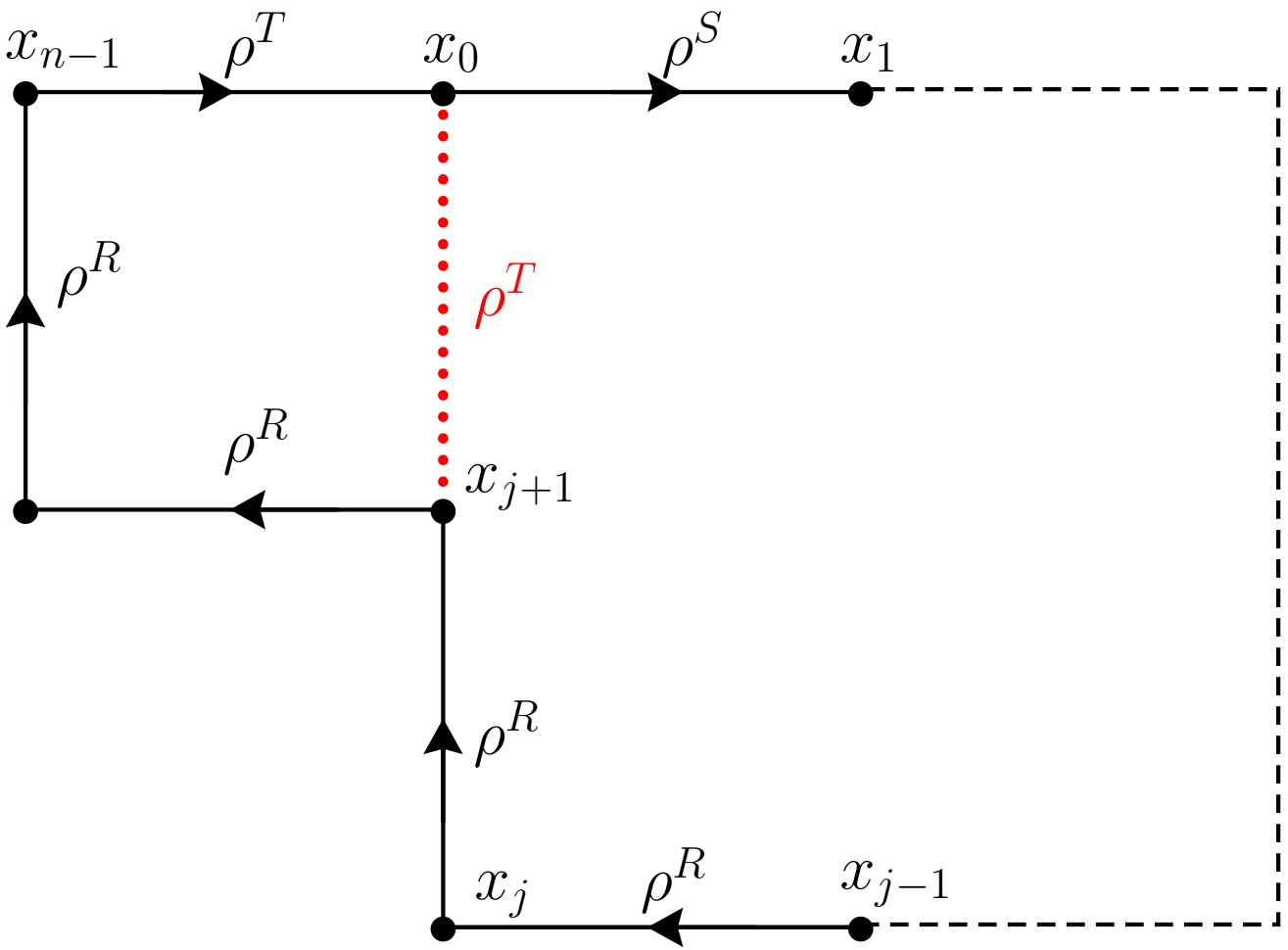
$$\rho^{S}_{x_{\mathbf{0}},\nu_{\mathbf{0}}} = \frac{A}{S_{x_{\mathbf{0}}}(\{k_{x_{\mathbf{0}},\bullet}\})\,S_{x_{\mathbf{1}}}(\{k_{x_{\mathbf{1}},\bullet}\})}\,\frac{L_{x_{\mathbf{0}},\nu_{\mathbf{0}}}(k^{trial}_{x_{\mathbf{0}},\nu_{\mathbf{0}}})}{L_{x_{\mathbf{0}},\nu_{\mathbf{0}}}(k_{x_{\mathbf{0}},\nu_{\mathbf{0}}})}$$

$$\rho^{R}_{x_j,\nu_j} = \frac{S_{x_j}(\{k^{new}_{x_j,\bullet}\})}{S_{x_{j+\mathbf{1}}}(\{k_{x_{j+\mathbf{1}},\bullet}\})}\,\frac{L_{x_j,\nu_j}(k^{trial}_{x_j,\nu_j})}{L_{x_j,\nu_j}(k_{x_j,\nu_j})}$$

$$\rho^{T}_{x_j,\nu_j}\Big|_{j=n-1} = \frac{S_{x_{n-\mathbf{1}}}(\{k^{new}_{x_{n-\mathbf{1}},\bullet}\})\,S_{x_{\mathbf{0}}}(\{k^{new}_{x_{\mathbf{0}},\bullet}\})}{A}\,\frac{L_{x_{n-\mathbf{1}},\nu_{n-\mathbf{1}}}(k^{trial}_{x_{n-\mathbf{1}},\nu_{n-\mathbf{1}}})}{L_{x_{n-\mathbf{1}},\nu_{n-\mathbf{1}}}(k_{x_{n-\mathbf{1}},\nu_{n-\mathbf{1}}})}$$

Dual formulation for ϕ^4

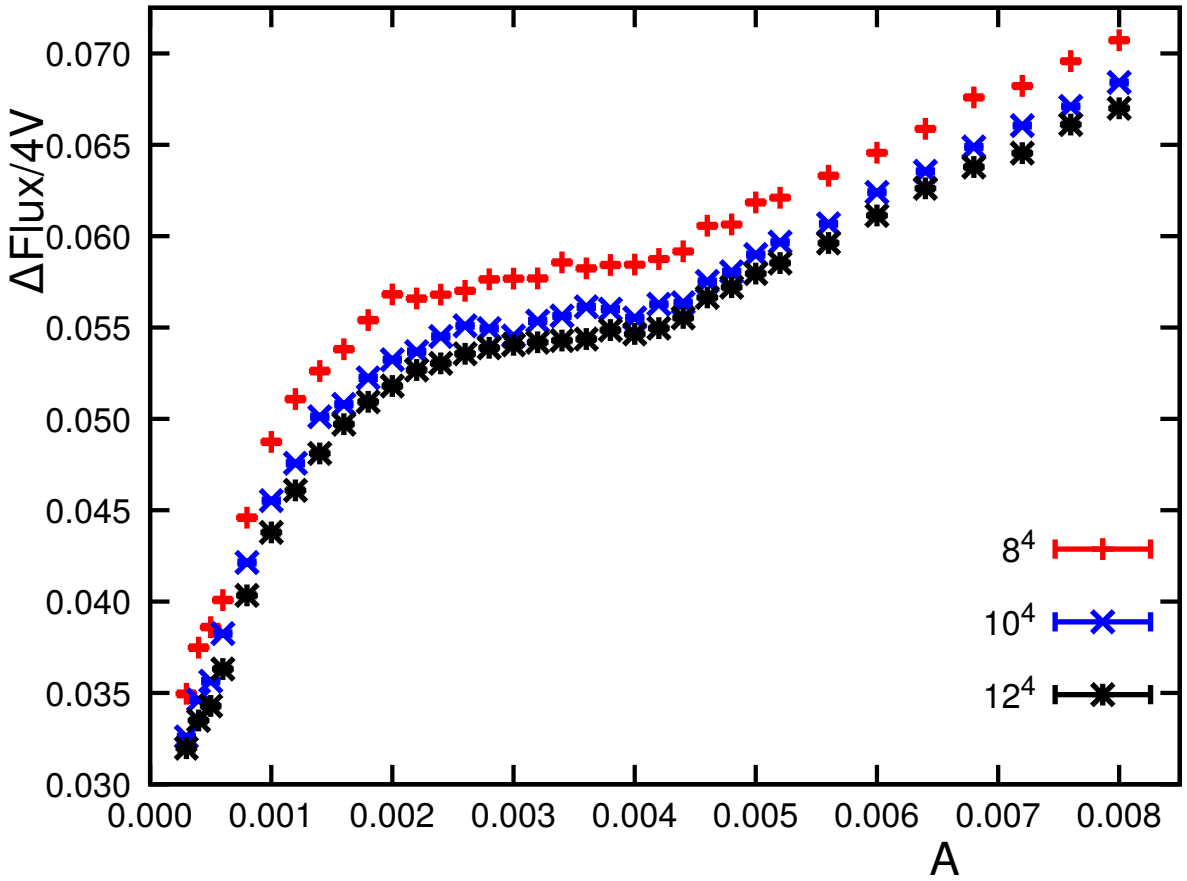
- The worm has the following process:



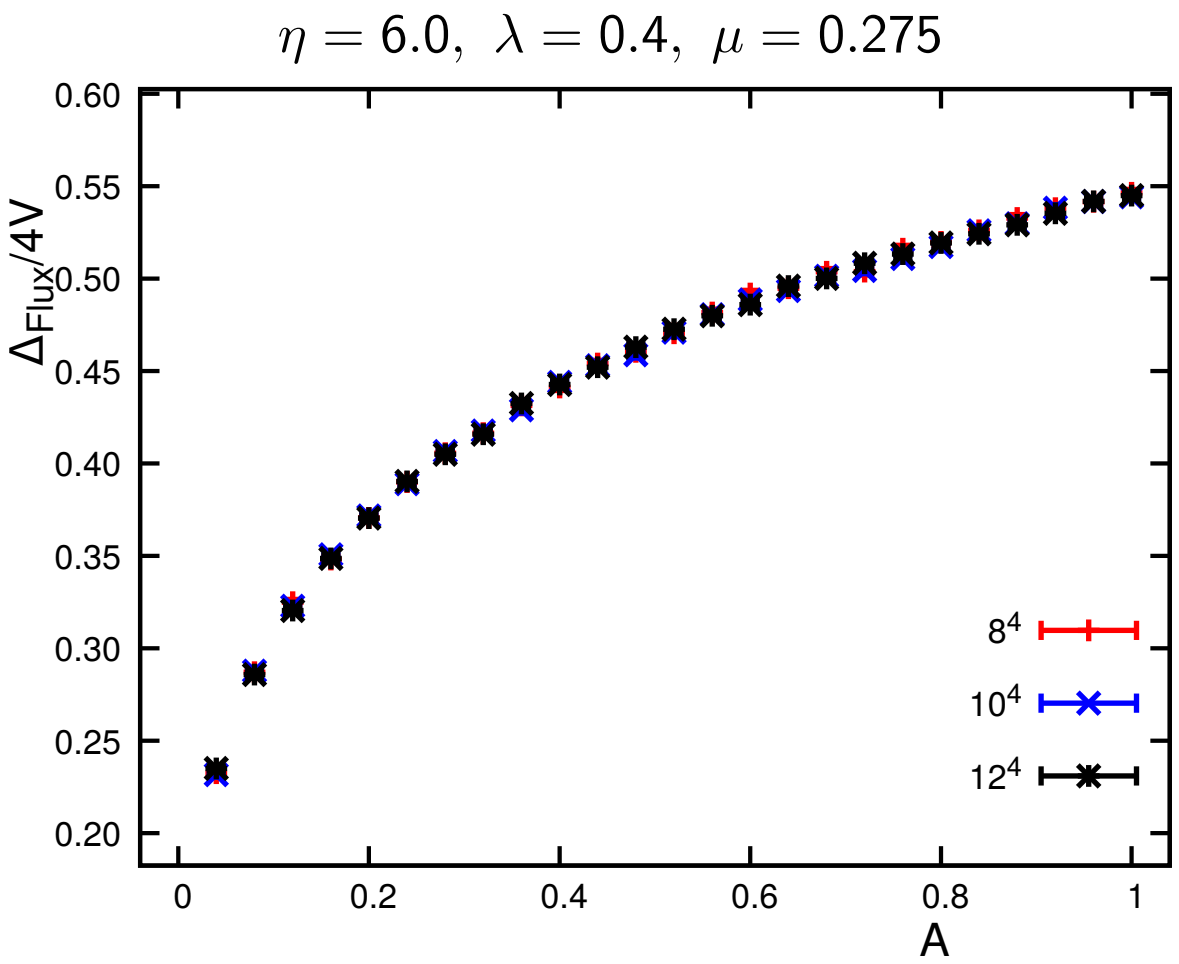
Dual formulation for ϕ^4

- The starting and terminating probabilities have a free parameter A
- It's possible to use this parameter to tune the average length of a worm

$$\eta = 7.44,\ \lambda = 1.0,\ \mu = 0.275$$



Dual formulation for ϕ^4

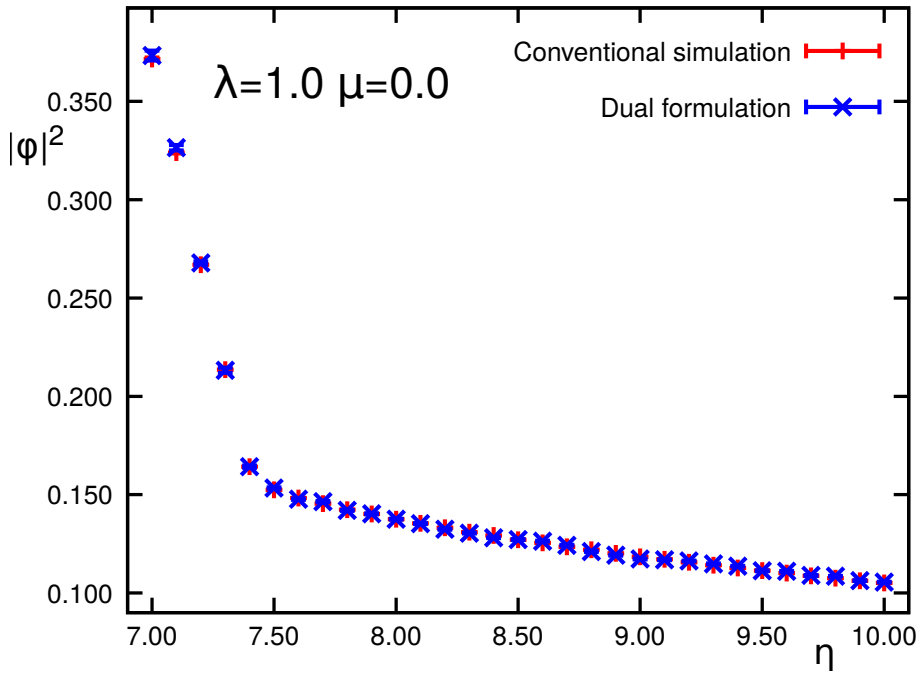


- It's important to have worm long enough if we want to change the winding number

Dual formulation for ϕ^4

- We check the correctness of the algorithm
- At $\mu = 0$ is possible to compute $|\phi|^2$ with a standard MC
- In the dual formulation the observable $|\phi|^2$ is defined as:

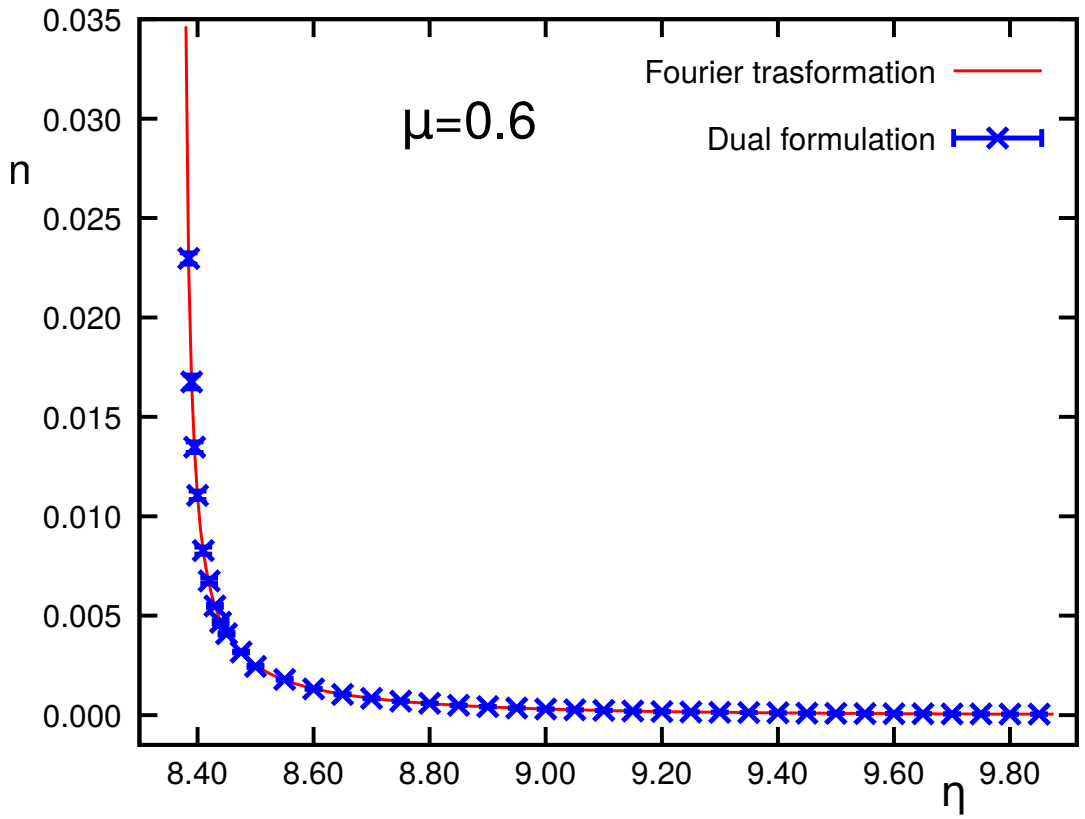
$$\langle |\phi|^2 \rangle = -\frac{T}{V}\frac{\partial \ln Z}{\partial \eta} = \frac{1}{L_s^3 L_T}\left\langle \sum_x \frac{W(f_{x+2})}{W(f_x)} \right\rangle$$



Dual formulation for ϕ^4

- For the coupling $\lambda = 0$ there is an analytical expression for the particle number density:

$$n = \frac{T}{V}\frac{\partial \log Z}{\partial \mu} = \frac{1}{L_s^3 L_T}\left\langle \sum_x k_{x,4} \right\rangle$$



Charge condensation method

- At finite volume and $T = 0$; 1,2,... particles sectors are separated by finite energy steps
- The energy steps are represented by threshold for the chemical potential $\mu_1, \mu_2, \ldots$
- Using Lüscher approach it is possible to address the finite volume effects
- From μ_1 we extract the mass of the lightest particle of the theory
- From $\mu_1 + \mu_2$ the interaction of two particles: scattering length

Charge condensation method

- The partition function of the system is described by the Grand canonical partition sum:

$$Z(\mu) = \mathrm{Tr}\left(e^{-\frac{(\hat{H}-\mu\hat{Q})}{T}}\right) = e^{-\frac{\Omega(\mu)}{T}}$$

- For $T \to 0$, every sector is dominated by the minimum of the Hamiltonian in that sector:

$$\Omega(\mu) \xrightarrow{T\to 0} \begin{cases} E_{min}^{Q=0} = 0 & \mu \in [0, \mu_1) \\ E_{min}^{Q=1} - \mu = m - \mu & \mu \in (\mu_1, \mu_2) \\ E_{min}^{Q=2} - 2\mu = W - 2\mu & \mu \in (\mu_2, \mu_3) \end{cases}$$

- The first transition occurs at:

$$0 = m - \mu_1 \;\rightarrow\; m = \mu_1$$

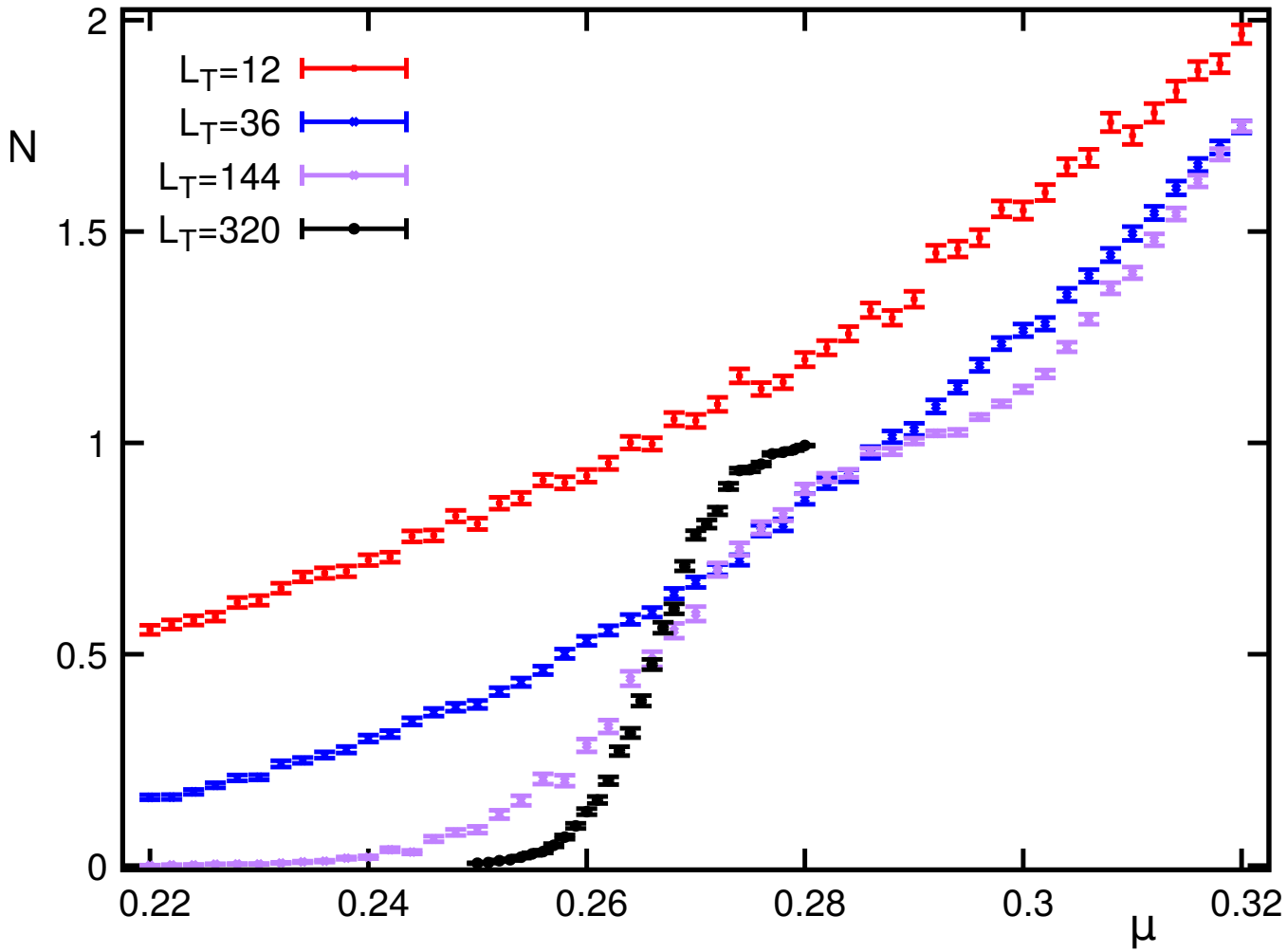
- The second transition occurs at:

$$m - \mu_2 = W - 2\mu_2 \;\rightarrow\; W = \mu_1 + \mu_2$$

Charge condensation method

- The limit to a step function can be showed going to larger L_T:

$$L_S = 4$$



Charge condensation method

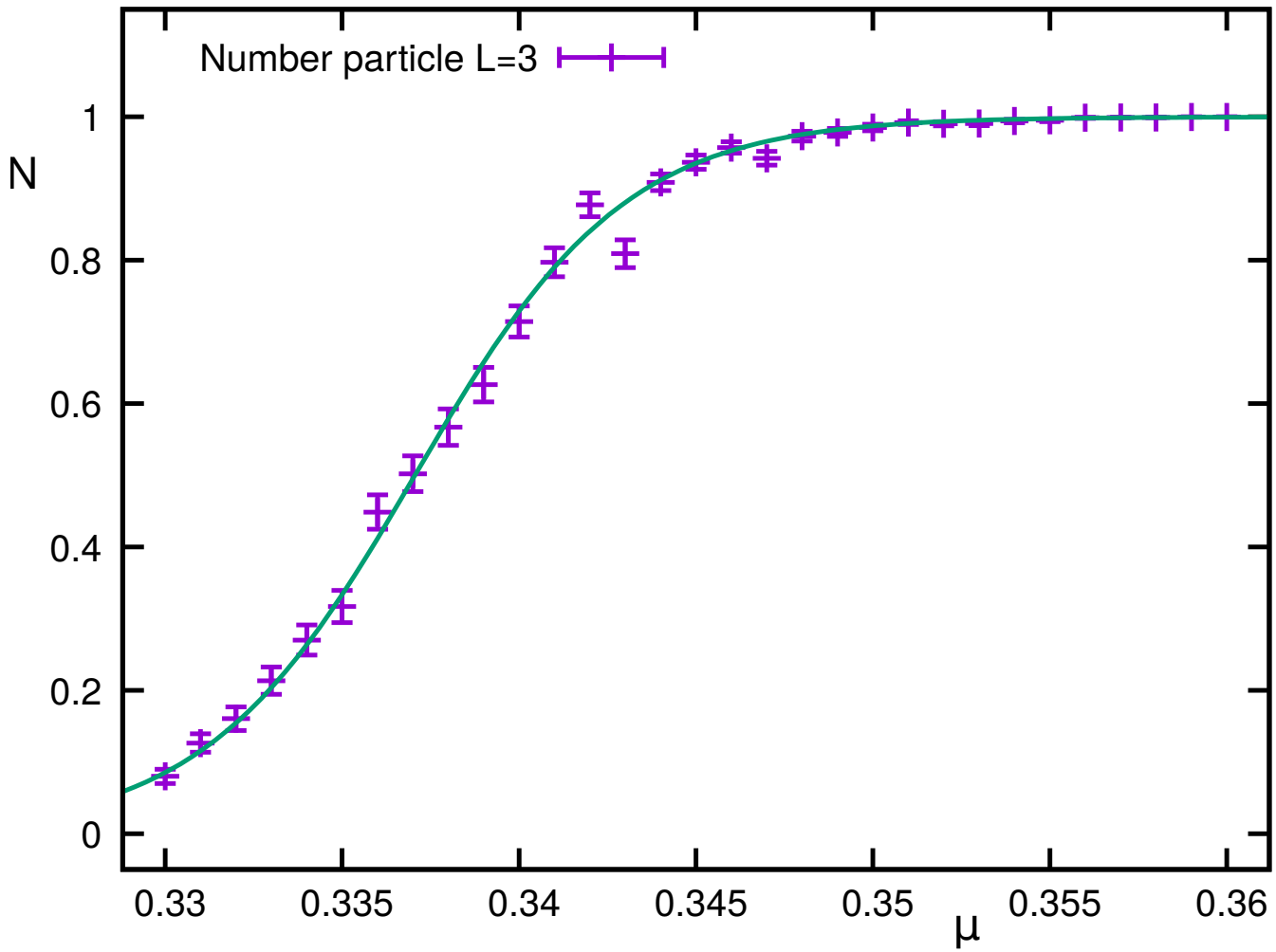
- In our case we simulate with parameters $\eta = 7.44$ and $\lambda = 1.0$
- In general it is sufficient to clearly separate the 1 and 2 particles sectors
- To find the threshold chemical potentials: $\mu_1,\ \mu_2, \ldots$, we fit the particle number with a sigmoid function
- In particular, the logistic function:

$$f(x) = \frac{1}{1 + e^{-k(\mu - \mu_0)}}$$

Charge condensation method

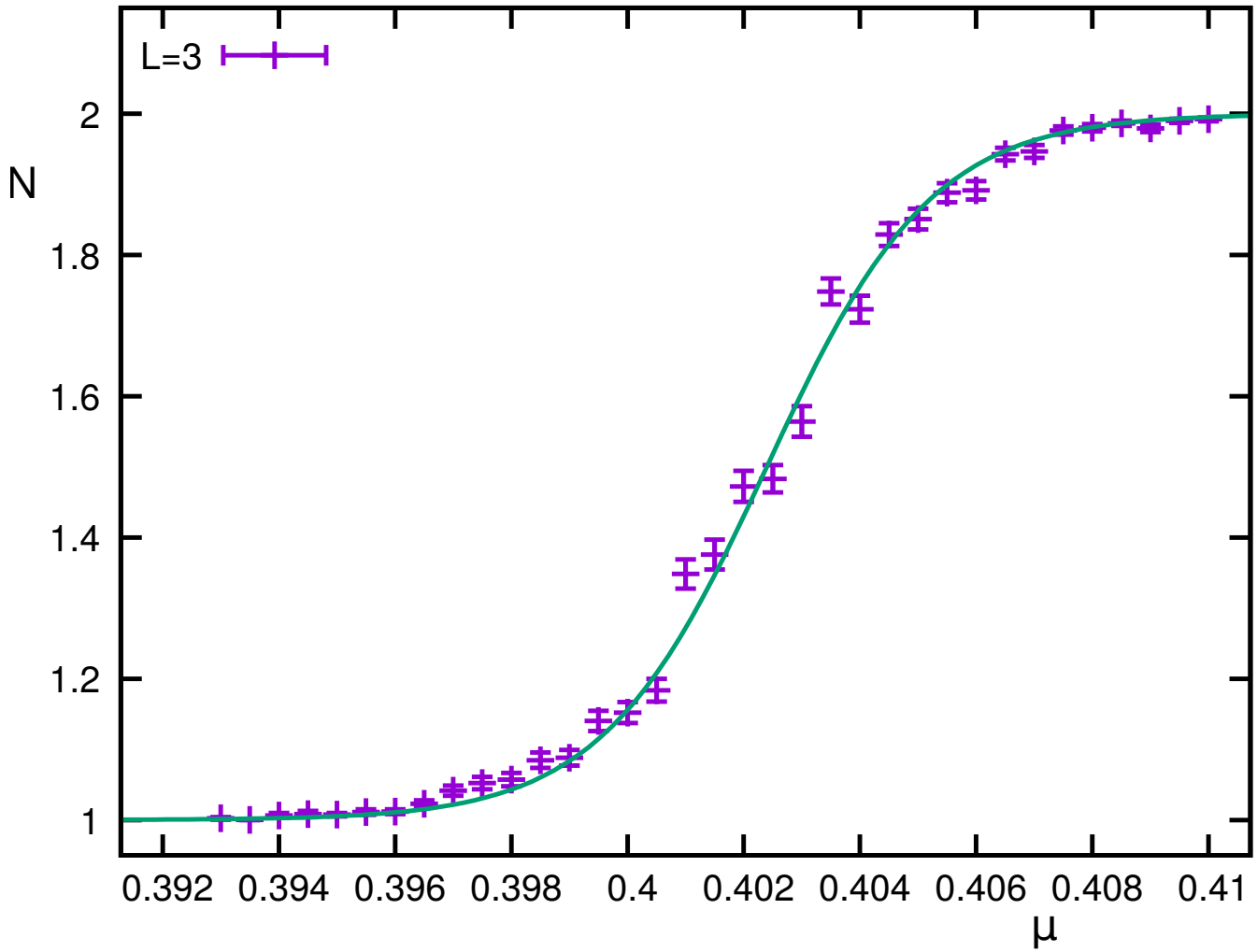
- We can show some examples for the particle number

1-particle, $L_s = 3, L_T = 320$



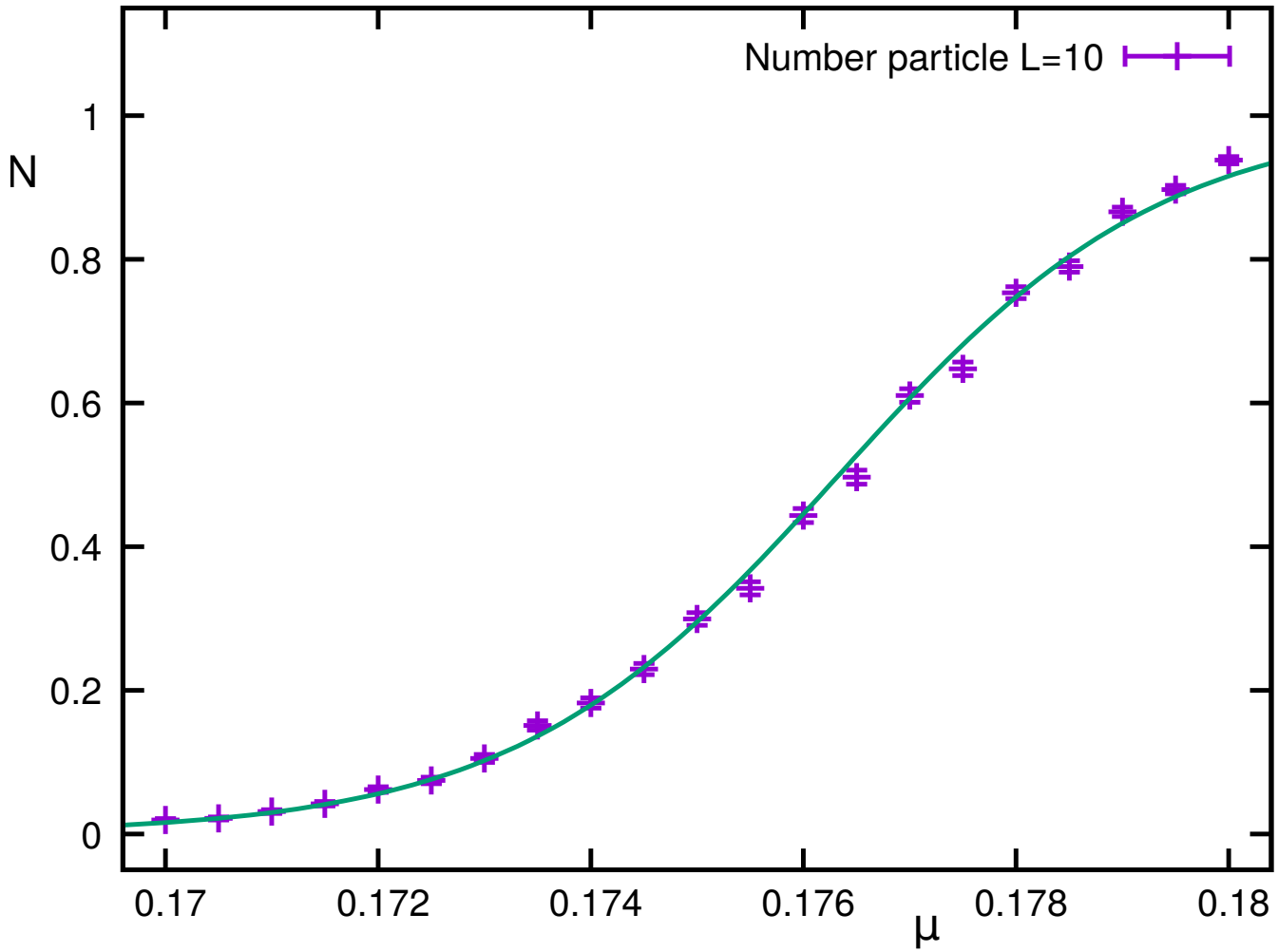
Charge condensation method

2-particles, $L_s = 3, L_T = 640$



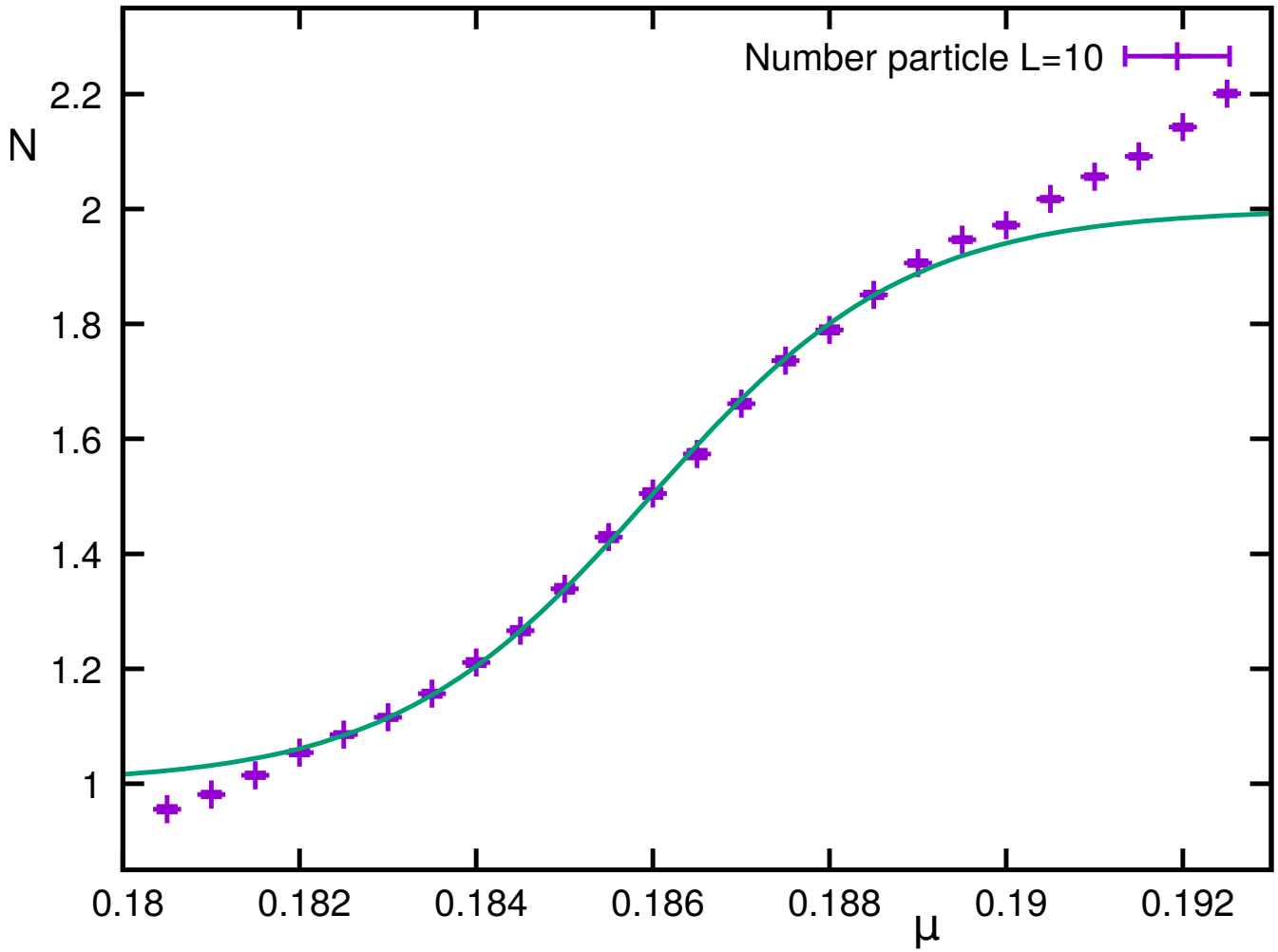
Charge condensation method

1-particle, $L_s = 10, L_T = 640$



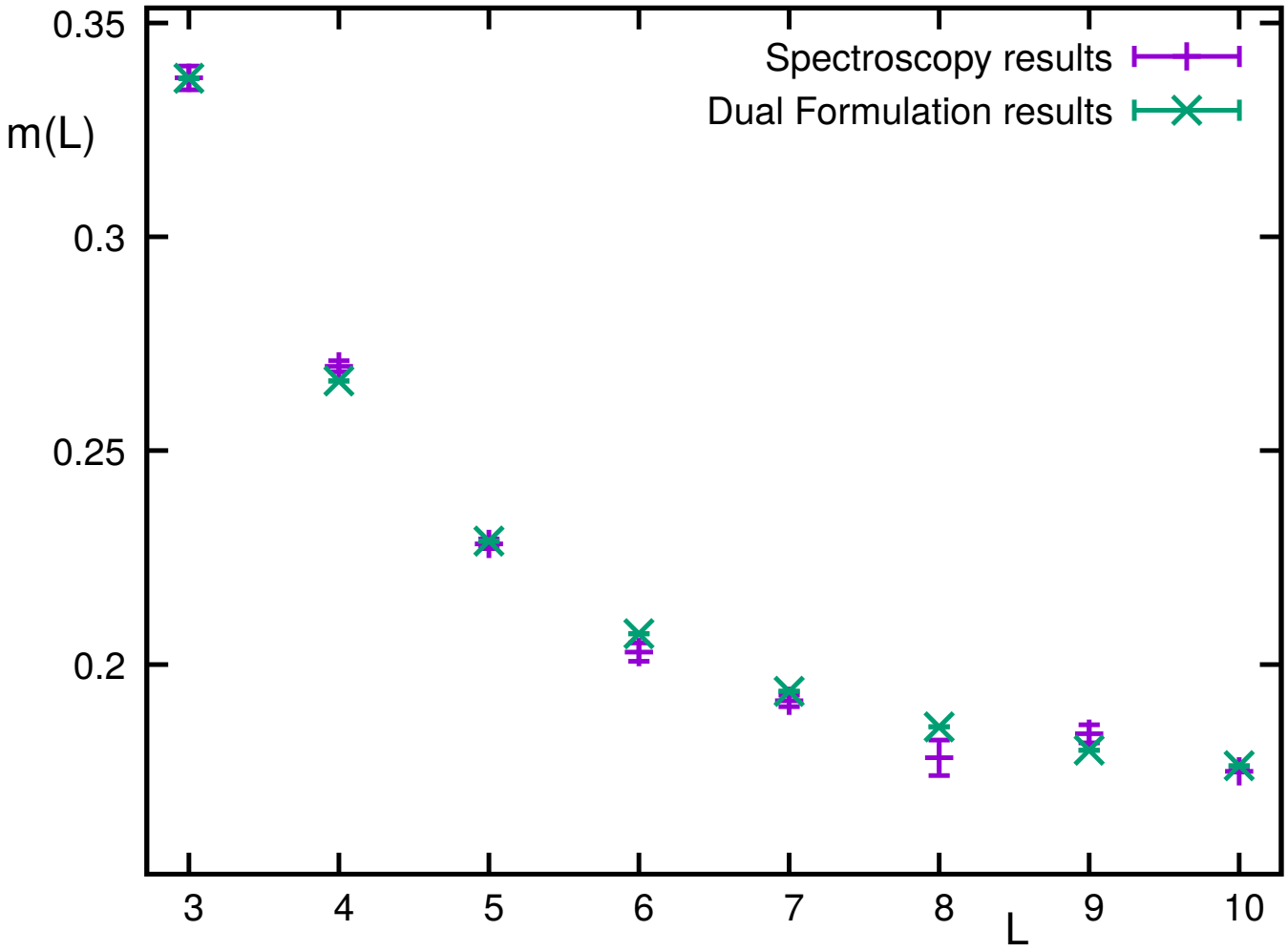
Charge condensation method

2-particles, $L_s = 10, L_T = 640$



Charge condensation method

- For one particle it's simple to compare our results with conventional spectroscopy results:



Charge condensation method

- We can use Lüscher's formulas to describe the finite volume scaling
- For 1 particle in a box we use:

$$m(L) = m_0 + c\, L^{-\frac{3}{2}} e^{-m_0 x}$$

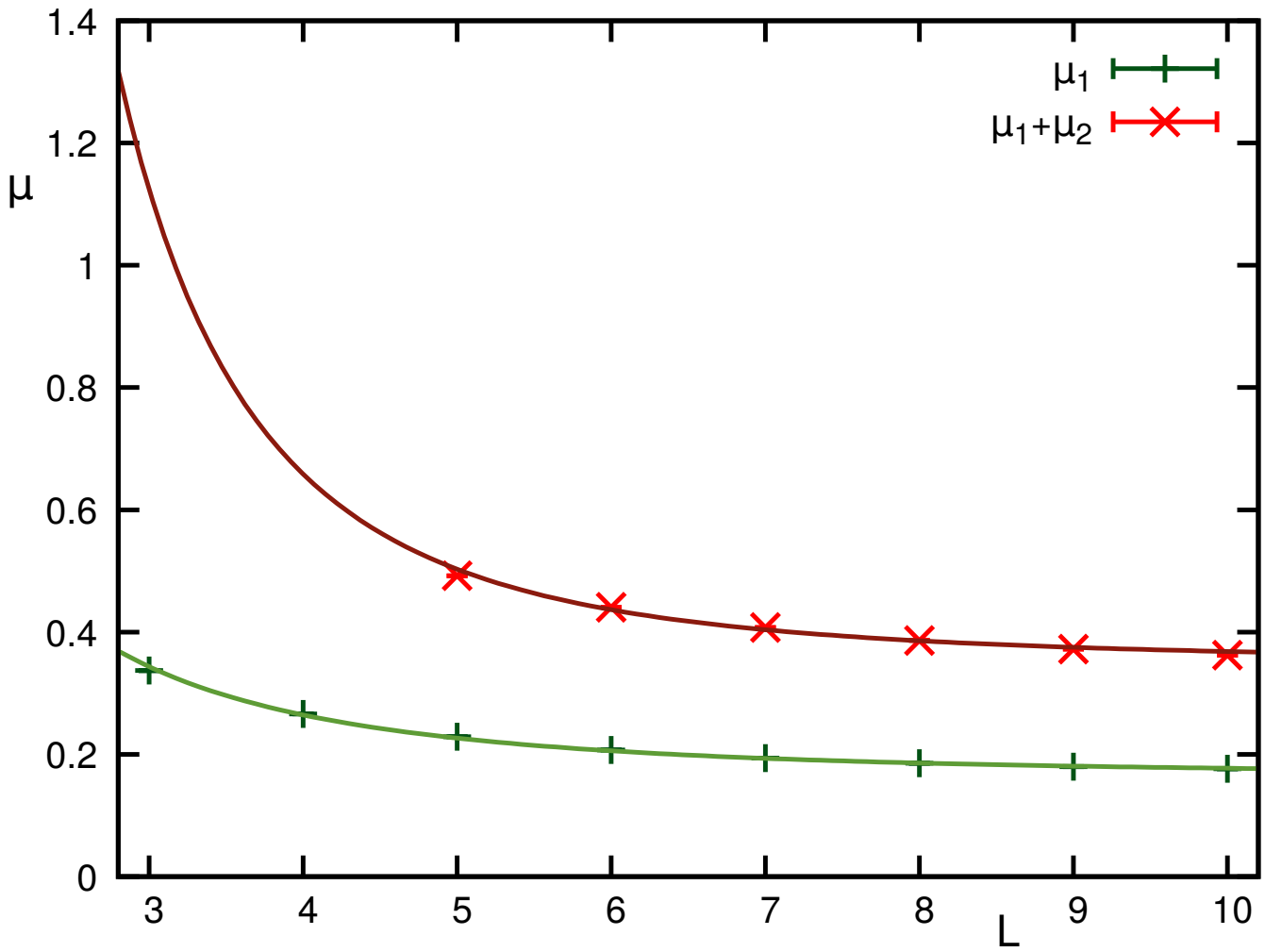
- For 2 particles in a box we use:

$$W(L) = 2m_0 - \frac{4\pi a_0}{mL^3}\left\{1 + c_1\frac{a_0}{L} + c_2\frac{a_0^2}{L^2}\right\} + \mathcal{O}\left(\frac{1}{L^6}\right)$$

$$c1 = -2.837297 \quad , \quad c2 = 6.375183$$

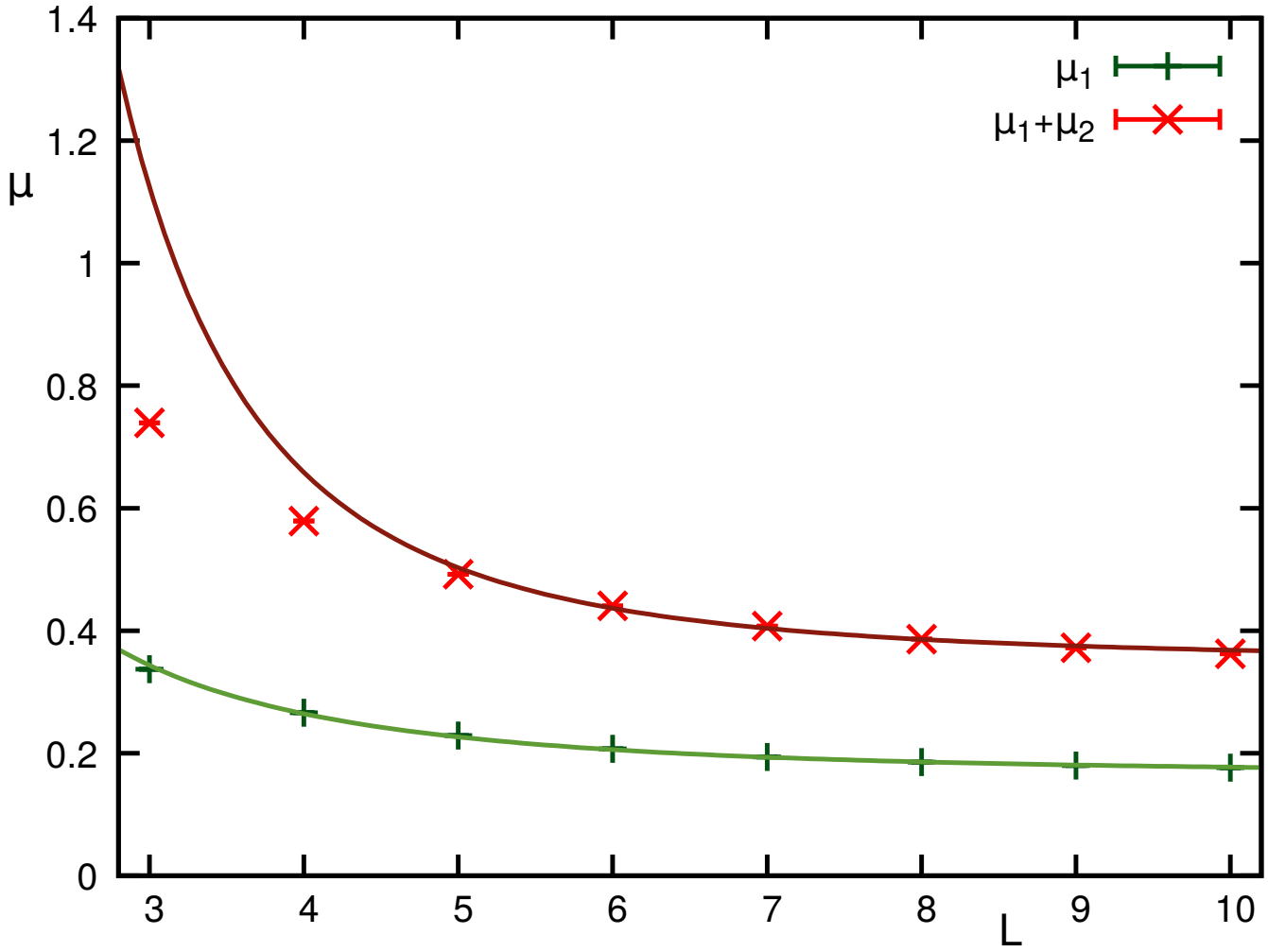
Charge condensation method

- The results of the fits are:



Charge condensation method

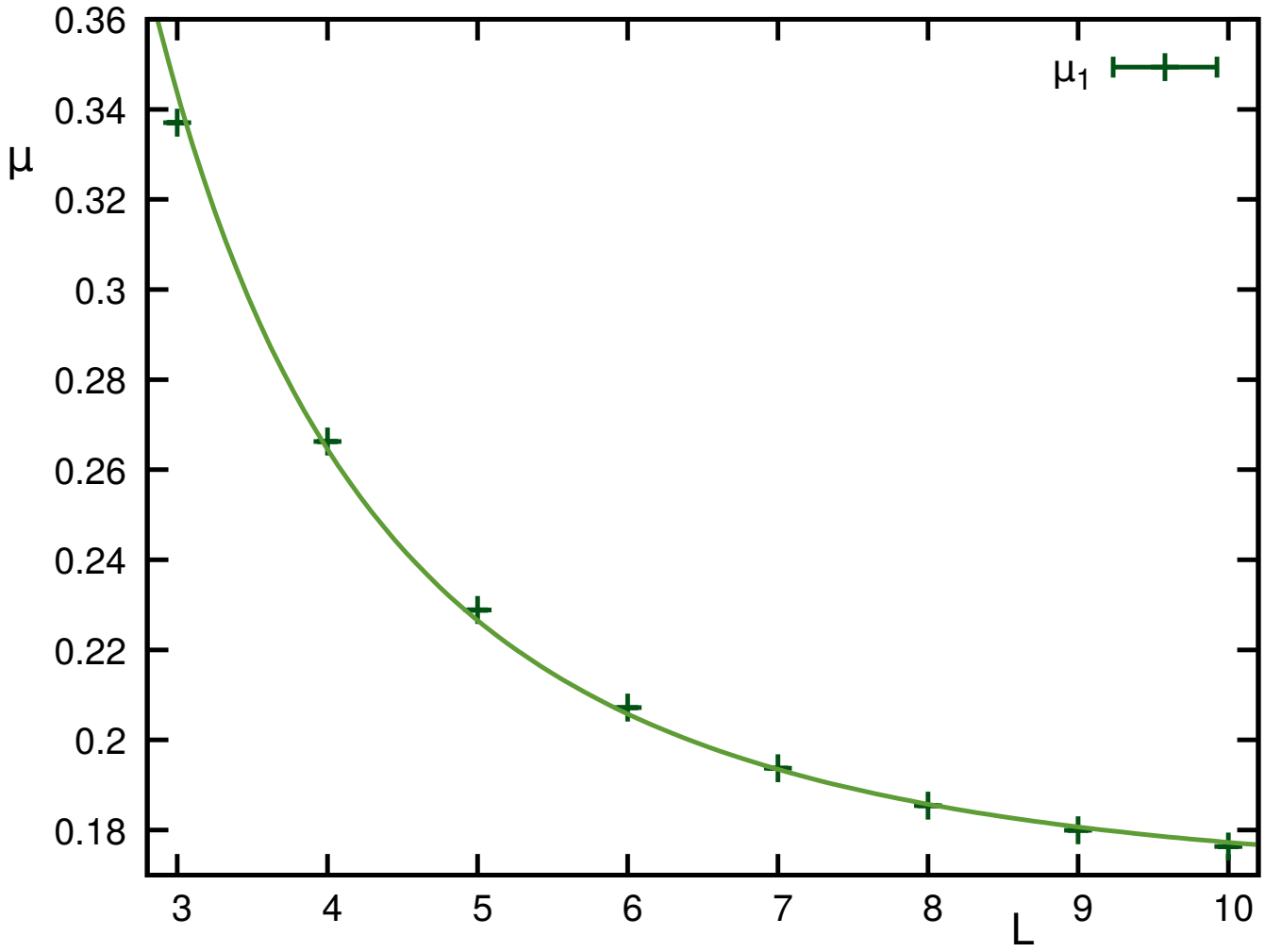
- Comparing with the point at $L_s = 3$ and $L_s = 4$:



- We suppose that order $\mathcal{O}\left(\frac{1}{L^6}\right)$ is not negligible in this case

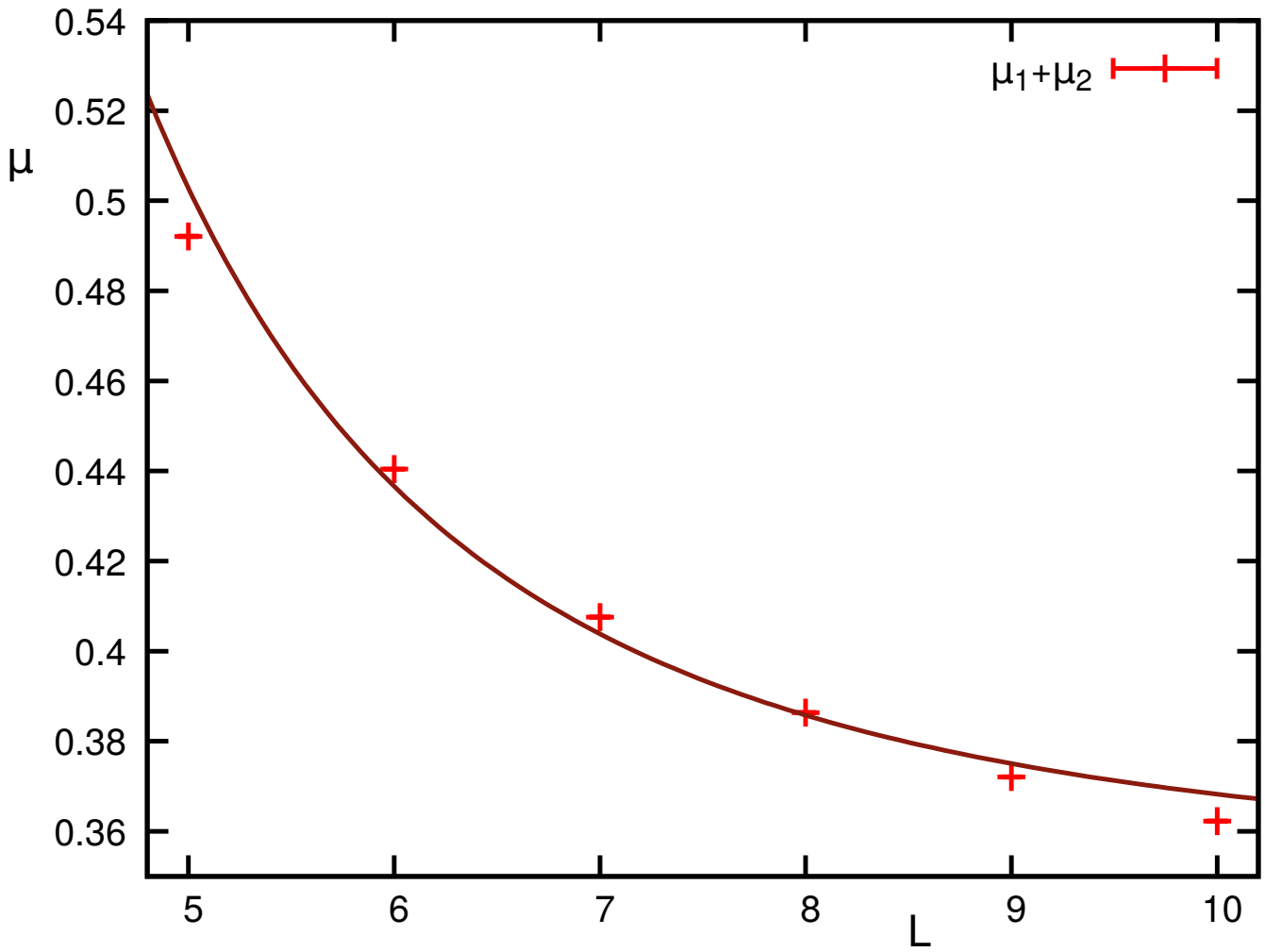
Charge condensation method

- Giving a more detailed overview of the 1 particle fit we have:



Charge condensation method

- For the two particle fit we have:



Charge condensation method

The results of the fits are:

- For $m(L)$ we have:

$$m_0 = 0.168 \pm 0.001$$
$$c = 1.508 \pm 0.025$$

- For $W(L)$ we find:

$$m_0 = 0.175 \pm 0.002$$
$$a_0 = -0.232 \pm 0.011$$

Conclusions

- Dual formulation is an approach not limited to the sign problem solution
- Spectroscopy results are available using dual formulation
- A large L_T is necessary to separate 1 and 2 particles sectors
- Up to today there is no dual formulation of a non-abelian theory