Resonance matrix elements

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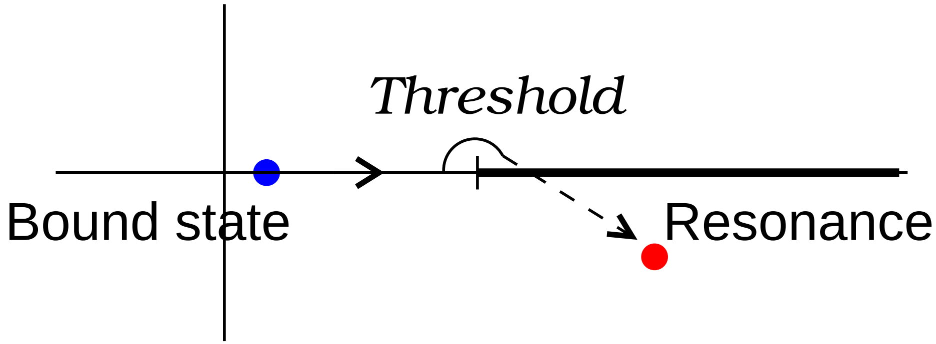
DESY Zeuthen, 16 January 2017

Plan

- Introduction: Matrix elements of the resonances
- Lattice QCD and the effective field theories in a finite volume
- Lüscher-Lellouch formula
- Continuation to the pole and fixing the photon virtuality
- Limit of an infinitely narrow resonance
- Conclusions, outlook

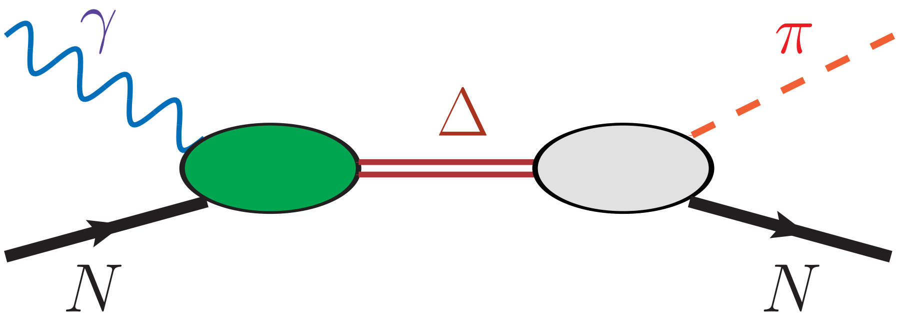
Resonances

- Resonances correspond to S-matrix poles on the unphysical Riemann sheets after continuation into the complex plane



- Resonances are characterized by their mass, their lifetime, . . . These are the *intrinsic* properties of a resonance that should not depend neither on a particular *experiment* nor a particular *theoretical model* which is used to describe the data

Resonance matrix elements



In complex plane, in the vicinity of the resonance pole...

$$\Rightarrow \quad i\int d^4x\, e^{iPx}\,\langle 0|T\Delta(x)\bar\Delta(0)|0\rangle \to \frac{Z_R}{s_R-P^2}+\cdots$$

$$\Rightarrow \quad i^2\int d^4x d^4y\, e^{iPx-iQy}\,\langle 0|T\Delta(x)J(0)\bar N(y)|0\rangle$$

$$\to \frac{Z_R^{1/2}}{s_R-P^2}\,\langle\Delta|J(0)|N\rangle\,\frac{Z_N^{1/2}}{m_N^2-Q^2}+\cdots$$

$\hookrightarrow$ model-independent definition of the quantity $\langle\Delta|J(0)|N\rangle$

The problem

- How does one determine the position of a pole in a complex plane, using lattice data on real energy spectrum?
- How are the matrix elements, measured on the lattice, continued into the complex plane?
- What is the virtuality of the photon q^2 at the pole?
- What is the relation to the decay amplitudes, measured at the real energies? How a comparison to the experiment is carried out?
- What is the volume-dependence of the measured matrix elements (Lüscher-Lellouch formula)?

Stable states in lattice field theory

The two-point function → spectrum

$$C_{\mathbf{p}}(t) = \int dU\, d\psi d\bar{\psi}\ \mathrm{e}^{-S_{QCD}(U,\psi,\bar{\psi})}\ \mathcal{O}_{\mathbf{p}}(t)\mathcal{O}_{\mathbf{p}}^{\dagger}(0) \to Z_{\mathbf{p}} \exp(-E(\mathbf{p})t)$$

Three-point function → form factors

$$\begin{aligned} C_{\mathbf{p},\mathbf{q}}(t,t') &= \int dU\, d\psi d\bar{\psi}\ \mathrm{e}^{-S_{QCD}(U,\psi,\bar{\psi})}\ \mathcal{O}_{\mathbf{p}}(t)\, j_{\mu}(0)\, \mathcal{O}_{\mathbf{q}}^{\dagger}(t') \\ &\to Z_{\mathbf{p}}^{1/2} Z_{\mathbf{q}}^{1/2} \exp(-E(\mathbf{p})|t| - E(\mathbf{q})|t'|)\, \langle p|j_{\mu}(0)|q\rangle \end{aligned}$$

The method does not apply to:

Resonances, e.g., the mass and width of the ρ meson ...

Decays, e.g., $K \to 2\pi$, $\Delta \to N\gamma^*$, $B \to K^*\ell^+\ell^-$ ...

→ Resonances do not correspond to an isolated energy level of the lattice Hamiltonian!

Lüscher's finite-volume approach to the scattering problems

M. Lüscher, lectures given at Les Houches (1988); NPB 364 (1991) 237, $\cdots$

The Lüscher equation for the two-particle elastic scattering:

- The box size L is much larger than the interaction range R
  $\to$ scattering phase shifts from the measured lattice levels

$$\det\left(\delta_{ll'}\delta_{mm'} - \tan\delta_l(p)\mathcal{M}_{lm,l'm'}\right) = 0$$

- $\mathcal{M}_{lm,l'm'}$ is a linear combination of Z_{lm} [partial-wave mixing]

$$Z_{lm}(1,q^2) = \lim_{s\to 1}\sum_{\mathbf{n}\in\mathbb{Z}^3}\frac{\mathcal{Y}_{lm}(\mathbf{n})}{(\mathbf{n}^2-q^2)^s}\,, \qquad q = \frac{pL}{2\pi}$$

- The resonance pole position in the complex plane is obtained by using certain parameterization of a scattering phase (e.g., effective range expansion)
- Generalizations: coupled-channel scattering (e.g., K-matrix parameterization)

Where are the resonance poles?

Assume, e.g., the effective range expansion (S-wave):

$$p\cot\,\delta(p) = A_0 + A_1 p^2 + \cdots\,,$$

Analytic continuation to the resonance pole:

$$p_R\cot\,\delta(p_R) = -ip_R$$

$\Rightarrow$ $A_0, A_1, \cdots$ are measured on the lattice (real)

$\Rightarrow$ Resonance pole p_R in the complex momentum plane

Can one perform effective-range expansion for the matrix elements?

Why effective field theories?

Most general effective Lagrangian: all terms with assumed symmetry

$\hookrightarrow$ Most general S-matrix consistent with analyticity, perturbative unitarity, cluster decomposition and symmetries (Weinberg, 1979)

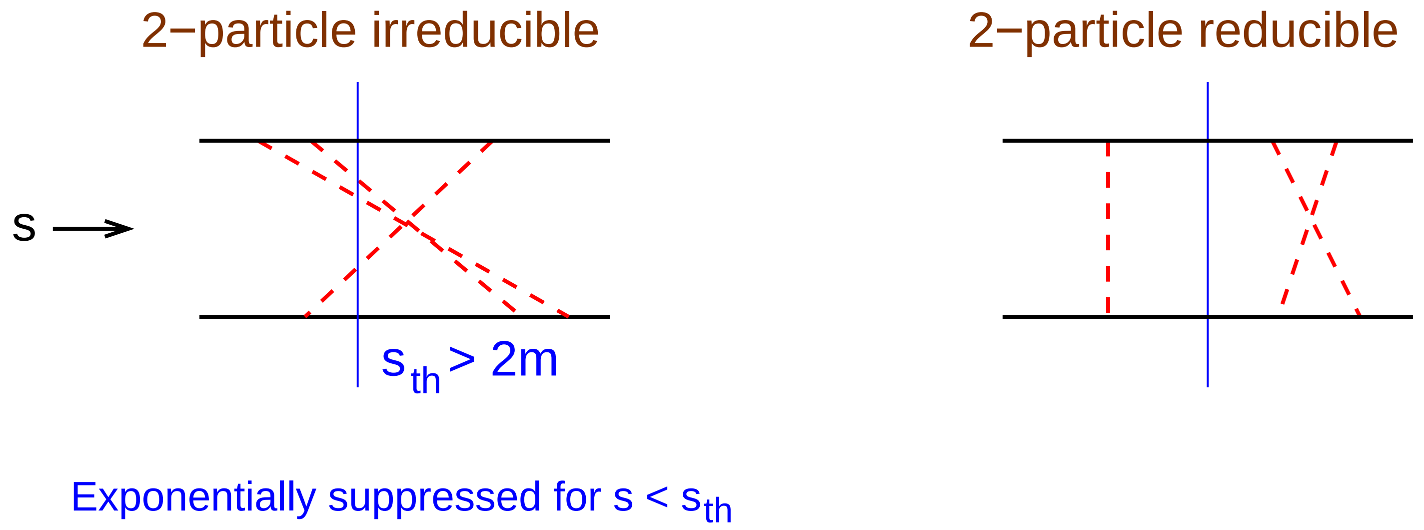
QCD $\longrightarrow$ ChPT $(p \sim M_\pi)$ $\longrightarrow$ non-rel. EFT $(p \ll M_\pi)$

- Lattice QCD $\longrightarrow$ EFT in a finite volume
- Short-distance behavior not changed: the same Lagrangian

$\hookrightarrow$ A bridge between the measured spectrum and scattering sector:

- Multichannel resonances
- Twisted boundary conditions
- Lüscher-Lellouch formula
- Few-body systems
- Resonance formfactors

Relativistic vs non-relativistic EFT



- Well below inelastic threshold, finite-volume corrections to the 2-particle irreducible diagrams are exponentially suppressed $\sim \exp(-\Delta E \cdot L)$
- 2-particle irreducible diagrams $\longrightarrow$ L-independent local vertices

$\hookrightarrow$ Particle number is conserved in intermediate states

Method: covariant NREFT in the infinite volume

G. Colangelo, J. Gasser, B. Kubis and A. Rusetsky, PLB 638 (2006) 187

J. Gasser, B. Kubis and A. Rusetsky, NPB 850 (2011) 96

The Lagrangian:

- Contains non-relativistic field operators
- Particle number is conserved
- Counting rules observed after applying "threshold expansion"
- Electromagnetic and weak interactions can be systematically included

$$\mathcal{L}_I = C_0 \Phi_1^\dagger \Phi_2^\dagger \Phi_1 \Phi_2 + \text{derivatives}$$

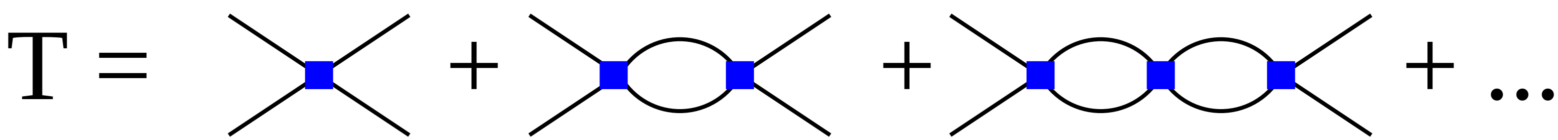
The propagator with the relativistic dispersion law:

$$D(p) = \frac{1}{2w(\mathbf{p})} \frac{1}{w(\mathbf{p}) - p_0 - i0}\,, \qquad w(\mathbf{p}) = \sqrt{M^2 + \mathbf{p}^2}$$

Lippmann-Schwinger equation

- Threshold expansion: in Feynman integrals, expand all integrands in powers of three-momenta, integrate in the dimensional regularization and sum up again

$$\hookrightarrow \text{loop} = \frac{ip}{8\pi\sqrt{s}}\,, \qquad p = \frac{\lambda^{1/2}(s, M_1^2, M_2^2)}{2\sqrt{s}}$$



$\hookrightarrow$ Scattering amplitude is Lorentz-invariant:

$$T_l = \frac{8\pi\sqrt{s}}{p\cot\delta_l(p) - ip}\,, \qquad p^{2l+1}\cot\delta_l(p) = -\frac{1}{a_l} + \frac{1}{2}r_l p^2 + \cdots$$

- Important in non-rest frames (formfactors, 3-body scattering)

Covariant NREFT in a finite volume

EFT in a finite volume with the *same* Lagrangian $\Rightarrow$ lattice QCD

Feynman loops are modified in a finite volume

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \to \frac{1}{L^3}\sum_{\mathbf{p}}$$ (finite box with periodic b.c.)

Lippmann-Schwinger equation in a finite volume

$$T_L(E) = V + VG_L(E)T_L(E)$$

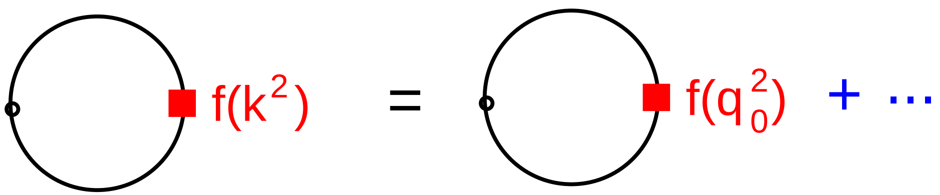
- The potential V is a low-energy polynomial, only exponentially suppressed finite-volume corrections $\to$ phase shift
- The finite-volume Green function $G_L(E)$ contains a tower of the real poles.
- Poles of $T_L(E)$ = spectrum in a finite volume

$\hookrightarrow$ Lüscher equation

Relativistic vs non-relativistic

- Matching is performed on mass shell. Consequently, only on-shell vertices appear in the perturbative expansion
- Threshold expansion automatically puts the amplitudes in the loops on energy shell:

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{f(\mathbf{k}^2)}{\mathbf{k}^2-q_0^2} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left(\frac{f(q_0^2)}{\mathbf{k}^2-q_0^2} + \underbrace{\text{regular}}_{=0,\ \text{threshold expansion}}\right)$$



- All results are the same as in the relativistic framework + skeleton expansion, come at a less effort

Example: $B \to K^*\ell^+\ell^-$ decays

- Rare B decays provide one of the best opportunities in search of BSM physics (anomalies)
- Theoretical uncertainties arise in the calculated form factors
  $\to$ lattice QCD
- In general, the decay products contain *resonances,* e.g. $B \to K^*\ell^+\ell^-$. A procedure for the extraction of the resonance form factors on the lattice should be defined
- Analytic continuation to the resonance pole should be studied. What is the photon virtuality q^2 *at the pole?*

The effective weak Hamiltonian for $B \to K^* \ell^+ \ell^-$ decay

Effective Hamiltonian:

$$\mathcal{H}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{10} C_i W_i$$

$$\begin{aligned}
W_7 &= \frac{m_b e}{16\pi^2} \bar{s} \sigma^{\mu\nu} \frac{1}{2} (1+\gamma_5) b \, F_{\mu\nu} \\
W_9 &= \frac{e^2}{16\pi^2} \bar{s} \gamma^{\mu} \frac{1}{2} (1-\gamma_5) b \, \bar{\ell} \gamma_{\mu} \ell \\
W_{10} &= \frac{e^2}{16\pi^2} \bar{s} \gamma^{\mu} \frac{1}{2} (1-\gamma_5) b \, \bar{\ell} \gamma_{\mu} \gamma^5 \ell
\end{aligned}$$

Seven form factors:

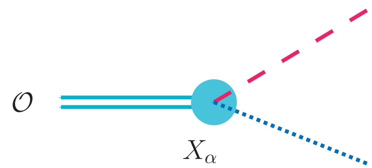
$$V(q^2), \quad A_0(q^2), \quad A_1(q^2), \quad A_2(q^2), \quad T_1(q^2), \quad T_2(q^2), \quad T_3(q^2)$$

Photon virtuality: $q^2 = (p_B - p_{K^*})^2$

Effective non-relativistic Lagrangian for the B-decays

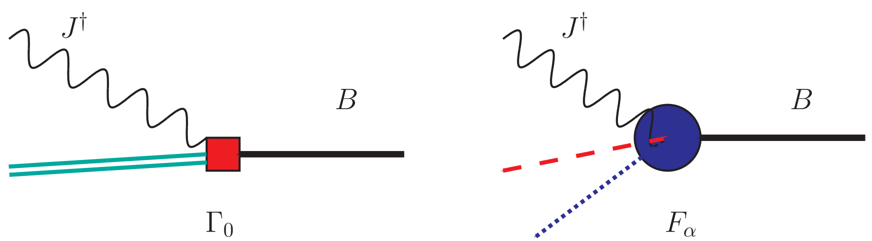
- Coupled-channel problem: $\underline{1)\ \pi K}$, $\underline{2)\ \eta K}$

$$\mathcal{L}_{\text{str}} = (\mathcal{O}^\dagger_{K^*} X_1 \pi K + \mathcal{O}^\dagger_{K^*} X_2 \eta K + \text{h.c.}) + \mathcal{L}_{4-\text{meson}}$$



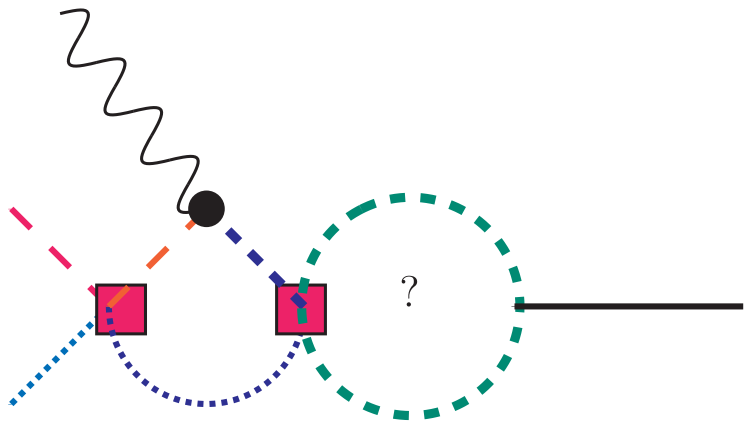
- Irreducible vertices describing the decay

$$\mathcal{L}_{\text{weak}} = (\mathcal{O}^\dagger_{K^*} J^\dagger \Gamma_0 B + \text{h.c.}) + (\pi^\dagger K^\dagger J^\dagger \bar{F}_1 B + \eta^\dagger K^\dagger J^\dagger \bar{F}_2 B + \text{h.c.})$$



Long-distance contributions?

There are no long-distance contributions in the matrix elements of the *flavor changing neutral corrents:*



$\hookrightarrow$ No vertices corresponding to $B \to K\pi$ or $B \to K\eta$ *without lepton-antilepton pair*

Lattice setting

- The K^* is unstable: need to scan the energy of the πK pair, photon 3-momentum fixed
- πK pair in the CM frame: no mixing
- Asymmetric box: $L \times L \times L'$, varying L, fixed L'
- Projection onto the irreps:

$$\mathcal{O}_{\mathbb{E}}^{(\pm)} = \frac{1}{\sqrt{2}}\,(\mathcal{O}_1 \mp \mathcal{O}_2)\,, \quad \mathcal{O}_{\mathbb{A}_1} = \mathcal{O}_3$$

- Extraction of the form factors:

$$\langle V(+)|\frac{1}{\sqrt{2}}\,\bar{s}(\gamma_1 + i\gamma_2)b|B\rangle = -\frac{2iE\,V(q^2)}{m_B + E}\,, \quad \text{etc}$$

Multi-channel amplitude in a finite volume

$$T_L = \frac{8\pi\sqrt{s}}{f(E)} \begin{pmatrix} \frac{1}{p_1}[t_1\tau_1(t_2+\tau_2)+s_\varepsilon^2\tau_1\tau_2 t] & -\frac{1}{\sqrt{p_1p_2}}c_\varepsilon s_\varepsilon \tau_1\tau_2 t \\ -\frac{1}{\sqrt{p_1p_2}}c_\varepsilon s_\varepsilon \tau_1\tau_2 t & \frac{1}{p_2}[t_2\tau_2(t_1+\tau_1)-s_\varepsilon^2\tau_1\tau_2 t] \end{pmatrix}$$

- $t_i = \tan\delta_i(p), \quad t = t_2 - t_1$: scattering phases
- $c_\varepsilon = \cos\varepsilon, \quad s_\varepsilon = \sin\varepsilon$: mixing angle
- $\tau_i = \tan\phi_i$: expressed through Lüscher zeta-functions

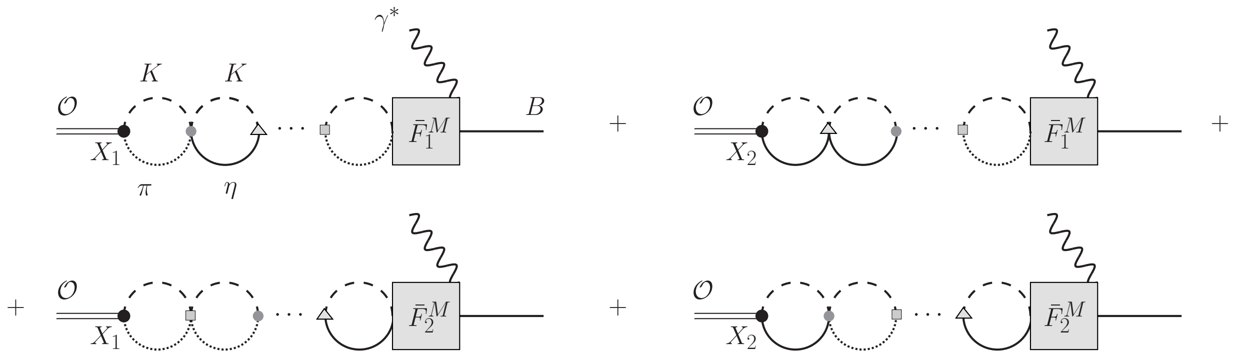
Lüscher equation in the two-channel case:

$$f(E) = (t_1+\tau_1)(t_2+\tau_2) + s_\varepsilon^2 t(\tau_2-\tau_1) = 0$$

Factorization of the amplitude near the eigenvalue:

$$T_L^{\alpha\beta}(E) \to \frac{f_\alpha f_\beta}{E_n - E} + \cdots$$

Lüscher-Lellouch formula, two-channel case



$$|F(E_n,|\mathbf{q}|)| = \frac{\mathcal{V}^{-1}}{8\pi E}\left|p_1\tau_1^{-1}f_1\,\bar F_1 + p_2\tau_2^{-1}f_2\,\bar F_2\right|\Bigg|_{E=E_n}$$

$$\mathcal{A}_1 = \bar F_1(c_\varepsilon^2\cos\delta_1 e^{i\delta_1} + s_\varepsilon^2\cos\delta_2 e^{i\delta_2}) + \sqrt{\frac{p_1}{p_2}}\,\bar F_2 c_\varepsilon s_\varepsilon(\cos\delta_1 e^{i\delta_1} - \cos\delta_2 e^{i\delta_2})$$

$$\mathcal{A}_2 = \bar F_2(c_\varepsilon^2\cos\delta_2 e^{i\delta_2} + s_\varepsilon^2\cos\delta_1 e^{i\delta_1}) + \sqrt{\frac{p_2}{p_1}}\,\bar F_1 c_\varepsilon s_\varepsilon(\cos\delta_1 e^{i\delta_1} - \cos\delta_2 e^{i\delta_2})$$

- $\bar F_1$, $\bar F_2$: Irreducible amplitudes, exponentially suppressed volume dependence
- Agrees with: M. Hansen and S. Sharpe, PRD 86 (2012) 016007

Extraction of the form factors at the pole

- T-matrix on the second Riemann sheet:

$$T_{II}^{\alpha\beta}(s) \to \frac{h_\alpha h_\beta}{s_R - s} + \cdots$$

- Form factors at the pole: residues in the pertinent Green functions

$$F_R(E_R, \mathbf{q}) = \frac{i}{8\pi E} \left(p_1 h_1 \bar{F}_1 - p_2 h_2 \bar{F}_2\right)\Big|_{E \to E_R}$$

- Universal: do not depend on the process!
- Uniquely defined!

Effective-range expansion for the matrix elements

- Extract irreducible amplitudes $\bar{F}_i(p,|\mathbf{Q}|)$ from the measured amplitude $F(p,|\mathbf{Q}|)$, multiplying by Lüscher-Lellouch factors (measurement at multiple energies/volumes, $|\mathbf{Q}|$ fixed)
- If there are no nearby resonances, $\bar{F}_i(p,|\mathbf{Q}|)$ are low-energy polynomials
- In the presence of a resonance, define:

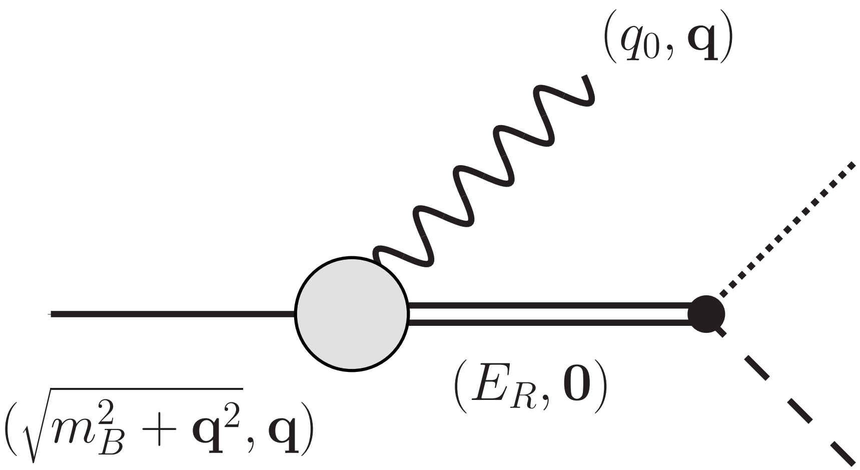
$$\tilde{u}_1 = t_1^{-1}(\sqrt{p_1}c_\varepsilon \bar{F}_1 + \sqrt{p_2}s_\varepsilon \bar{F}_2)$$
$$\tilde{u}_2 = t_2^{-1}(\sqrt{p_2}c_\varepsilon \bar{F}_2 - \sqrt{p_1}s_\varepsilon \bar{F}_1)$$

  Effective-range expansion: $\tilde{u}_i(p,|\mathbf{Q}|) = A_i(|\mathbf{Q}|) + p^2 B_i(|\mathbf{Q}|) + \cdots$

  Fit the real coefficients $A_i(|\mathbf{Q}|), B_i(|\mathbf{Q}|), \ldots$ to the data

- Find the form factor at the pole by substituting $p \to p_R$ in the effective range expansion

Photon virtuality



$$q^2 = (E_R - \sqrt{m_B^2 + \mathbf{q}^2})^2 - \mathbf{q}^2\,, \qquad \text{complex!}$$

- Keeping q^2 real and performing the limit $s \to E_R^2$ does not correspond to the resonance pole!
- Experimental data should be analysed in a similar manner!

Infinitely small width

- 1-channel case: $F_R = F$ up to normalization in the limit $\Gamma \to 0$
- 2-channel case: normalization different in different channels...

$\to$ Define

$$\tilde{u}_1 = t_1^{-1}(\sqrt{p_1}c_\varepsilon \bar{F}_1 + \sqrt{p_2}s_\varepsilon \bar{F}_2)$$
$$\tilde{u}_2 = t_2^{-1}(\sqrt{p_2}c_\varepsilon \bar{F}_2 - \sqrt{p_1}s_\varepsilon \bar{F}_1)$$

- $\tilde{u}_1,\ \tilde{u}_2$ are low energy polynomials: do not contain the small energy scale Γ
- At the resonance, the decay matrix element factorizes
  $\to \quad \tilde{u}_1 = O(\Gamma^{-1/2})$ and $\tilde{u}_2 = O(1)$

The form factor in the limit $\Gamma \to 0$

$$F_R(E_R,\mathbf{q}) = \frac{1}{\sqrt{4\pi}}\,(r_1\tilde{u}_1 + r_2\tilde{u}_2)\Big|_{E\to E_R}$$

$$r_1^2 = t_1^2\,\frac{t_2 + i - 2is_\varepsilon^2}{h'(E)}\,, \qquad r_2^2 = t_2^2\,\frac{t_1 - i + 2is_\varepsilon^2}{h'(E)}$$

$$h(E) = (t_1 - i)(t_2 + i) + 2is_\varepsilon^2(t_2 - t_1)$$

Resonance emerges in one channel:

t_1 diverges, t_2 stays finite, $t_1' = O(\Gamma^{-1})$

$$\hookrightarrow \quad F_R(E_R,\mathbf{q})\Big|_{\Gamma\to 0} = \frac{\sqrt{2E_n}}{\mathcal{V}}\,F(E_n,\mathbf{q}) + O(\Gamma^{1/2})$$

Conclusions, outlook

- Using non-relativistic EFT in a finite volume, a framework for the extraction of the resonance matrix elements from the lattice data is constructed. The case of the $B \to K^* \ell^+ \ell^-$ form factors is considered in great detail
- The multi-channel Lüscher-Lellouch formula is reproduced.
- The extraction of the form factors at the resonance pole is carried out by using effective-range expansion for thematrix elements
- It is shown that, at the resonance, the photon virtuality q^2 must be complex
- The limit $\Gamma \to 0$ is studied in detail in the multi-channel case