Loop formulation of supersymmetric models on the lattice

On the relevance of the sign problem

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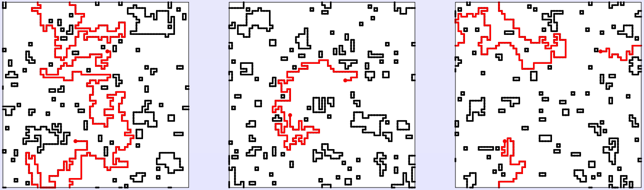


- Supersymmetry is thought to be a crucial ingredient in
  - the unification of the SM interactions,
  - the solution of the hierarchy problem.
- Low energy physics is, however, not supersymmetric:
  - SUSY must be broken spontaneously,
  - this can not be described in perturbation theory.
- Lattice provides a non-perturbative regularisation:
  - discretisation breaks Poincaré symmetry explicitely,
  - Leibniz' rule is absent,
  - fermion doubling,

  $\longrightarrow$ recovered in the continuum limit?
- Fermionic sign problem hampers MC simulations.

- Accidental symmetries may emerge from a non-symmetric lattice action:
  - lattice action enjoys some exact symmetries,
  - allows only irrelevant symmetry breaking operators,
  - they become unimportant in the IR,

  $\rightarrow$ full symmetry emerges in the continuum limit

- (Euclidean) Poincaré symmetry in lattice QCD.
- Supersymmetry in $\mathcal{N} = 1$ SYM:
  - only relevant operator is the gaugino mass term $m\bar{\xi}\xi$,
  - violates Z_{2N} chiral symmetry,
  - chirally symmetric lattice action forbids this term,

$\rightarrow$ SUSY automatically recovered in the continuum limit.

- For SUSY theories involving scalar fields this is not easily possible:
  - scalar mass term $m^2|\phi|^2$ breaks SUSY,
  - no other symmetry available to forbid that term.
- Some symmetries can be fine tuned with counterterms:
  - chiral symmetry for Wilson fermions,
  - might be feasible in lower dimensions if theories are superrenormalisable.
- Look for subalgebras of the SUSY algebra [Catterall; Kaplan; Ünsal; etc '01-'09]:
  - combine Poincaré and flavour group (twisted SUSY),
  - leads to Dirac-Kähler (staggered) fermions,
  - consistent with orbifolding approach.

- Consider the Lagrangian for supersymmetric quantum mechanics

$$\mathcal{L} = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}P'(\phi)^2 + \overline{\psi}\left(\frac{d}{dt} + P''(\phi)\right)\psi\,,$$

  - real commuting bosonic 'coordinate' ϕ,
  - complex anticommuting fermionic 'coordinate' ψ,
  - superpotential, e.g. $P(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{4}g\phi^4$.

- Two supersymmetries in terms of Majorana fields $\psi_{1,2}$:

$$\begin{array}{rclrcl} \delta_A\phi &=& \psi_1\varepsilon_A, & \delta_B\phi &=& \psi_2\varepsilon_B, \\ \delta_A\psi_1 &=& \frac{d\phi}{dt}\varepsilon_A, & \delta_B\psi_1 &=& -iP'\varepsilon_B, \\ \delta_A\psi_2 &=& iP'\varepsilon_A, & \delta_B\psi_2 &=& \frac{d\phi}{dt}\varepsilon_B. \end{array}$$

- Define fields on lattice sites $x = na$, $n = 0, \ldots, L-1$.
- To eliminate fermion doubling in 1D use forward or backward derivative

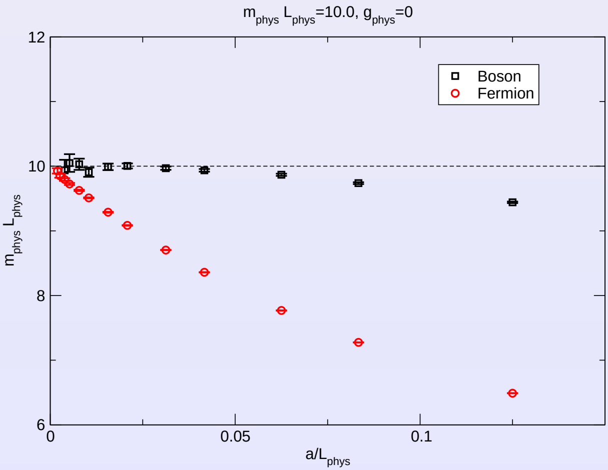
$$\nabla\phi(x) = \phi(x+a) - \phi(x), \quad \nabla^*\phi(x) = \phi(x) - \phi(x-a).$$

- SUSY variation δ_A leads to

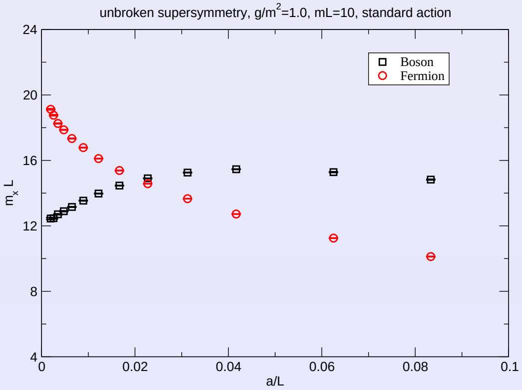
$$\delta_A S_L = i\varepsilon_A \sum_x \psi_2 \left(-\nabla P' + P''\nabla^*\phi\right),$$

  - due to the absence of the Leibniz rule on the lattice,
  - term is $\mathcal{O}(a)$ and vanishes in the naive continuum limit,
  - vanishes at finite a if $g = 0$.

Standard discretisation at $g = 0$

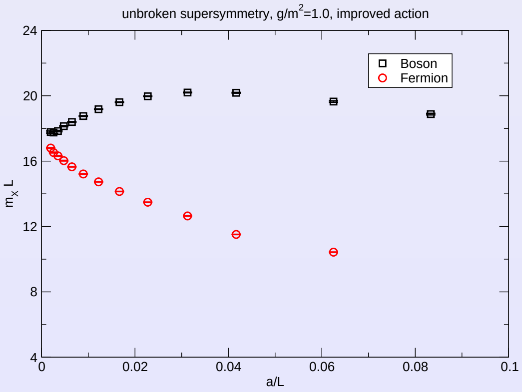


- Introduce interaction term $P''(\phi)\overline{\psi}\psi \propto g\overline{\psi}\psi\phi^2$:



radiative corrections spoil continuum limit...

- Introduce interaction term $P''(\phi)\overline{\psi}\psi \propto g\overline{\psi}\psi\phi^2$:



$\Rightarrow$ can be tuned away by adding counter term $\frac{1}{2}P''(\phi) \propto g\phi^2$

- Find a combination of supersymmetries which can be transferred to the lattice.
- Recall symmetry breaking of the lattice action:

$$\begin{aligned} \delta_A S_L &= i\varepsilon_A \sum_x \psi_2 \left(-\nabla P' + P''\nabla^* \phi\right) \\ &= -i\delta_B \sum_x P'\,\nabla^*\phi. \end{aligned}$$

- Notice the similar term for δ_B,

$$\delta_B S_L = i\delta_A \sum_x P'\,\nabla^*\phi,$$

so the linear combination $\delta \equiv \delta_A + i\delta_B$ gives

$$\delta S_L = -\delta \sum_x P'\,\nabla^*\phi.$$

- Correction term $P' \nabla^* \phi$ is a surface term vanishing in the limit $a \to 0$.
- Corrected lattice action is invariant under the 'twisted' supersymmetry δ:

$$S_L^{\text{exact}} = \sum_x \frac{1}{2} (\nabla^* \phi)^2 + \frac{1}{2} P'^2 + \overline{\psi} (\nabla^* + P'') \psi + P' \nabla^* \phi$$

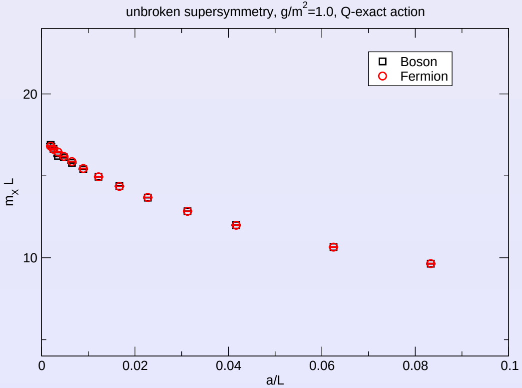
- Note that the bosonic action can also be written as

$$S_B^{\text{exact}} = \sum_x \frac{1}{2} (\nabla^* \phi + P')^2$$

which exposes the relation to a (local) Nicolai map

- variable transformation $\phi \to \mathcal{N} = \nabla^* \phi + P'(\phi)$,
- action becomes Gaussian,
- Jacobian cancels exactly the fermion determinant.

- Now simulate this SUSY-exact (or Q-exact) action:



- Witten index provides a necessary but not sufficient condition for SSB:

$$W \equiv \lim_{\beta\to\infty} \mathrm{Tr}(-1)^F \exp(-\beta H) \quad \Rightarrow \begin{cases} = 0 & \text{SSB may occur} \\ \neq 0 & \text{no SSB} \end{cases}$$

- Index counts the difference between the number of bosonic and fermionic zero energy states:

$$W \equiv \lim_{\beta\to\infty} \left[\mathrm{Tr}_B \exp(-\beta H) - \mathrm{Tr}_F \exp(-\beta H)\right] = n_B - n_F$$

- Index is equivalent to partition function with periodic b.c.:

$$W = \int_{-\infty}^{\infty} \mathcal{D}\phi \;\det\left[\not{D}(\phi)\right]\; e^{-S_B[\phi]} = Z_{\mathrm{per}}$$

$\Rightarrow$ Determinant (or Pfaffian) must be indefinite for SSB.

- Recall the Lagrangian for supersymmetric quantum mechanics

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}P'(\phi)^2 + \overline{\psi}\left(\frac{\partial}{\partial t} + P''(\phi)\right)\psi\,,$$

- The (regulated) fermion determinant can be calculated exactly:

$$\det\left[\frac{\partial_t + P''(\phi)}{\partial_t + m}\right] = \sinh\int_0^T \frac{P''(\phi)}{2}dt \quad\Longrightarrow\quad Z_0 - Z_1$$

- If under some symmetry $\phi \to \phi'$ of $S_B(\phi)$ we have

$$\int_0^T \frac{P''(\phi')}{2}dt \quad = \quad \begin{cases} +\int_0^T \frac{P''(\phi)}{2}dt & \text{no SSB} \quad\Rightarrow\quad Z_0 \neq Z_1 \\ -\int_0^T \frac{P''(\phi)}{2}dt & \text{SSB} \quad\Rightarrow\quad Z_0 = Z_1 \end{cases}$$

- On the lattice we find with Wilson type fermions

$$\det\left[\nabla^* + P''(\phi)\right] = \prod_t \left[1 + P''(\phi_t)\right] - 1\,.$$

- For even potentials, e.g. $P(\phi) = \frac{1}{2} m\,\phi^2 + \frac{1}{4} g\phi^4$ we have

$$\det\left[\nabla^* + P''\right] = \prod_t \left[1 + m + 3g\phi_t^2\right] - 1 \geq 0$$

for $m \geq 0, g \geq 0$.

- As a side remark, note that

$$\lim_{a\to 0} \det\left[\nabla^* + P''\right] \sim \exp \int_0^T \frac{P''(\phi)}{2} dt \;\; \det\left[\partial_t + P''(\phi)\right]\,,$$

so this term needs 'fine tuning'.

- For odd potentials, e.g. $P(\phi) = \frac{m^2}{2\lambda}\phi^2 + \frac{1}{3}\lambda\phi^3$ we have

$$\det\left[\nabla^* + P''\right] = \prod_t \left[1 + 2\lambda\phi_t\right] - 1$$

no longer positive...

$\Rightarrow$ sign problem!

- Every supersymmetric model which allows SSB must have a sign problem:
  - SUSY QM with odd potential,
  - $\mathcal{N} = 16$ Yang-Mills quantum mechanics [Catterall, Wiseman '07],
  - $\mathcal{N} = 1$ Wess-Zumino model in 2D [Catterall '03],
  - $\mathcal{N} = (2,2)$ Super-Yang-Mills in 2D [Giedt '03].

- We propose a novel approach circumventing these problems [Wenger '08]:
  - based on the exact hopping expansion of the fermion action,
  - eliminates critical slowing down,
  - allows simulations directly in the massless limit,

  $\Rightarrow$ solves the fermion sign problem.

- Applicable to the
  - Gross-Neveu model in $d = 2$ dimensions,
  - Schwinger model in the strong coupling limit in $d = 2$ and 3,
  - SUSY QM,
  - $\mathcal{N} = 1$ and 2 supersymmetric Wess-Zumino model.

- Consider now the $\mathcal{N} = 1$ Wess-Zumino model in 2D:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}P'(\phi)^2 + \frac{1}{2}\overline{\psi}(\not\partial + P''(\phi))\psi$$

  - with ψ a Majorana field,
  - and ϕ real bosonic field,
  - superpotential, e.g. $P(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{3}g\phi^3$.

- Integrating out Majorana fermions yields indefinite Pfaffian.

- For Majorana fermions use exact reformulation in terms of loops [old idea]:

$\Rightarrow$ sign of Pfaffian under perfect control

- Can also be done for bosonic fields [Prokof'ev, Svistunov '01].

- Using Wilson lattice discretisation for the fermionic part:

$$\mathcal{L} = \frac{1}{2}\xi^T \mathcal{C}(\gamma_\mu \tilde{\partial}_\mu - \frac{1}{2}\partial^* \partial + P''(\phi))\xi \,,$$

  - ξ is a real, 2-component Grassmann field,
  - $\mathcal{C} = -\mathcal{C}^T$ is the charge conjugation matrix.

- Using the nilpotency of Grassmann elements we expand the Boltzmann factor

$$\int \mathcal{D}\xi \prod_x \left(1 - \frac{1}{2}M(x)\xi^T(x)\mathcal{C}\xi(x)\right) \prod_{x,\mu} \left(1 + \xi^T(x)\mathcal{C}\Gamma(\mu)\xi(x+\hat{\mu})\right)$$

where $M(x) = 2 + P''(\phi)$ and $\Gamma(\pm\mu) = \frac{1}{2}(1 \mp \gamma_\mu)$.

- At each site, the fields $\xi^T \mathcal{C}$ and ξ must be exactly paired to give a contribution to the path integral:

$$\int \mathcal{D}\xi \prod_x \left(M(x)\xi^T(x)\mathcal{C}\xi(x)\right)^{m(x)} \prod_{x,\mu} \left(\xi^T(x)\mathcal{C}\Gamma(\mu)\xi(x+\hat{\mu})\right)^{b_\mu(x)}$$

  with occupation numbers
  - $m(x) = 0, 1$ for monomers,
  - $b_\mu(x) = 0, 1$ for fermion bonds (or dimers),

  satisfying the constraint

$$m(x) + \frac{1}{2}\sum_\mu b_\mu(x) = 1.$$

- Only closed, non-intersecting paths survive the integration.

- Analogous treatment for the bosonic field [Prokof'ev, Svistunov '01]:
  - $(\partial_\mu \phi)^2 \quad \rightarrow \quad \phi_{x+\hat{\mu}}\phi_x$,
  - expand hopping term to all orders:

$$\int \mathcal{D}\phi \prod_{x,\mu} \sum_{n_\mu(x)} \frac{1}{n_\mu(x)!} \left(\phi_x \phi_{x+\hat{\mu}}\right)^{n_\mu(x)} \exp\left(-V(\phi_x)\right) \, M(\phi_x)^{m(x)}$$

with bosonic bond occupation numbers $n_\mu(x) = 0, 1, 2, \ldots$

- Integrating out $\phi(x)$ yields bosonic site weights

$$Q(N) = \int d\phi \; \phi^N \exp\left(-V(\phi)\right)$$

where N includes powers from $M(\phi)$.

- Loop representation in terms of fermionic monomers and dimers and bosonic bonds.

- Partition function summing over all non-oriented, self-avoiding fermion loops

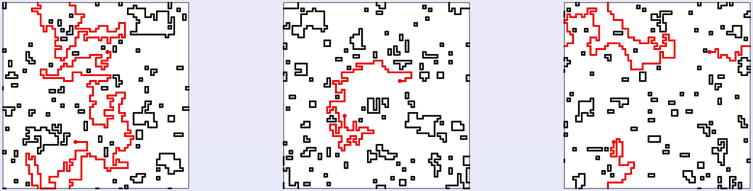
$$Z_{\mathcal{L}} = \sum_{\{\ell\} \in \mathcal{L}} \sum_{\{n_\mu\}} |\omega[\ell, n_\mu(x), m(x)]|, \qquad \mathcal{L} \in \mathcal{L}_{00} \cup \mathcal{L}_{10} \cup \mathcal{L}_{01} \cup \mathcal{L}_{11}$$

  represents a system with unspecified fermionic b.c. [Wolff '07].

- Simulate bosons with worm algorithm [Prokof'ev, Svistunov '01].

- Simulate fermions by enlarging the configuration space by one open fermionic string [Wenger '08].

- The open fermionic string corresponds to the insertion of a Majorana fermion pair $\{\xi^T(x)\mathcal{C}, \xi(y)\}$ at position x and y:



- It samples the relative weights between $Z_{\mathcal{L}_{00}}, Z_{\mathcal{L}_{10}}, Z_{\mathcal{L}_{01}}, Z_{\mathcal{L}_{11}}$.
- Reconstruct the Witten index a posteriori

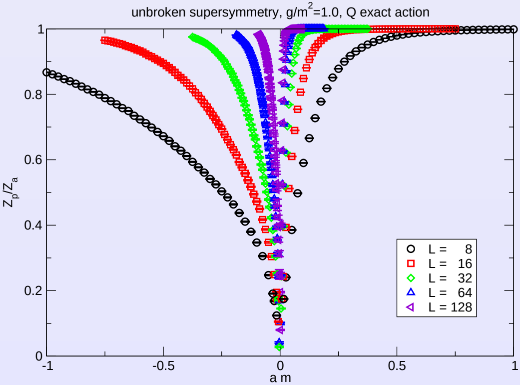
$$W \equiv Z^{pp} = Z_{\mathcal{L}_{00}} - Z_{\mathcal{L}_{10}} - Z_{\mathcal{L}_{01}} - Z_{\mathcal{L}_{11}},$$

or a system at finite temperature

$$Z^{pa} = Z_{\mathcal{L}_{00}} - Z_{\mathcal{L}_{10}} + Z_{\mathcal{L}_{01}} + Z_{\mathcal{L}_{11}}.$$

- Especially simple for supersymmetric QM:

$$Z^p = Z_{\mathcal{L}_0} - Z_{\mathcal{L}_1} \quad \Rightarrow \text{Witten index}$$
$$Z^a = Z_{\mathcal{L}_0} + Z_{\mathcal{L}_1} \quad \Rightarrow \text{finite temperature}$$



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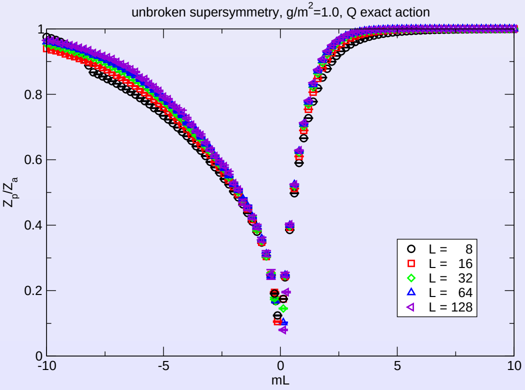
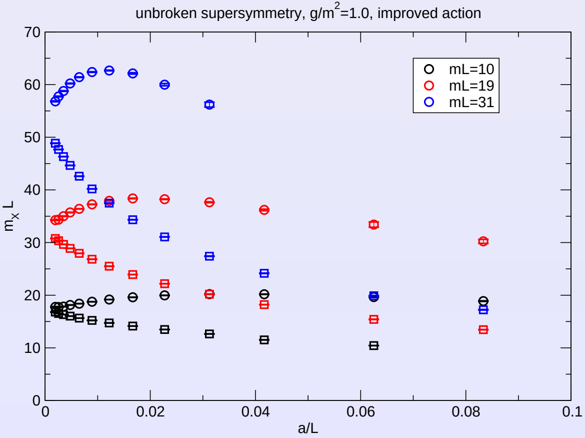


- Standard discretisation at $g \neq 0$ with counterterm:



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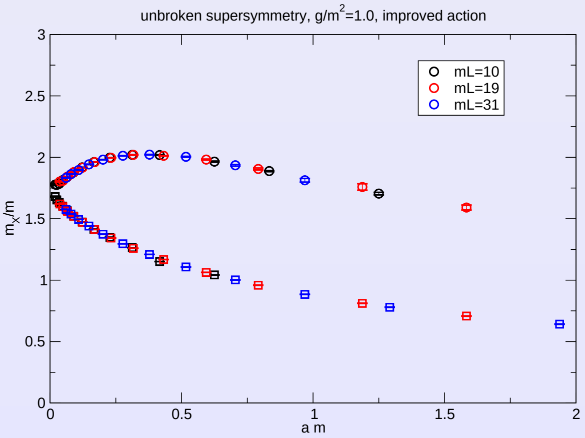

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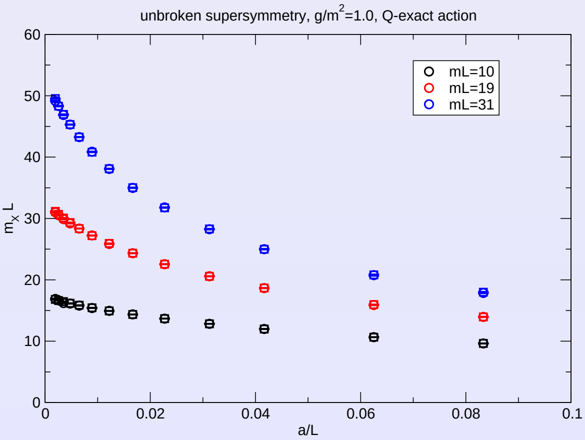

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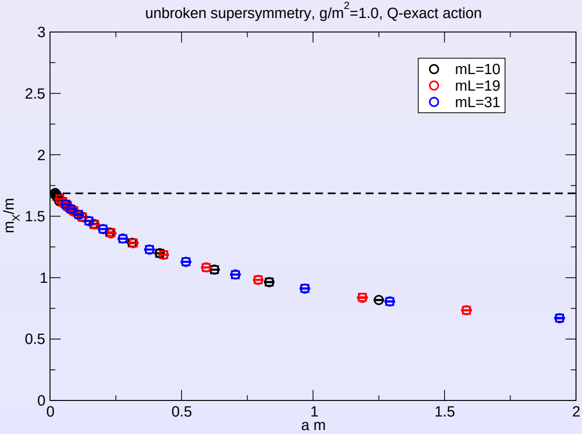
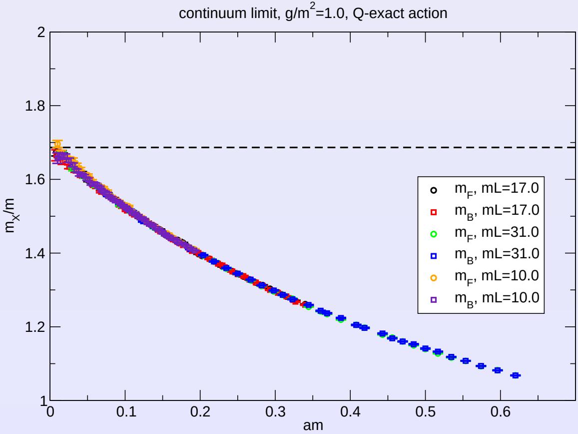
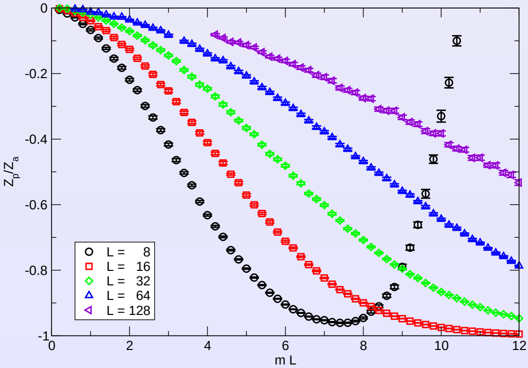


- High precision consistency check:



- Broken supersymmetry:

  standard, perturbatively improved action, $\lambda^2/m^3 = 1.0$:



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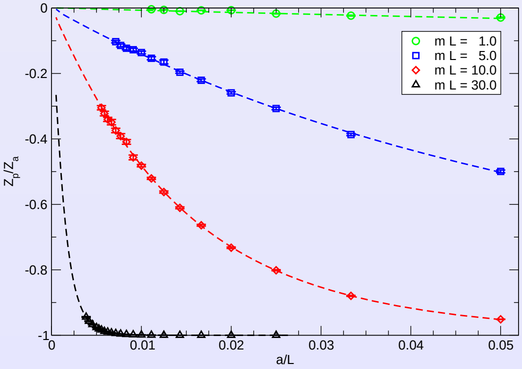


- Exact calculations can be done via the transfermatrix.
- In the loop formulation:
  - states are characterised by the bond occupation numbers:

$$|b(x), n(x)\rangle$$

  - transfer matrix on the dual lattice:

$$T_{|b(x-1),n(x-1)\rangle,\langle b(x),n(x)|}$$

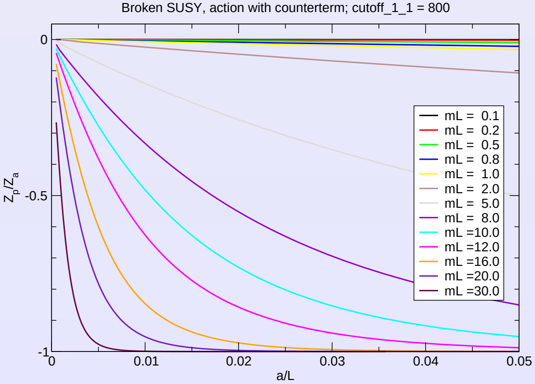
- Transfermatrix block diagonalises into bosonic and fermionic parts, $T^0_{|0,n(x-1)\rangle,\langle 0,n(x)|}$ and $T^1_{|1,n(x-1)\rangle,\langle 1,n(x)|}$.
- Specifically,

$$T^1_{|1,n(x-1)\rangle,\langle 1,n(x)|} = \frac{Q(n(x-1)+n(x))}{\sqrt{n(x-1)!n(x)!}}$$

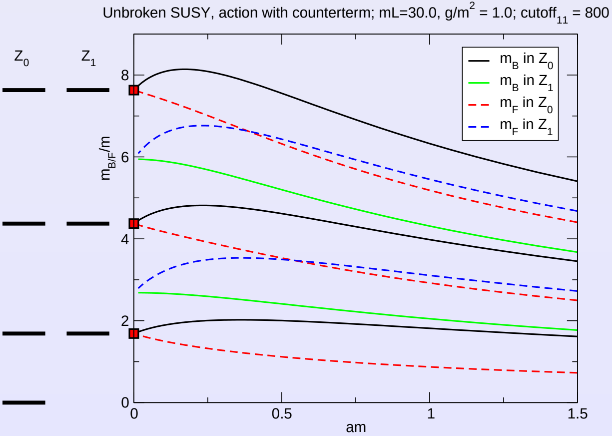
and T^0 analogously.

- Partition functions on a L_t lattice are given by $Z_i = \left(T^i\right)^{L_t}$.
- Witten index

$$W \equiv Z_p = Z_0 - Z_1 = \left(T^1\right)^{L_t} - \left(T^0\right)^{L_t}$$



- Mass gaps can be calculated from ratios of eigenvalues of T^0 and T^1.



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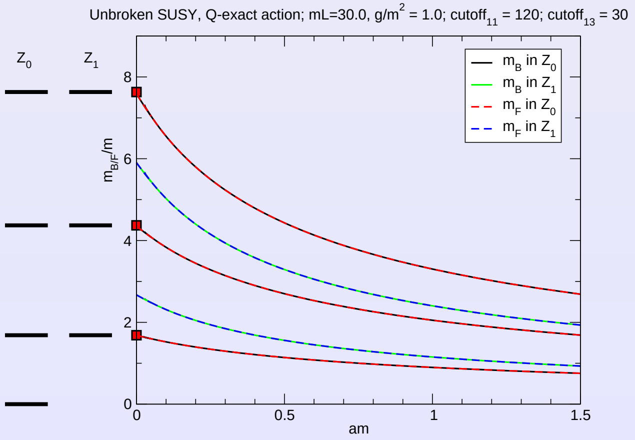

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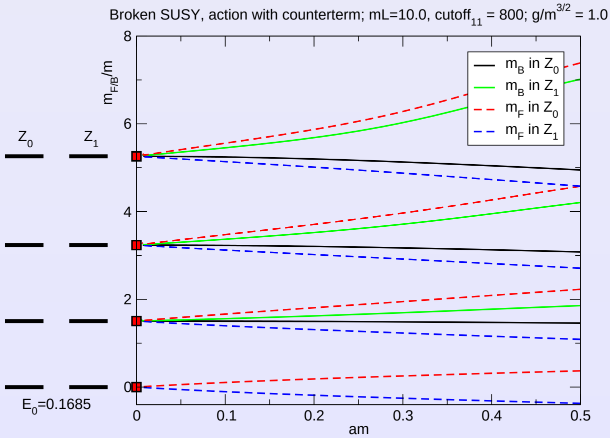

- Construction of Q-exact discretisations on the lattice.
- The fermionic sign problem and its relevance to the Witten index.
- Representation of $\mathcal{N} = 2$ SUSY QM and $\mathcal{N} = 1$ Wess-Zumino model in terms of interacting bosonic and fermionic loops.
- Use of topological boundary conditions for the solution of the sign problem.
- Results for the Witten index and the spectrum in SUSY QM for both the broken and unbroken case.
- Goldstino mode is clearly exposed.
- $\mathcal{N} = 1$ $2d$ Wess-Zumino model with SSB under way.