The leading hadronic vacuum polarization on the lattice

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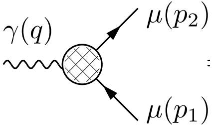
Outline

- Motivations: The muon $g-2$, three σ between experiment and theory.
- Challenges of the lattice approach.
- Field theoretical description of connected and disconnected quark diagrams. PqQCD and PqχPT.
- Improved momentum resolution of the connected contribution by using twisted boundary conditions.
- Numerical results (Wilson clover, $N_{\rm f} = 2$).
- Conclusions

The anomalous magnetic moment of the muon $a_\mu = \frac{g_\mu - 2}{2}$ is defined through the $q^2 \to 0$ limit of the vertex

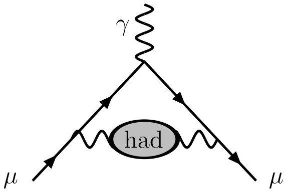


- ⋆ It is one of the most precisely measured quantity; 0.5 ppm at Brookhaven. Theory is at the same level.
- ⋆ It mediates helicity flips transition and therefore has a sensitivity $O\left(\frac{m_\mu^2}{M^2}\right)$ to particles of mass M beyond the SM. Best sensitivity to nearby NP.
- ⋆ 3 sigmas discrepancy between exp and theo [Jegerlehner and Nyffeler, 2009] :

$$a_\mu^{exp} = 1.16592080(63) \times 10^{-3}$$

$$a_\mu^{the} = 1.16591790(65) \times 10^{-3}$$

Contribution	Value	Error
QED incl. 4-loops+LO 5-loops	116 584 718.1	0.2
Leading hadronic vacuum polarization	6 903.0	52.6
Subleading hadronic vacuum polarization	-100.3	1.1
Hadronic light–by–light	116.0	39.0
Weak incl. 2-loops	153.2	1.8
Theory	116 591 790.0	64.6



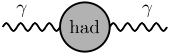
The *theory* number is obtained by estimating the hadronic contribution to the photon propagator from the hadronic e^+e^- annihilation cross-section. The latter comes from:

- experimental e^+e^- data.
- Hadronic τ decays. The hadronic matrix elements are related to those in e^+e^- annihilation by isospin (broken symmetry). This would give agreement with the experimental number.

$\Rightarrow$ need for a purely theoretical number.

The Euclidean hadronic vacuum polarisation tensor is defined as

$$\Pi_{\mu\nu}^{(N_f)}(q) = i \int d^4 x e^{iqx} \langle J_\mu^{(N_f)}(x) J_\nu^{(N_f)}(0) \rangle$$



Euclidean invariance and current conservation imply

$$\Pi_{\mu\nu}^{(N_f)}(q) = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi^{(N_f)}(q^2)$$

The relation between $\Pi_{\mu\nu}^{(N_f)}(q^2)$ and a_μ^{HLO} is [T. Blum, 2002]

$$a_\mu^{HLO} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dq^2\, f(q^2) \hat{\Pi}(q^2)$$

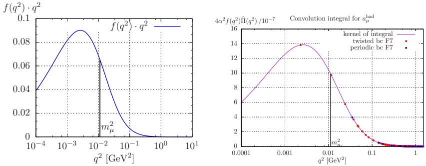
with

$$f(q^2) = \frac{m_\mu^2 q^2 Z^3 (1 - q^2 Z)}{1 + m_\mu^2 q^2 Z^2}\,, \qquad Z = -\frac{q^2 - \sqrt{q^4 + 4 m_\mu^2 q^2}}{2 m_\mu^2 q^2}$$

and $\hat{\Pi}(q^2) = 4\pi^2 \left[\Pi(q^2) - \Pi(0)\right]$.

Main challenges

- The integral is dominated by the $q^2 \simeq m_\mu^2$ region



  The lowest Fourier mode $2\pi/L$ is typically much larger than m_μ, and the VP has to be extrapolated to the relevant region [FSE].
- Correlator of flavor-diagonal currents $\rightarrow$ disconnected diagrams.
- The vector resonances (ρ , ω , ϕ) give a large contribution. Those have to be properly included in the lattice computation.
- Previous lattice efforts
  $a_\mu^{HLO} = 748(21) \times 10^{-10}$ [Aubin and Blum, 2007] $N_f = 2 + 1$ ($\sqrt{\text{staggered}}$).
  $a_\mu^{HLO} = 446(23) \times 10^{-10}$ [QCDSF, 2003] quenched.
  computation in progress by ETMC [Renner, 2009,2010] $N_f = 2$.

Let's consider the 2 flavor case and the contribution form the u quark only

$$C^{uu}_{\mu\nu,QCD}(q) = \frac{4}{9}\int d^4x e^{iqx}\langle J^{uu}_\mu(0)J^{uu}_\nu(x)\rangle$$

After Wick contractions the correlator $\langle J^{uu}_\mu(0)J^{uu}_\nu(x)\rangle$ is written in terms of connected and disconnected quark diagrams

We add a quark r degenerate with u and quenched it away (valence quark)

$$L^{(2)}_{QCD} \to L^{(2)}_{QCD} + \bar{r}(\not{D} + m_u)r + \tilde{r}^\dagger(\not{D} + m_u)\tilde{r} = L_{PqQCD}$$

$\tilde{r}$ is a commuting spin 1/2 filed, a ghost. PqQCD is an unphysical theory. However the partition function is the same as in QCD and in this theory the (dis)connected diagrams above can be written as correlation functions.

$$
\begin{aligned}
C^{uu}_{\mu\nu,QCD}(q) &= \frac{4}{9}\int d^4x e^{iqx}\langle J^{ur}{}_{\mu}(0) J^{ru}{}_{\nu}(x)\rangle_{PqQCD} + \\
&\quad \frac{4}{9}\int d^4x e^{iqx}\langle J^{uu}_{\mu}(0) J^{rr}{}_{\nu}(x)\rangle_{PqQCD} \\
&= C^{conn}_{\mu\nu,PqQCD}(q) + C^{disc}_{\mu\nu,PqQCD}(q)
\end{aligned}
$$

– A low energy effective theory (PqχPT) exists, which should describe correlation functions in PqQCD. This theory is based on an extended (graded) flavor-symmetry group.

– The connected part is not flavor-diagonal anymore in PqQCD. Twisting [Sachrajda and Villadoro, 2005] can be used there to improve the momentum resolution.

For the case of 2 degenerate flavors there's no need for the quark r, making the charge factors explicit and using iso-spin invariance

$$\begin{aligned}\Pi^{(2)}_{\mu\nu}(q^2) = &\quad i\frac{5}{9}\int d^4x e^{iqx}\langle J^{ud}_\mu(0)J^{du}_\nu(x)\rangle + \\ &\quad i\frac{1}{9}\int d^4x e^{iqx}\langle J^{uu}_\mu(0)J^{dd}_\nu(x)\rangle \\ = &\quad \Pi^{(2),conn}_{\mu\nu}(q^2) + \Pi^{(2),disc}_{\mu\nu}(q^2)\end{aligned}$$

For a computation in SU(2) χPT it is convenient to rewrite the currents using the generators

$$\begin{aligned}J^{uu}_\mu &= \tfrac{1}{2}\bar\psi(\sqrt{2}\sigma_0+\sigma_3)\gamma_\mu\psi\,, & J^{dd}_\mu &= \tfrac{1}{2}\bar\psi(\sqrt{2}\sigma_0-\sigma_3)\gamma_\mu\psi\,,\\ J^{ud}_\mu &= \tfrac{1}{2}\bar\psi(\sigma_1+i\sigma_2)\gamma_\mu\psi\,, & J^{du}_\mu &= \tfrac{1}{2}\bar\psi(\sigma_1-i\sigma_2)\gamma_\mu\psi\,,\end{aligned}$$

where $\psi^T=(u,d)$, and $\sigma_0=\sqrt{1/2}\,I_{2\times 2}$.

$O(p^2)$ SU(2) chiral Lagrangian [Gasser and Leutwyler, '85]

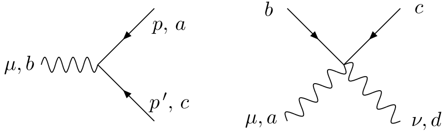
$$\mathcal{L}^{(2)} = \frac{F_0}{4}\mathrm{Tr}(D_\mu U D^\mu U^\dagger) + \frac{1}{2}\mathrm{Tr}(MU^\dagger + M^\dagger U)\ , \quad U = exp(i\lambda_a\phi_a/F_0)$$

The vector current is described by external fields $v_\mu = \frac{\lambda_a}{2}\cdot v_\mu^a$ coupled through the covariant derivative

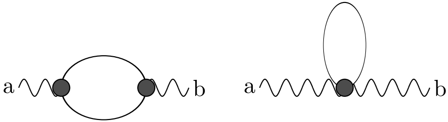
$$D_\mu U = \partial_\mu U + iv_\mu U - iUv_\mu$$

Note: Green functions of singlet current do not receive contributions at LO. The expansion produces the vertices

$$\mathcal{L}_{int} = -f_{abc}\phi_c\partial_\mu\phi_a v_\mu^b + \frac{1}{2}f_{abg}f_{cdg}\phi_b\phi_c v_\mu^a v_\mu^d$$



These give the 1-loop diagrams contributing to the vector-vector correlator



they both diverge. The divergences are removed by tree level insertions of the $O(p^4)$ terms prop. to l_5 and h_1, which get scale dependent. In addition a coupling h_s is introduced to parameterise the dynamics of flavor-singlet contributions [Kaiser, 2001]. At this order it is scale-independent.

$$\Pi^{(2)}(q^2) = - \left(\Lambda^{(2)}(M_\pi) + \frac{2}{9} h_s + i4\bar{B}_{21}(q^2, M_\pi^2)\right),$$

$$\Pi^{(2)}_{conn}(q^2) = - \frac{10}{9}\left(\Lambda^{(2)}(M_\pi) + i4\bar{B}_{21}(q^2, M_\pi^2)\right),$$

$$\Pi^{(2)}_{disc}(q^2) = \frac{1}{9}\left(\Lambda^{(2)}(M_\pi) - 2h_s + i4\bar{B}_{21}(q^2, M_\pi^2)\right)$$

with $\Lambda^{(2)}(M_\pi) = 2(l_5(M_\pi) + h_1(M_\pi))$

This leads to the simple result

$$\frac{\hat{\Pi}^{(2)}_{disc}(q^2)}{\hat{\Pi}^{(2)}_{conn}(q^2)} = -\frac{1}{10}$$

at NLO in mesonic χPT.

For the 2+1 or 2+ quenched strange cases one has to introduce one and two valence quarks at least (there's no degenerate partner of the strange quark) and therefore consider $SU(4|1)$ and $SU(4|2)$ PqχPT.

The flavor-symmetry group of PqQCD is $SU(N_s + N_v|N_v)_L \times SU(N_s + N_v|N_v)_R$. An element U of $SU(N_s + N_v|N_v)$ can be written as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

A is $(N_s + N_v) \times (N_s + N_v)$ commuting numbers, D is $N_v \times N_v$ commuting numbers, B and C are made of anti-commuting numbers. One introduces

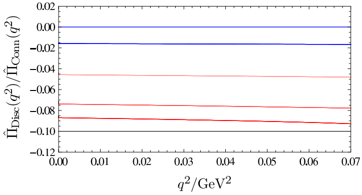
$$Str(U) = Tr(A) - Tr(D) \ \Rightarrow Sdet(U) = e^{(Str\ log(U))} = \frac{det(A - BD^{-1}C)}{det(D)}$$

s.t. the Str has the cyclic property, if in addition $(ab)^* = b^* a^*$ then the U matrices are unitary and with $Sdet(U) = 1$. The Lie algebra is defined through commutation and anti-commutation relations

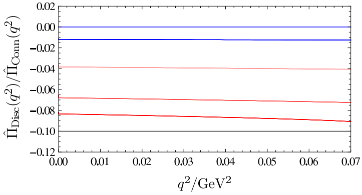
$$t_a t_b - (-1)^{\eta_a \eta_b} t_b t_a = i \sum_c C^c_{ab} t_c$$

$\eta_a = 1$ if t_a mixes valence or sea with ghosts.

For our purposes $Pq\chi PT$ is obtained by $Tr \to Str$ in the χPT Lagrangian



(a)



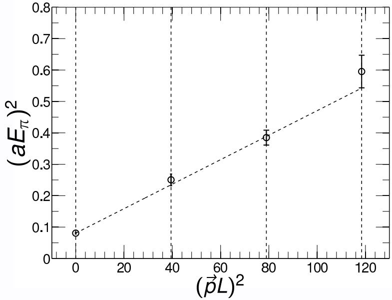
(b)

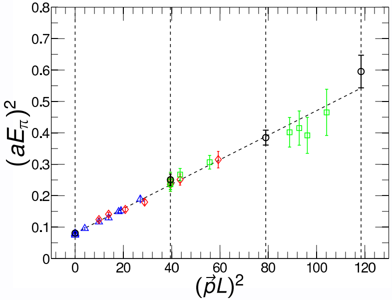
(a) $N_f = 2$ with a quenched strange quark,
(b) $N_f = 2+1$. From top to bottom: $M_\pi = M_K$, then fixed $M_K = 495$MeV and $M_\pi = 400$, 300, 200, 139 MeV. The bottom most-line at -1/10 is the result for $N_f = 2$.

Partially Twisted boundary conditions

Valence quarks periodic in space up to a phase $\longrightarrow$ spatial momentum to hadrons up to exponentially small finite volume effects [Sachrajda and Villadoro, 2005]

$$E_\pi = \sqrt{m_\pi^2 + (\vec{n}\frac{2\pi}{L})^2} \qquad E_\pi = \sqrt{m_\pi^2 + (\vec{n}\frac{2\pi}{L} + \frac{\vec{\theta}_u - \vec{\theta}_d}{L})^2}$$





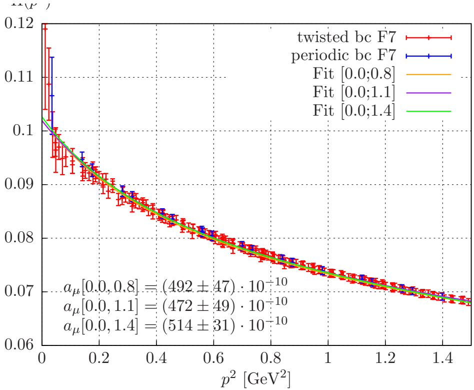
[Flynn, Jüttner and Sachrajda, 2006]

Ensembles of gauge configurations for $N_f = 2$ NP O(a) improved Wilson fermions generated within the CLS community effort by using the DD-HMC [Lüscher, 2005] algorithm.

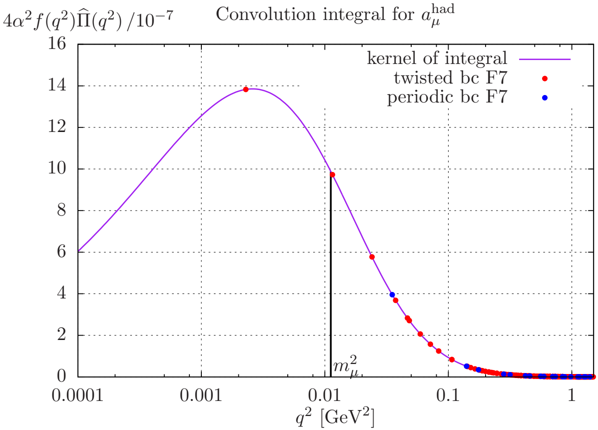
run	κ_{sea}	β	L [fm]	N_{cfg}	m_π [MeV]
$A2$	0.13590	5.2	2.6	125	~ 580
$A3$	0.13605	5.2	2.6	133	~ 460
$A4$	0.13610	5.2	2.6	200	~ 350
$A5$	0.13625	5.2	2.6	108	~ 300
$E2$	0.13590	5.3	2.2	158	696.5(0.9)
$E3$	0.13605	5.3	2.2	156	593.4(1.1)
$E4$	0.13610	5.3	2.2	162	554.2(1.1)
$E5$	0.13625	5.3	2.2	168	414.4(1.4)
$F6$	0.13635	5.3	3.3	200	297.9(0.9)
$F7$	0.13638	5.3	3.3	205	~ 250
$N5$	0.13660	5.5	2.5	161	~ 410

$0.05 \leq a \leq 0.08$ fm.

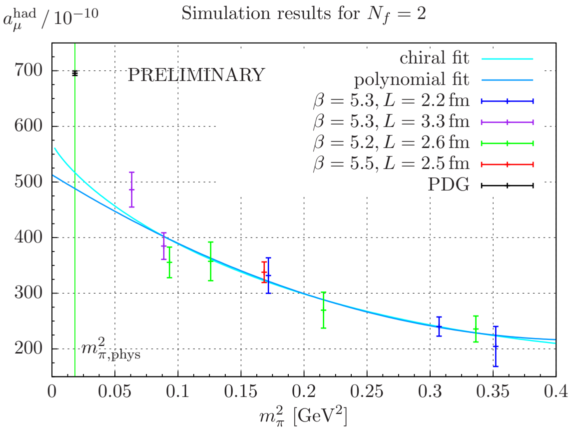
We twist the d quark in $C_{\mu\nu}^{conn}(q^2) \propto \int d^4x e^{iqx} \langle J_\mu^{ud}(0) J_\nu^{du}(x) \rangle$.
We use the conserved 1-point split vector current.



- Dispersion relation fits from a model for $\sigma_{e^+e^- \to hadrons}(s)$:
  $\Pi(q^2) = B \log(a^2 q^2 + a^2 s_0) - \frac{A}{(q^2 + m_V^2)} + K$ [QCDSF, '03]
- Polynomial
- Padé

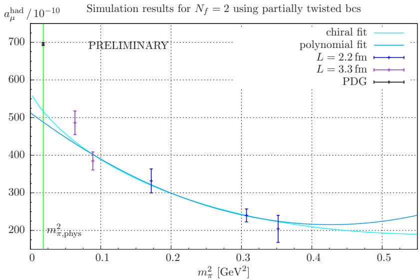


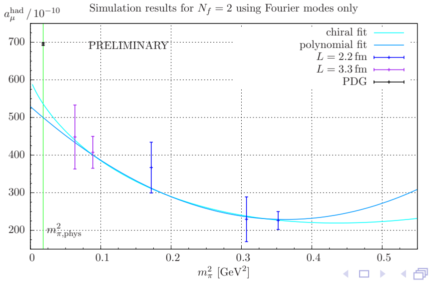
$4\alpha^2 f(q^2)\widehat{\Pi}(q^2)\,/10^{-7}$
Convolution integral for $a_\mu^{\rm had}$
16
14
12
10
8
6
4
2
0
kernel of integral
twisted bc F7
periodic bc F7
m_μ^2
0.0001
0.001
0.01
0.1
1
q^2 [GeV^2]



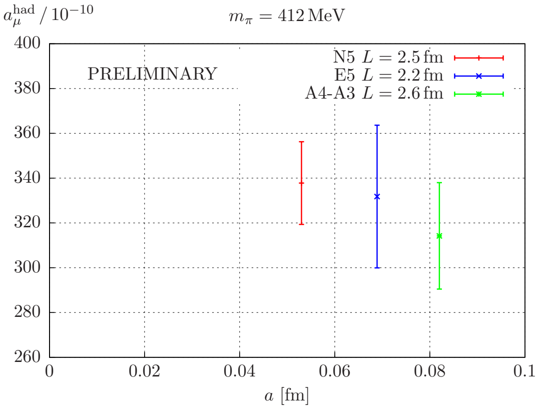
Fits to $\beta = 5.3$ data. Chiral fit: $a_\mu(m_\pi^2) = a + b m_\pi^2 (1 + c \log(m_\pi^2))$

Reduction of systematic errors from twisting ($\beta = 5.3$)



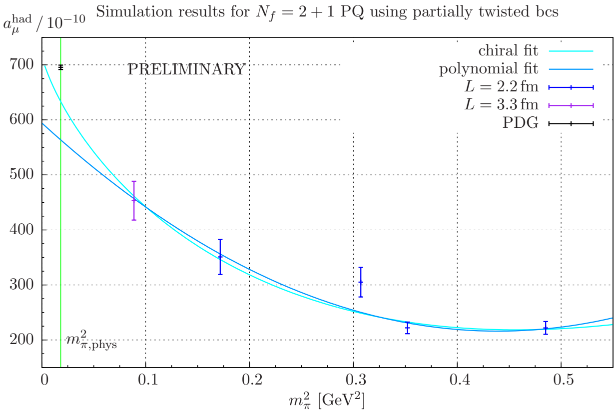


We expect O(a) cutoff effects as the VP receives off-shell contributions.



[The value at $\beta = 5.3$ is interpolated]

Adding a quenched strange quark ($\beta = 5.3$)

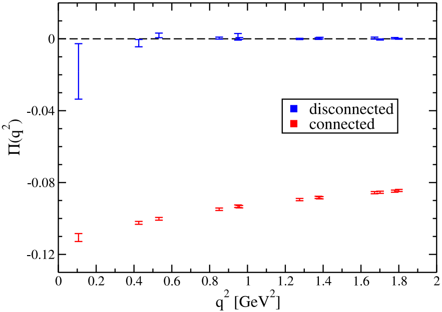


Numbers get about 20% larger wrt $N_f = 2$

Conclusions and outlook

- The anomalous magnetic moment of the muon is a phenomenologically very interesting quantity. Lattice determinations of the leading hadronic vacuum polarization can play an important role in producing a solid theoretical number.
- The appearance of disconnected diagrams and the poor momentum resolution represent the main difficulties in a lattice computation.
- PqχPT can be used to estimate the size of the disconnected contribution. Interesting way to look also at other quantities where disconnected diagrams appear.
- We use the χPT result for VP only to estimate the size of the disconnected piece, we will include vectors to see how stable the result is and we will compute disconnected contribution numerically.
- However this estimate tells us we should control the disconnected contribution to a 10-20% level to have a overall result which is as accurate as the present 'theoretical' one. That is challenging.

- Numerically these disconnected contributions seem to be even smaller than predicted by PqχPT at NLO. That is good.



[D. Renner, LAT10]

- The target precision appear to be reachable for the connected contribution also thanks to the use of twisted boundary conditions, which help reducing the systematic errors from the fits to the momentum dependence of the VP.
- We want to be able to control these effects to the desired level with 2 dynamical fermions, before considering $N_f > 2$. We see that the inclusion of a quenched strange quark is an effect O(10%).