

# Topological Lattice Actions

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Humboldt University, Berlin, January 24, 2011

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JHEP (2010) 1012:020

# Outline

The 1-d  $O(2)$  Model

The 2-d  $O(3)$  Model

Conclusions and Outlook

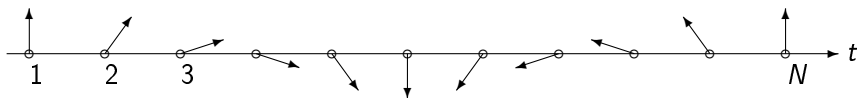
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## Configuration of the 1-d $O(2)$ model



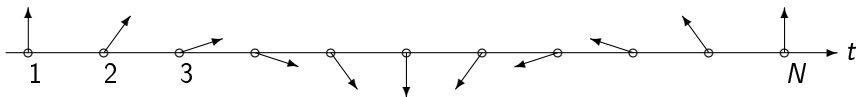
## Hamiltonian of a particle on a circle

$$H(\theta) = -\frac{1}{2I} \left( \partial_\varphi - i \frac{\theta}{2\pi} \right)^2$$

## Moment of inertia

$$I = MR^2$$

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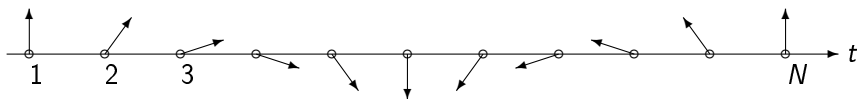
## Moment of inertia

$$I = MR^2$$

## Eigenfunctions and eigenvalues

$$\langle \varphi | m \rangle = \frac{1}{\sqrt{2\pi}} \exp(im\varphi), \quad E_m(\theta) = \frac{1}{2I} \left( m - \frac{\theta}{2\pi} \right)^2$$

## Configuration of the 1-d $O(2)$ model



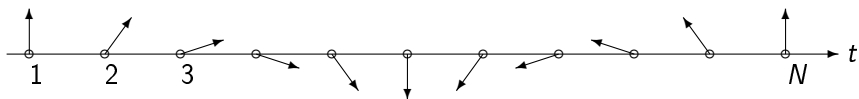
### Partition function

$$Z(\theta) = \text{Tr} \exp(-\beta H(\theta)) = \int \mathcal{D}\varphi \exp(-S[\varphi])$$

### Euclidean action

$$S[\varphi] = \int_0^\beta dt \frac{1}{2} \dot{\varphi}^2 - i\theta Q[\varphi]$$

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### Topological charge

$$Q[\varphi] = \frac{1}{2\pi} \int_0^\beta dt \dot{\varphi} \in \Pi_1[S^1] = \mathbb{Z}$$

### Schwarz inequality

$$S[\varphi] \geq \frac{2\pi^2 I}{\beta} Q[\varphi]^2$$

## Topological charge distribution

$$p(Q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta Z(\theta) \exp(-i\theta Q) = \sqrt{\frac{2\pi l}{\beta}} \exp\left(-\frac{2\pi^2 l}{\beta} Q^2\right)$$

## Topological susceptibility

$$\chi_t = \frac{\langle Q^2 \rangle}{\beta} = \frac{1}{\beta} \frac{\sum_{Q \in \mathbb{Z}} Q^2 p(Q)}{\sum_{Q \in \mathbb{Z}} p(Q)} \rightarrow \frac{1}{4\pi^2 l} \text{ for } \beta \rightarrow \infty$$

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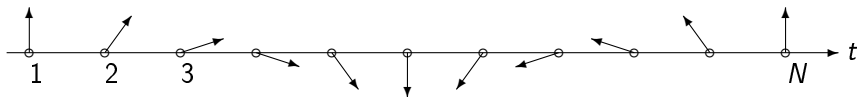
## Correlation length

$$\xi = \frac{1}{E_1(0) - E_0(0)} = 2l$$

## Dimensionless ratio

$$\chi_t \xi = \frac{1}{2\pi^2}$$

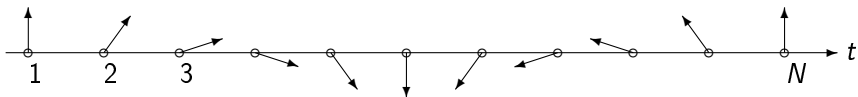
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## Topological lattice action with angle constraint

$\exp(-s(\varphi_t, \varphi_{t+a})) = 1$  for  $|(\varphi_{t+a} - \varphi_t) \bmod 2\pi| < \delta$ , and zero otherwise

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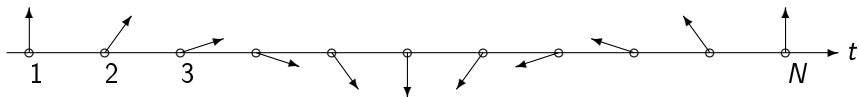
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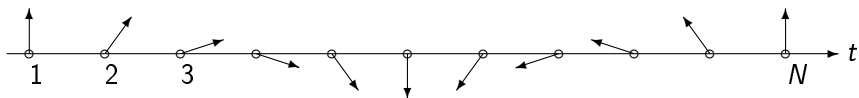
### Partition function

$$Z(\theta) = \text{Tr } T(\theta)^N, \quad Na = \beta$$

### Transfer matrix

$$\langle m | T(\theta) | m' \rangle = \delta_{mm'} \frac{1}{2\pi} \int_{-\delta}^{\delta} d\varphi \exp \left( -i \frac{\theta}{2\pi} \varphi + im\varphi \right)$$

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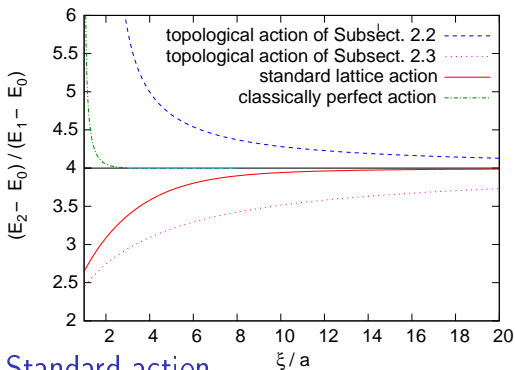
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### Energy spectrum

$$E_m(\theta) - E_0(0) = \frac{\delta^2}{6a} \left( m - \frac{\theta}{2\pi} \right)^2 = \frac{1}{2l} \left( m - \frac{\theta}{2\pi} \right)^2, \quad l = \frac{3a}{\delta^2}$$

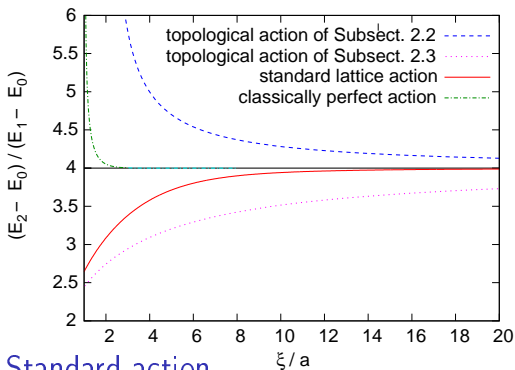
## Cut-off effects of energy gaps



### Standard action

$$\frac{E_2^s(0) - E_0^s(0)}{E_1^s(0) - E_0^s(0)} = 4 \left( 1 - \frac{a^2}{\xi_s^2} - 3 \frac{a^3}{\xi_s^3} + \dots \right), \quad \xi_s = \frac{1}{E_1^s(0) - E_0^s(0)}$$

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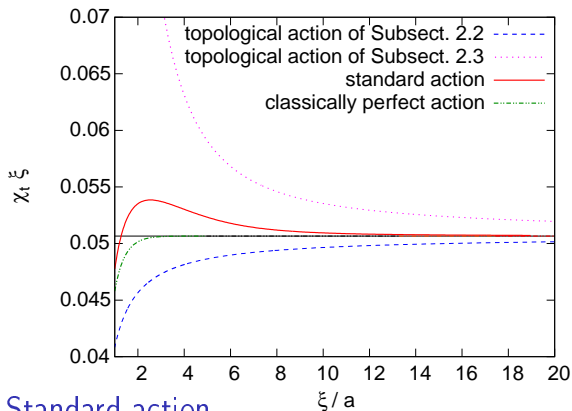
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$$\frac{E_2(0) - E_0(0)}{E_1(0) - E_0(0)} = 4 \left( 1 + \frac{3a}{5\xi} + \dots \right)$$

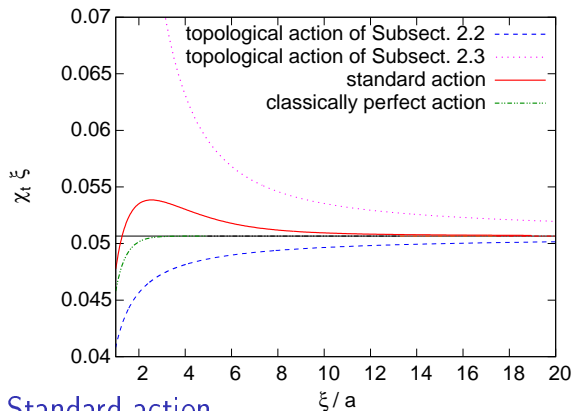
## Cut-off effects of topological susceptibility



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$$\chi_t^s \xi_s = \frac{1}{2\pi^2} \left( 1 + \frac{a^2}{3\xi_s^2} + \dots \right)$$

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$$\chi_t \xi = \frac{1}{4\pi^2} \frac{\delta^2}{3} \left[ \log \frac{\delta}{\sin \delta} \right]^{-1} = \frac{1}{2\pi^2} \left( 1 - \frac{a}{5\xi} + \dots \right)$$

Lattice action with topological charge suppression

$$S[\varphi] = \lambda \sum_t |(\varphi_{t+a} - \varphi_t) \bmod 2\pi|$$

Lattice Schwarz inequality

$$S[\varphi] \geq 2\pi\lambda |Q[\varphi]|$$

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$$\begin{aligned} \langle \varphi_t | T(\theta) | \varphi_{t+a} \rangle &= \exp(-\lambda |(\varphi_{t+a} - \varphi_t) \bmod 2\pi|) \\ &\times \exp\left(-i \frac{\theta}{2\pi} (\varphi_{t+a} - \varphi_t) \bmod 2\pi\right) \end{aligned}$$

## Energy spectrum

$$\begin{aligned} E_m(\theta) - E_0(0) &= \frac{1}{a} \log \left[ 1 + \frac{1}{\lambda^2} \left( m - \frac{\theta}{2\pi} \right)^2 \right] \\ &\rightarrow \frac{1}{a\lambda^2} \left( m - \frac{\theta}{2\pi} \right)^2, \quad l = \frac{a\lambda^2}{2} \end{aligned}$$

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$$\chi_t \xi = \frac{1}{2\pi^2 \lambda^2} \left[ \log \left( 1 + \frac{1}{\lambda^2} \right) \right]^{-1} = \frac{1}{2\pi^2} \left( 1 + \frac{a}{2\xi} + \dots \right)$$

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Euclidean action

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Topological charge

$$Q[\vec{e}] = \frac{1}{8\pi} \int d^2x \varepsilon_{\mu\nu} \vec{e} \cdot (\partial_\mu \vec{e} \times \partial_\nu \vec{e}) \in \Pi_2[S^2] = \mathbb{Z}$$

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Exact mass gap

$$m = \frac{8}{e} \Lambda_{\overline{MS}}$$

P. Hasenfratz, M. Maggiore, and F. Niedermayer, Phys. Lett. B245 (1990) 522

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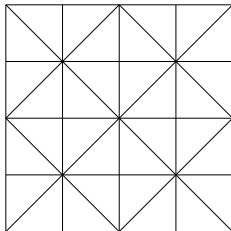
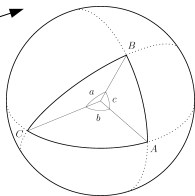
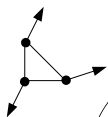
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## Step-scaling function

$$\sigma(2, u_0) = 2Lm(2L), \quad u_0 = Lm(L) = L/\xi(L)$$

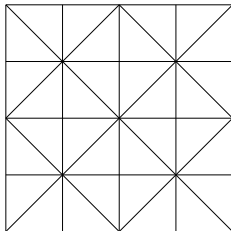
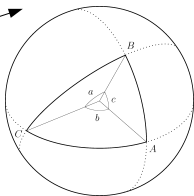
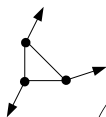
M. Lüscher, P. Weisz, and U. Wolff, Nucl. Phys. B359 (1991) 221

J. Balog and A. Hegedus, J. Phys. A: Math. Gen. 37 (2004) 1881



Standard lattice action  $S^2$

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Lattice topological charge

$$Q[\vec{e}] = \frac{1}{4\pi} \sum_{t_{xyz}} A_{xyz} \in \mathbb{Z}$$

Area of spherical triangle

$$A_{123} = 2\varphi \in [-2\pi, 2\pi], \quad X + iY = r \exp(i\varphi),$$

$$X = 1 + \vec{e}_1 \cdot \vec{e}_2 + \vec{e}_2 \cdot \vec{e}_3 + \vec{e}_3 \cdot \vec{e}_1, \quad Y = \vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)$$

## Topological lattice action with an angle constraint

$\exp(-s(\vec{e}_x, \vec{e}_y)) = 1$  for  $\vec{e}_x \cdot \vec{e}_y > \cos \delta$ , and zero otherwise

A. Patrascioiu and E. Seiler, Nucl. Phys. Proc. Suppl. 30 (1993) 184; J. Stat. Phys. 106 (2002) 811

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## Lattice Schwarz inequality

$$S[\vec{e}] \geq \lambda \left| \sum_{t_{xyz}} A_{xyz} \right| = 4\pi\lambda |Q[\vec{e}]|$$

For this action an efficient Wolff-type embedding cluster algorithm does not exist. Hence, one is limited to Metropolis-type algorithms.

## Correlation function

$$G(x - y) = \langle \vec{e}_x \cdot \vec{e}_y \rangle, \quad \tilde{G}(p) = \sum_x G(x) \exp(ipx)$$

## Second moment correlation length

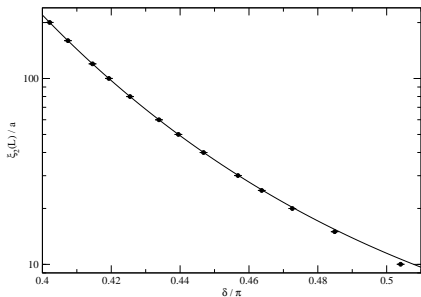
$$\xi_2(L) = \left( \frac{\chi - F}{4F \sin^2(\pi/L)} \right)^{1/2}, \quad \chi = \tilde{G}(p=0), \quad F = \tilde{G}(p=(2\pi/L, 0))$$

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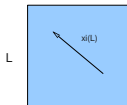
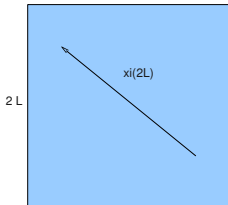
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$$\frac{\xi_2(L)}{a} = Ag^2 \exp\left(\frac{2\pi}{g^2}\right), \quad \frac{1}{g^2} = \frac{b}{\delta^2} + c$$

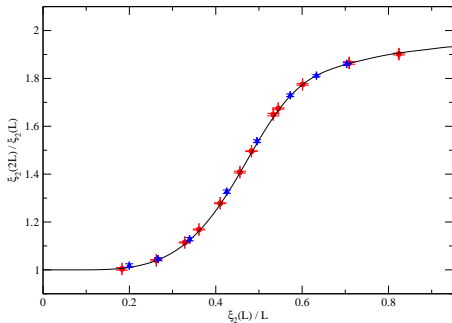
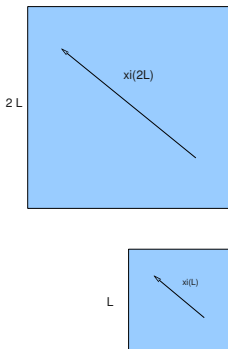
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S. Caracciolo, R. G. Edwards, A. Pelissetto, and A. D. Sokal, Phys. Rev. Lett. 75 (1995) 1891

## Step-scaling function

$$\Sigma(2, u_0, a/L) = 2Lm(2L), \quad u_0 = Lm(L) = 1.0595$$

## Cut-off effects of the step-scaling function

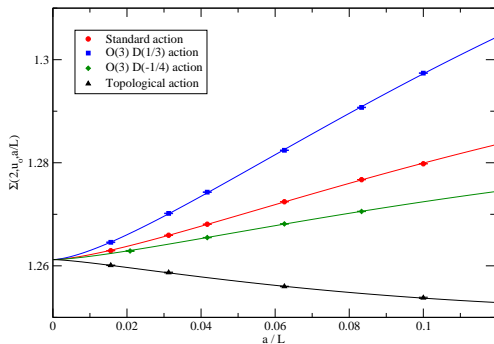
$$\Sigma(2, u_0, a/L) = \sigma(2, u_0) + \frac{a^2}{L^2} \left[ B \log^3(L/a) + C \log^2(L/a) + \dots \right]$$

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## Topological susceptibility

$$\chi_t = \frac{\langle Q^2 \rangle}{V}$$

## Expected scaling violation due to dislocations

$$\chi_t \xi^2 \propto \exp(-S_d) \exp\left(\frac{4\pi}{g^2}\right) = \exp\left(\frac{4\pi - c}{g^2}\right) \propto \left(\frac{\xi}{a}\right)^{2-c/2\pi}$$

M. Lüscher, Nucl. Phys. B200 (1982) 61

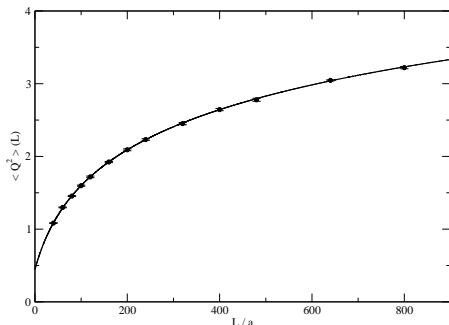
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## Topological charge density

$$q(x) = \frac{1}{8\pi} \epsilon_{\mu\nu} \vec{e}(x) \cdot [\partial_\mu \vec{e}(x) \times \partial_\nu \vec{e}(x)]$$

## Point-to-time-slice correlator

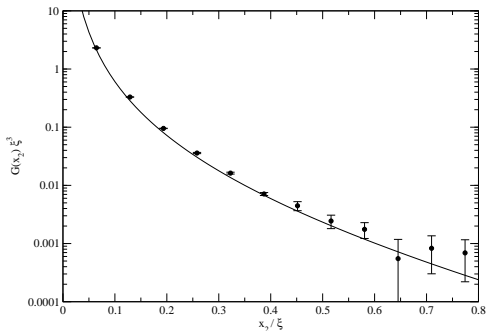
$$G(x_2) = \int_0^L dx_1 \langle q(0) q(x) \rangle, \quad x = (x_1, x_2)$$

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- **Topological lattice actions** are invariant against small continuous deformations of the lattice field.
- Topological lattice actions are **tree-level impaired**. Hence they **do not have the correct classical continuum limit**, one **cannot treat them perturbatively**, and they **may violate a Schwarz inequality**.

## Conclusions

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- The correlation function  $G(x_2)$  of the topological charge density is in **reasonable agreement with analytic predictions**, but a more detailed investigation of cut-off effects would be welcome.

## Outlook

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- The **meron-cluster algorithm** (an extension of the Wolff cluster algorithm) is capable of addressing the **complex action problem at non-zero vacuum angle  $\theta$** . This is currently under investigation.
- It is straightforward to define topological lattice actions for **Abelian and non-Abelian gauge theories**. It is conceivable that one may be able to take algorithmic advantages from such unconventional lattice actions. This is perhaps worth investigating.