The QCD Rotator

Peter Hasenfratz

Institute of Theoretical Physics, University of Bern, Switzerland

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Introduction and summary

analytic results on the low lying spectrum of QCD
next-to-next-to-leading (NNL) chiral perturbation theory (ChPT)
special environment: delta-regime
created by the 'would be Goldstone bosons'
in a box of size $L_s \times L_s \times L_s \times (L_t \to \infty)$, $L_s \gtrsim 2.5 fm$

The low lying spectrum is a quantum mechanical rotator
whose inertia recieves small, calculable corrections.
leading order(L):
Fisher, Privman, 1983; Brezin, Zinn-Justin 1983;
Leutwyler, 1987
(a note on ChPT in condensed matter vs. high energy physics)

next-to-leading(NL)
P.H., Niedermayer, 1993

next-to-next-to-leading(NNL)
P.H., 2009

Up to NNL order the low lying spectrum is expressed
in terms of only 3 constants of ChPT in the chiral limit.

This spectrum will never be measured in a real experiment.
However, the same low lying spectrum can be studied in
numerical experiments (lattice QCD).

→ precise constraints on the low energy constants.

Notes:

the low lying stable energy spectrum is the simplest and cleanest numerical problem on the lattice.

The condition $L_s \gtrsim 2.5fm$ is not trivial. The lattice community is close to that today and will be there tomorrow.

The NNL symmetry breaking contribution: Manuel Weingart.

2-flavor QCD in the chiral limit; $SU(2) \times SU(2) \sim O(4)$;
dimensional regularization (DR) is used in this work;
the low lying spectrum up to NNL order in ChPT reads:

$$E_j = \frac{1}{2\Theta} j(j+2)\,,\ j = 0, 1, 2, \dots\,,$$

where the inertia Θ depends on the low energy constants F, Λ_1, Λ_2:

$$\begin{aligned}\Theta = F^2 L_s^3 \Bigg\{ 1 - \frac{2}{F^2 L_s^2} (D^*(0) L_s^2) \\ + \frac{1}{(F^2 L_s^2)^2} \Bigg[0.088431628 \\ + (\partial_0 \partial_0 D^*(0) L_s^4) \frac{1}{3\pi^2} \left(\frac{1}{4} \ln(\Lambda_1 L_s)^2 + \ln(\Lambda_2 L_s)^2 \right) \Bigg] \Bigg\}\end{aligned}$$

$$D^*(0) L_s^2 = -0.2257849591\,; \qquad \partial_0 \partial_0 D^*(0) L_s^4 = -0.8375369106\,.$$

The result is simple, the underlying ChPT is, however, not.
Is the result correct?
F. Niedermayer and Ch. Weyermann (PhD):
result with a different technique using lattice regularization;

the connection between DR vs. lattice regularization is needed;
for *effective* field theories this problem is almost untouched;
in this case the NL result is known (P.H., Niedermayer, 1993).

interested?

The chiral action

Use 'magnetic language': we have a field with 4 components in the internal space. The field is described in terms of microscopic magnets. The lagrangean up to NNL order reads:

$$L = L^2_{\mathrm{eff}} + L^4_{\mathrm{eff}} \,,$$

where

$$\begin{aligned} L^2_{\mathrm{eff}} &= \frac{F^2}{2} \partial_\mu \mathbf{S} \, \partial_\mu \mathbf{S} \ , \\ L^4_{\mathrm{eff}} &= -l_1 \, (\partial_\mu \mathbf{S} \, \partial_\mu \mathbf{S})(\partial_\nu \mathbf{S} \, \partial_\nu \mathbf{S}) - l_2 \, (\partial_\mu \mathbf{S} \, \partial_\nu \mathbf{S})(\partial_\mu \mathbf{S} \, \partial_\nu \mathbf{S}) \, . \end{aligned}$$

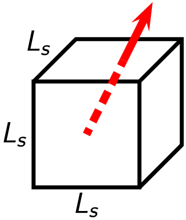
Here F, l_1, l_2 are the bare low energy constants. Further,

$$\mathbf{S}(x) = (S_0(x), S_1(x), S_2(x), S_3(x)), \ \ \mathbf{S}^2(x) = 1 \,,$$

and x lives in $d = 4 = (d-1) + 1$ (space and euclidean time)

The leading (L) rotator

Consider the spatial box at fixed time t. The microscopic magnets are closely parallel in the $L_s \times L_s \times L_s$ box. In leading order we ignore the small fluctuations. The arrow is the sume of the parallel micromagnets at t.

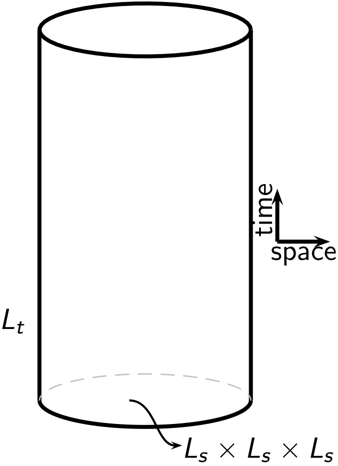


In leading order, on each time slice, the length of the magnetisation is constant, but the direction is changing slowly. Let $\mathbf{e}(t)$ the direction of the total magnetization at t. The leading action reads

$$A_{\mathrm{eff}}^2 = \frac{F^2}{2} \int \, dx \partial_\mu \mathbf{S}(x) \, \partial_\mu \mathbf{S}(x) \; \rightarrow$$

$$\frac{F^2 V_s}{2} \int dt \, \dot{\mathbf{e}}(t) \dot{\mathbf{e}}(t) \, , \quad \mathbf{e}(t)^2 = 1$$

This is a quantum mechanical rotator with inertia $\Theta = F^2 V_s$.



Separating the slow and fast modes

The diraction of the magnetization $\mathbf{e}(t)$ moves much slower than the single microscopic magnets. We integrate out these fast modes and obtain a generalized rotator in terms of the slow modes $\mathbf{e}(t)$. Then remains a simple problem in quantum mechanics.
We start with the path integral

$$Z = \prod_x \int d\mathbf{S}(x)\delta(\mathbf{S}^2(x) - 1)\exp\left(-A_{\text{eff}}(\mathbf{S})\right) ,$$

where A_{eff} is built from the lagrangean $L^2_{\text{eff}} + L^4_{\text{eff}}$.
Insert '1' in the path integral

$$1 = \prod_t \int d\mathbf{m}(t)\delta(\mathbf{m}(t) - \frac{1}{V_s}\sum_{\mathbf{x}} \mathbf{S}(t,\mathbf{x})) , \quad \mathbf{m}(t) = m(t)\,\mathbf{e}(t) .$$

The vector $\mathbf{e}(t)$ is the direction of the 'magnetisation' on the time slice t. These are the slow modes.

At every time slice the microscopic magnets are fluctuating around their sum $\mathbf{m}(t)$. We want to integrate over these fast modes.

The slow modes, however, are moving in the time. It would be very difficult to integrate out the fast modes in this rotating system. The solution of this problem is technical and we do not go in to these details.
The remaining modes are the fast modes

$$\mathbf{R}(x) = \Big((1 - \boldsymbol{\Pi}^2(x))^{\frac{1}{2}}, \boldsymbol{\Pi}(x)\Big)$$

which can be treated in perturbation theory. In the pairing

$$< \Pi(x)_i, \Pi(0)_j >= \delta_{i,j} \frac{1}{F^2} D^*(x)$$

the $k = (k_0, \mathbf{k} = \mathbf{0})$ part is subtracted, since those are the slow modes.

Counting rules

The small expansion parameter in the δ-regime is $1/F^2L_s^2 = O(\delta^2)$. The expansion of the rotator action has the form

$$\int_t \frac{F^2}{2} V_s \dot{\mathbf{e}}(t)\dot{\mathbf{e}}(t)(1+ \sim \frac{1}{F^2L_s^2} + \sim \frac{1}{(F^2L_s^2)^2} \dots) \qquad (1)$$

in the leading L, NL, NNL,... order.

In the expansion there are terms also with quadratic and higher powers of the slow mode $\dot{\mathbf{e}}$. It turns out that an additional $\sim \dot{\mathbf{e}}\dot{\mathbf{e}}$ term in the equation above is $O(\delta^6)$, i.e. beyond our NNL calculation.

The inertia Θ up to NNL order

The standard $O(4)$ rotator is obtained, where only the inertia is modified

$$\begin{aligned}\Theta = \quad & F^2 V_s \Big\{ 1 - \tfrac{N-2}{F^2} D^*(0) + \tfrac{N-2}{F^4} \Big(D^*(0) D^*(0) \\ & + 2 \int_x \partial_0 \partial_0 D^*(x) D^*(x) D^*(x) \Big) + \tfrac{1}{F^4} (8 l_1 + 16 l_2) \partial_0 \partial_0 D^*(0) \Big\}\end{aligned}$$

The only unknown part is the integral above. This integral is singular and needs some work. The result reads

$$\begin{gathered}\int dx\, \partial_0 \partial_0 D^*(x) D^*(x) D^*(x) = \\ -\tfrac{1}{L_s^4} \Big\{ d0d0G^* \tfrac{1}{8\pi^2} \tfrac{5}{3} \Big[\tfrac{1}{d-4} + \ln\big(\tfrac{1}{L_s}\big) \Big] + 0.029492025146\, . \Big\}\end{gathered}$$

The singularities in the low energy constants l_1, l_2 cancel the singularities above. We obtain the result on page 4.

$$E_j = \frac{1}{2\Theta} j(j+2)\,,\ j = 0, 1, 2, \ldots\,,$$

$$\begin{aligned}
\Theta = F^2 L_s^3 \Bigg\{ & 1 - \frac{2}{F^2 L_s^2}(D^*(0) L_s^2) \\
& + \frac{1}{(F^2 L_s^2)^2} \Bigg[0.088431628 \\
& + (\partial_0 \partial_0 D^*(0) L_s^4) \frac{1}{3\pi^2} \left(\frac{1}{4} \ln(\Lambda_1 L_s)^2 + \ln(\Lambda_2 L_s)^2 \right) \Bigg] \Bigg\}
\end{aligned}$$

Corrections to the first excitation

Consider the NL and NNL corrections at different L_s values. Take for the low energy constants:

$$F = 90 Mev, \qquad \Lambda_1 = 110 MeV, \qquad \Lambda_2 = 1200 MeV$$

and measure L_s in fermi. The first excitation is

$$E_1 = \frac{3}{2\Theta}, \qquad \Theta = F^2 L_s^3 (1 + c_1 + c_2)$$

	c_1	c_2
$L_s = 1.0$	2.1709	-0.1264
$L_s = 1.5$	0.9647	-0.1556
$L_s = 2.0$	0.5427	-0.0787
$L_s = 2.5$	0.3473	-0.0416
$L_s = 3.0$	0.2412	-0.0237

The NNL corrections are ten times smaller than that of the NL corrections.