Minimally doubled fermions and their renormalization

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Introduction

Minimally doubled fermions (2 flavors):
realize the minimal doubling allowed by the Nielsen-Ninomiya theorem

Preserve an exact chiral symmetry for a degenerate doublet of quarks

chiral symmetry protects mass renormalization

Remain at the same time also strictly local

fast for simulations

`A cost-effective realization of chiral symmetry at nonzero lattice spacing`

We can construct a conserved axial current, which has a simple expression

Compared with staggered fermions:

- same kind of $U(1)$ chiral symmetry
- 2 flavors instead of 4
  $\Rightarrow$ no uncontrolled extrapolation to 2 physical light flavors
- no complicated intertwining of spin and flavor

Introduction

Ideal for $N_f = 2$ simulations: no rooting needed!

Much cheaper and simpler than Ginsparg-Wilson fermions
(overlap, domain-wall, fixed-point)

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Minimally doubled fermions: *'new'* . . . but also *'old'*

Revival in the last 2 years – initiated by studies on graphenes by Creutz , in December 2007

Here we consider two realizations of minimally doubled fermions: Boriçi-Creutz and Karsten-Wilczek fermions – and derive their Feynman rules

We then compute the self-energy of the quark and the renormalization of the Dirac bilinears

Mixings of a new kind arise, as a consequence of the breaking of the hypercubic symmetry $\longrightarrow$ preferred direction in euclidean spacetime

One of the main aims of our work is the investigation of the mixing patterns that appear in radiative corrections

Introduction

We also construct the **conserved** vector and **axial** currents

They have simple expressions which involve only nearest-neighbors sites

Apart from staggered fermions, these are then the only discretizations in which one can give a simple expression for a conserved axial current

This conserved axial current is even ultralocal

These features could turn out to be very useful also in numerical simulations

We also compute the vacuum polarization of the gluon

Here the breaking of hypercubic symmetry does not generate any kind of power divergences

In principle these divergences could have arisen with a $1/a^2$ or $1/a$ dependency

All this is also an example of the usefulness of perturbation theory in helping to unfold theoretical aspects of (new) lattice formulations

Chiral fermions on the lattice

Simplest discretization of the Dirac action: naive fermions

The massless propagator of naive fermions is $a \dfrac{-i\sum_\mu \gamma_\mu \sin ap_\mu}{\sum_\mu \sin^2 ap_\mu}$

Each pole of the propagator corresponds to a massless fermion in the theory

This propagator has a pole at $ap = (0,0,0,0)$, as expected

But: $\sin ap_\mu$ vanishes whenever any component p_μ is either 0 or π/a

Then, there are poles also at $(\pi,0,0,0)$, $(0,\pi,0,0)$, . . . , $(\pi,\pi,0,0)$, . . . , (π,π,π,π) (at the edges of the first Brillouin zone)

One would then have to take into account all these 16 Dirac particles when doing lattice computations

Although they are a lattice artifact, they are pair produced as soon as interactions are switched on

They appear in internal loops and contribute to intermediate processes

$\Rightarrow$ $2^4 = 16$ particles are propagating on our lattice

Chiral fermions on the lattice

On the lattice:
It is impossible to eliminate the doublers in any fermion action without at the same time breaking chiral symmetry or some important property of field theory

This is a special case of a very important no-go theorem, established by Nielsen and Ninomiya many years ago

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No-go theorem: it is impossible to construct a lattice fermion formulation without fermion doubling and with an explicit continuous chiral symmetry – unless one gives up some other fundamental property, like locality, unitarity, . . .

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This statement only applies to the "standard" chiral symmetry, which acts on the spinor fields according to the transformations

$$\begin{aligned} \psi &\to \psi + \epsilon \cdot \gamma_5 \psi \\ \overline{\psi} &\to \overline{\psi} + \epsilon \cdot \overline{\psi} \gamma_5 \end{aligned}$$

One of the major theoretical advances of the last years: there are other transformation laws that can define a lattice chiral symmetry – and which do not necessarily imply fermion doubling

$\Rightarrow$ Ginsparg-Wilson fermions

Chiral fermions on the lattice

No-go theorem of Nielsen & Ninomiya (1981)

Any massless Dirac operator $D = \gamma_\mu D_\mu \equiv D(x-y)$ in a lattice fermionic action cannot satisfy the following properties at the same time:

- $D(x)$ is local (in the sense that is bounded by $Ce^{-\gamma|x|}$)

  i.e. D couples only fields $\overline{\psi}(x)$, $\psi(y)$ with $(x-y) = O(a)$

  (avoids interactions over macroscopic distances)

- its Fourier transform has the right continuum behavior for small p: $\widetilde{D}(p) = i\gamma_\mu p_\mu + O(ap^2)$
- $\widetilde{D}(p)$ is invertible for any $p \neq 0$

  $\Rightarrow$ avoidance of additional poles

  $\Rightarrow$ there are no massless doublers

- $\gamma_5 D + D\gamma_5 = 0$: it is invariant under chiral transformations (a realization of the chiral symmetry)

This is always true – there is no exception

Chiral fermions on the lattice

These 4 conditions cannot be fulfilled at the same time by whatever lattice formulation

Therefore, for any lattice action that one can think of, at least one of these conditions has to fail

$\Rightarrow$ either fermion doubling, or explicit chiral symmetry breaking, or . . .

All this can be seen already at the level of FREE fermions ($U_\mu = 1$)

So:

- `Naive fermions`: 16-fold degeneracy
- `Wilson fermions`: degeneracy completely removed, but they break chiral symmetry
- `staggered fermions`: 4-fold degeneracy; entanglement of flavor and spin

  only a $U(1) \otimes U(1)$ subgroup of the full $SU(N_f) \otimes SU(N_f)$ chiral group remains unbroken; the doublers are removed only partially, and taken as different flavors (*tastes*)
- `SLAC fermions`: non-local

Chiral fermions on the lattice

We can understand why all this happens from general arguments regarding the free fermion propagator on the lattice, and the energy-momentum relation in the Brillouin zone

Requirements: periodicity, continuum-like behavior around $p = 0$, and possibly continuity

The general form of a massless fermion propagator on the lattice which is compatible with continuous chiral invariance ($=$ anticommutes with γ_5) is

$$\frac{1}{i \sum_\mu \gamma_\mu P_\mu(p)}$$

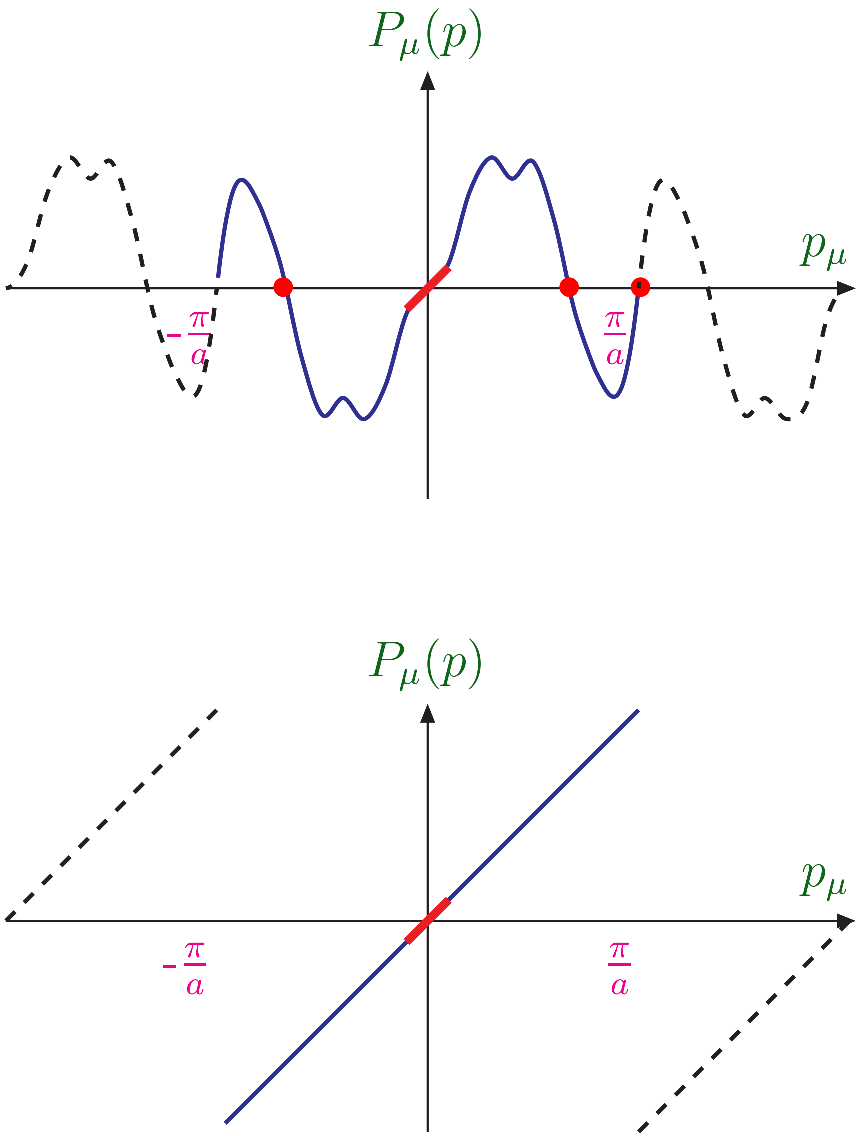
For naive fermions: $P_\mu(p) = \frac{1}{a} \sin a p_\mu$

Let us assume at first that $P_\mu(p)$ is a continuous function

Since there must be a zero of first order at $p_\mu = 0$, and because of periodicity also at $a p_\mu = 2\pi n_\mu$, there must be another zero somewhere else in the first Brillouin zone

This other crossing is a doubler, and must have a derivative of opposite sign, which means opposite chirality

Chiral fermions on the lattice



Chiral fermions on the lattice

Thus: equal number of left-handed and right-handed fermions

It is unavoidable to have these extra particles in the theory

In four dimensions: $2^4 = 16$ doublers

This argument is independent of the particular shape of the function $P_\mu(p)$, as long as this is continuous

The only possibility to avoid the second crossing: consider a discontinuous function $P_\mu(p)$

Most famous example of this: the SLAC propagator [Drell, Weinstein and Yankielowicz, 1976], for which $P_\mu(p) = p_\mu$ throughout the whole Brillouin zone

However, this choice implies a nonlocality in the lattice action – it corresponds to a nonlocal lattice derivative:

$$\partial_\mu = \text{ infinite series in } \left(\nabla_\mu + \nabla^\star_\mu\right)^n$$

$\Rightarrow$ many problems: the very existence of the continuum limit is in doubt

(∂_μ: continuum derivative; ∇_μ, $\nabla^\star_\mu$: lattice finite differences)

Chiral fermions on the lattice

At the end of the day the origin of the fermion doubling lies in the fact that the Dirac equation is first order

A scalar propagator does not have this kind of problems, as it is the solution of a second-order differential equation

$\Rightarrow$ linear crossings around $p = 0$ are replaced by second-order zeros

The scalar does not pass again through zero, because at the origin is $O(p^2)$, and thus does not need to go into negative values!

$\Rightarrow$ no further crossings $\Rightarrow$ no doublers

How do Wilson fermions fit in the pictures discussed before?

The above considerations are not valid anymore; now we have

$$\frac{1}{i \sum_\mu \gamma_\mu P_\mu(p) + Q(p)} \qquad \left(P_\mu(p) = \frac{1}{a} \sin a p_\mu ; \;\; Q(p) = \frac{2r}{a} \sum_\mu \sin^2 a p_\mu / 2 \right)$$

and at π/a the denominator, instead of going to zero, is proportional to r/a

The functional form has changed; the additional term, without gammas, breaks chiral symmetry

Chiral fermions on the lattice

Contrary to what one would naively expect from the Nielsen-Ninomiya theorem, it is still possible to construct a Dirac operator which satisfies the first three conditions and it is also chirally invariant

Solution to this apparent paradox : the corresponding chiral symmetry is not the one associated with a Dirac operator which anticommutes with γ_5

The fourth condition is instead replaced by the Ginsparg-Wilson relation: $\gamma_5 D + D\gamma_5$ is not zero, but proportional to $aD\gamma_5 D$

Thus, the actual lattice chiral symmetry is not what one naively expects

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Lüscher [1998] has shown that Ginsparg-Wilson fermions possess an exact global chiral symmetry at finite lattice spacing, of the form

$$\begin{aligned} \psi \quad &\rightarrow \quad \psi + \epsilon \cdot \gamma_5 \left(1 - \frac{a}{\rho} D\right) \psi \\ \overline{\psi} \quad &\rightarrow \quad \overline{\psi} + \epsilon \cdot \overline{\psi}\,\gamma_5 \end{aligned}$$

It is a sort of *“escape”* from the Nielsen-Ninomiya theorem

The Nielsen-Ninomiya theorem is still valid, but – in spite of this – one can still get a nonpathological formulation of chiral fermions with no doublers

Chiral fermions on the lattice

When the condition that the Dirac operator anticommutes with γ_5 is released (at $a \neq 0$), the lattice quark propagator is not restricted to be of the form

$$\frac{1}{i \sum_\mu \gamma_\mu P_\mu(p)}$$

Then, the considerations about the presence of the doublers that we derived from it are not anymore valid

In fact, one finds more general forms of the fermion propagator – for instance the overlap propagator

Non-trivial solutions of the Ginsparg-Wilson relation (1982 $\rightarrow$ 1997/98) :

- domain-wall fermions (Kaplan, Shamir & Furman, 1992/93)
- overlap fermions (Neuberger & Narayanan, 1992 $\rightarrow$ Neuberger, 1998)
- fixed-point fermions *[Perfect actions]* (Hasenfratz & Niedermayer, 1993)

Theoretically superior, but extremely costly in practice – and non local

Ginsparg-Wilson fermions are much more complicated and computationally expensive than Wilson or staggered fermions

Graphene

Graphene: a *"cousin"* of **diamond** and **graphite**

Nanoscopic material, ultra-thin sheet of matter – a form of the element carbon that is just a single atom thick

Is a single layer of graphite consisting of a 2-dimensional hexagonal lattice of carbon atoms

Graphite (*pencil, 1564*): essentially a jumbled mass of tiny scraps of graphene

Graphene

Writing with a pencil on paper actually produces graphene stacks

. . . somewhere among them, there could be individual graphene layers

Graphene was identified as a theoretical possibility as early as 1947 *(Wallace)*

However, for many years it was thought that it couldn't exist in nature – no one expected graphene to exist in the free state

Graphene is presumably produced every time someone writes with a pencil

however, no experimental tools existed to search for macroscopic one-atom-thick flakes among the pencil debris

Only in 2004 the existence of graphene as a real separate material was first demonstrated *(University of Manchester, UK)*

In graphene, electrons behave **as if** they were **relativistic massless** particles

$\Rightarrow$ ultra-high mobilities exhibited by graphene devices

$\Rightarrow$ a variety of unique, and potentially very useful, characteristics

Graphene

Its unique electrical characteristics could make graphene the successor to silicon in a whole new generation of microchips

$\Rightarrow$ further development of ever-smaller, ever-faster silicon chips

Because of its single-atom thickness, pure graphene is transparent , and can be used to make transparent electrodes for light-based applications such as LEDs or improved solar cells

Graphene could also substitute for copper to make the electrical connections between computer chips and other electronic devices, providing much lower resistance and thus generating less heat

It has also potential uses in quantum-based electronic devices that could enable a new generation of computation and processing

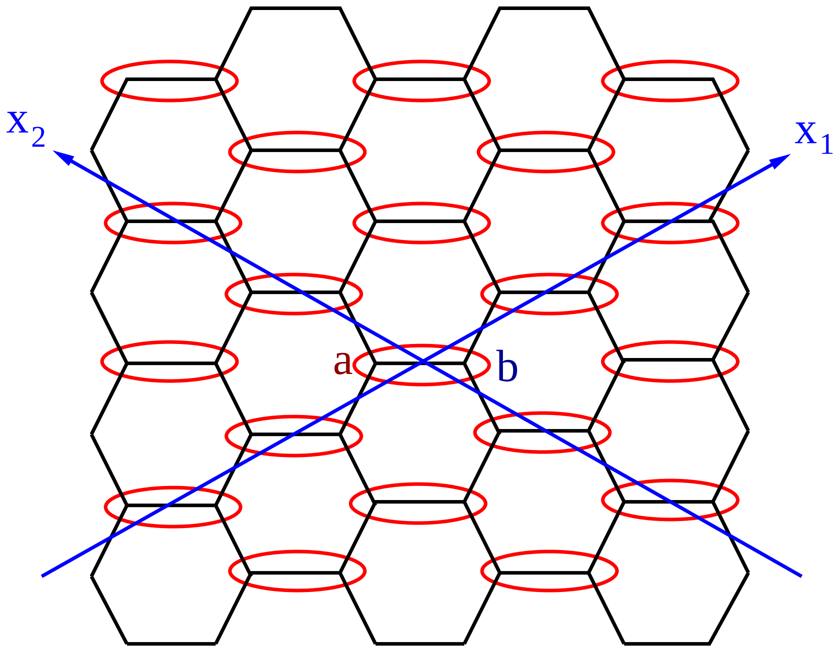
This field is really in its infancy

There isn't any other material like graphene

Its strength is 200 times that of steel

The mobility of electrons in graphene is by far the highest of any known material

Graphene



Michael Creutz, JHEP0804:017, 2008

Clever choice of coordinates

Orient one third of the bonds horizontal, one third sloping up at 60 degrees, and one third sloping down

Organize the graphene structure into two-atom "sites" involving "collapsed" horizontal bonds *(as enclosed in ellipses)*

Use a non-orthogonal coordinate system with axes sloping up and down at 30 degrees intersecting the corresponding sites

Graphene

The Hamiltonian contains only nearest-neighbor hoppings between a and b type sites:

$$H = K \sum_{x_1,x_2} \Big(a^\dagger_{x_1,x_2} b_{x_1,x_2} + b^\dagger_{x_1,x_2} a_{x_1,x_2} + a^\dagger_{x_1+1,x_2} b_{x_1,x_2} + b^\dagger_{x_1-1,x_2} a_{x_1,x_2} + a^\dagger_{x_1,x_2-1} b_{x_1,x_2} + b^\dagger_{x_1,x_2+1} a_{x_1,x_2} \Big)$$

In momentum space

$$H = K \left[\tilde{a}^\dagger_{p_1,p_2} \tilde{b}_{p_1,p_2} \left(1 + e^{-ip_1} + e^{ip_2}\right) + \tilde{b}^\dagger_{p_1,p_2} \tilde{a}_{p_1,p_2} \left(1 + e^{ip_1} + e^{-ip_2}\right) \right]$$

can be represented by a matrix $K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$, where $z = 1 + e^{-ip_1} + e^{ip_2}$

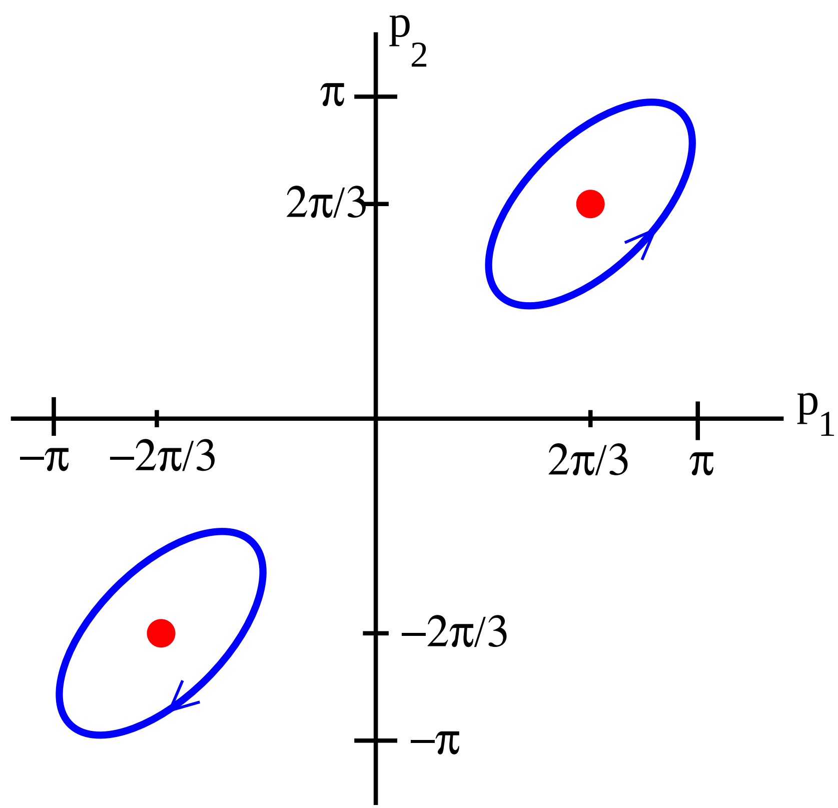
Eigenvalues of the energy : $\pm K|z|$

The energy vanishes when $|z|$ does

$\Rightarrow$ <u>only 2 zeros</u>: $p_1 = p_2 = \pm 2\pi/3$

Consider contours of <u>constant $|z|$</u> around the zeros

Graphene

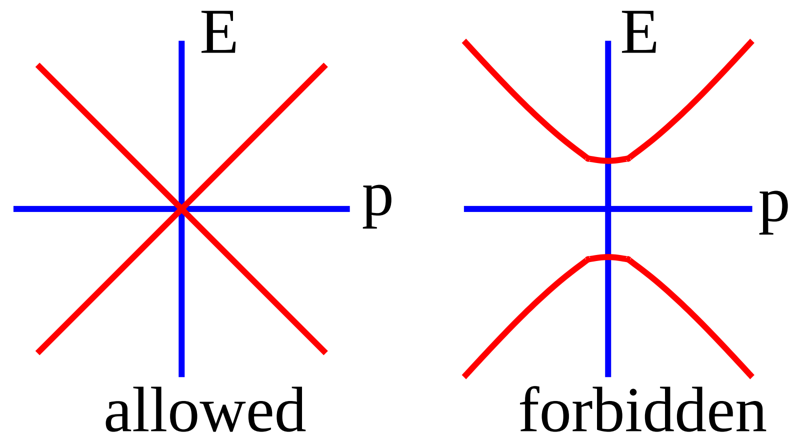


The phase of $|z|$ wraps non-trivially around such a contour of constant energy

We can then understand that when one reduces the energy and shrinks the contour to a point, the magnitude of the energy at this point must vanish

The massless structure is robust , thanks to this topological stability

Graphene



This mechanism prevents a band gap from opening in the spectrum

$\Rightarrow$ linear dispersion relation

$\Rightarrow$ graphite is black and a conductor

$\Rightarrow$ Dirac equation

Striking: it contains 2 massless Dirac particles
(Hou, Chamon and Mudry, 2006; Jackiw and Pi, 2007)

Creutz's original motivation: the low energy electronic excitations are described by the massless relativistic Dirac equation

Graphene

The massless structure is robust, for topological reasons related to chirality: map of circles onto circles

These electrons mimic Dirac fermions, but actually move in graphene with a speed

$$\frac{v_F}{c} \approx \frac{1}{300}$$

(comparable to that in half-filled metals)

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Creutz then constructed an action with similar properties in four dimensions *(in the same paper, JHEP0804:017, 2008)*

Afterwards (2008): further developed by Boriçi , and then again by Creutz

$\rightarrow$ **Boriçi-Creutz** fermions

can also be constructed without referring to graphenes (see later)

Karsten-Wilczek fermions: another instance of minimally doubled fermions

proposed already in 1981 and 1987

Minimally doubled fermions

Connection with chiral symmetry: $b \to -b$ changes the sign of H

The matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ anticommutes with $H(p_1, p_2) = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$

in four dimensions it would correspond to γ_5

Four-dimensional extension of the graphene (*Creutz*):

complex numbers $\longrightarrow$ **quaternions**

Look for an analogous form $H(p_\mu) = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$ in four dimensions

$H(p_\mu)$ is now a 4x4 matrix

$z = z(p_1, p_2, p_3, p_4)$ are 2x2 matrices in a quaternionic space:

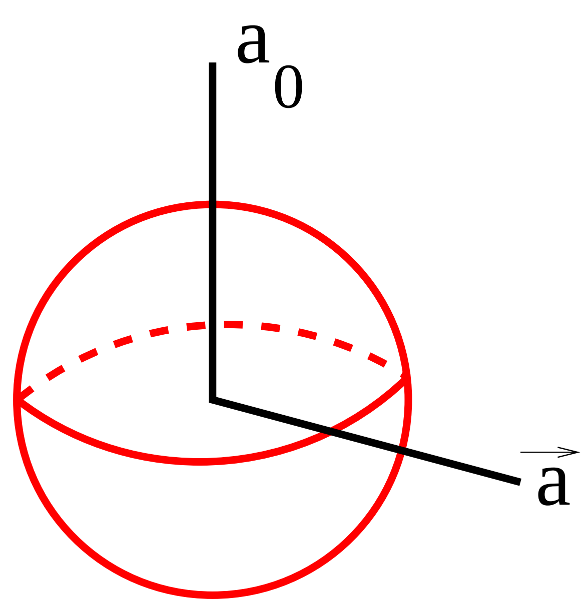
$$z = a_0 + i\,\vec{a}\,\vec{\sigma}, \qquad \text{with } |z|^2 = \sum_\mu a_\mu^2$$

where a_μ is a real 4-vector

Minimally doubled fermions

Eigenvalues of the energy: still $\pm K|z|$

Generalize topology to mapping 3-spheres onto 3-spheres



Constant energy surfaces must involve non-trivial mappings in the quaternionic space near the zeros ($= a_\mu$ *vanishing as a 4-vector*)

$\Rightarrow$ topological stability of the massless structure

Minimally doubled fermions

Gamma matrices:

$$\vec{\gamma} = \sigma_1 \otimes \vec{\sigma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

$$\gamma_4 = -\sigma_2 \otimes 1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\gamma_5 = \sigma_3 \otimes 1 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The lattice implementation of $\gamma_5 D = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$ is not unique – we only need a $z(p)$ with two zeros

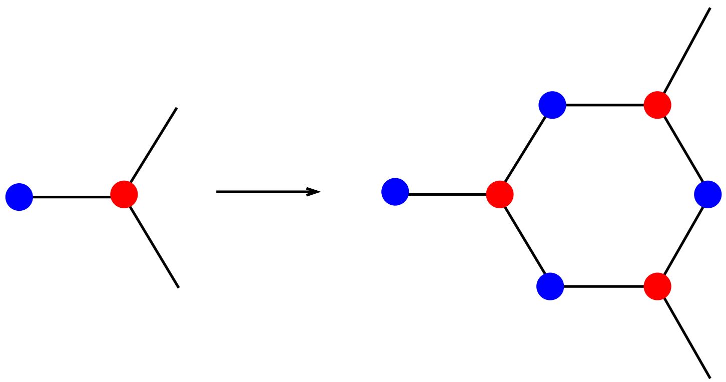
Creutz's proposal:

$$\begin{aligned} z \quad = \quad & B\,(4\,C - \cos p_1 + \cos p_2 - \cos p_3 - \cos p_4) \\ & + i\sigma_x\,(\sin p_1 + \sin p_2 - \sin p_3 - \sin p_4) \\ & + i\sigma_y\,(\sin p_1 - \sin p_2 - \sin p_3 + \sin p_4) \\ & + i\sigma_z\,(\sin p_1 - \sin p_2 + \sin p_3 - \sin p_4) \end{aligned}$$

B and C control anisotropic distortions

Minimally doubled fermions

Graphene (2 d): one bond splits into two



and iterate

smallest loops are hexagons

Minimally doubled fermions

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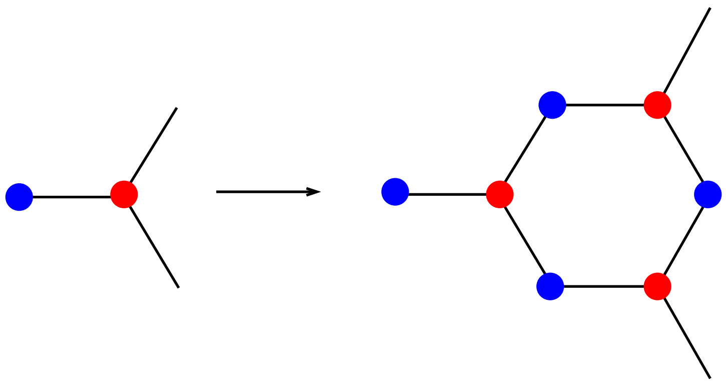
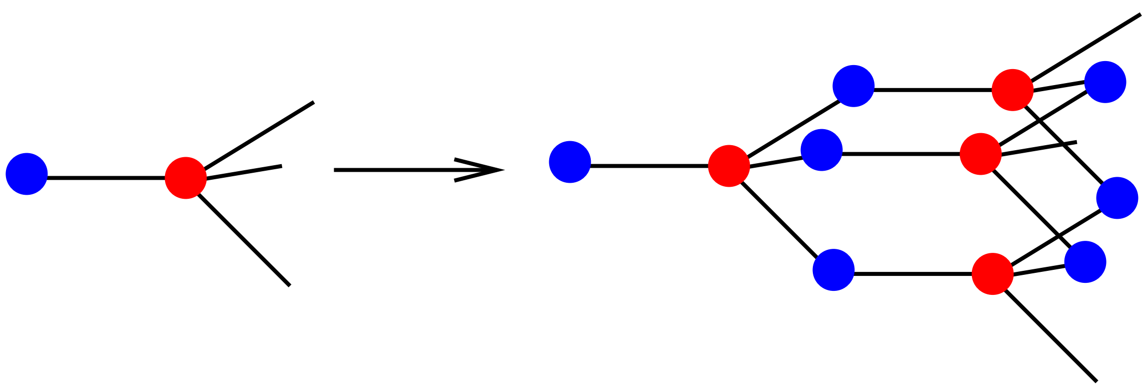


and iterate

smallest loops are hexagons

Diamond (3 d): one bond splits into three

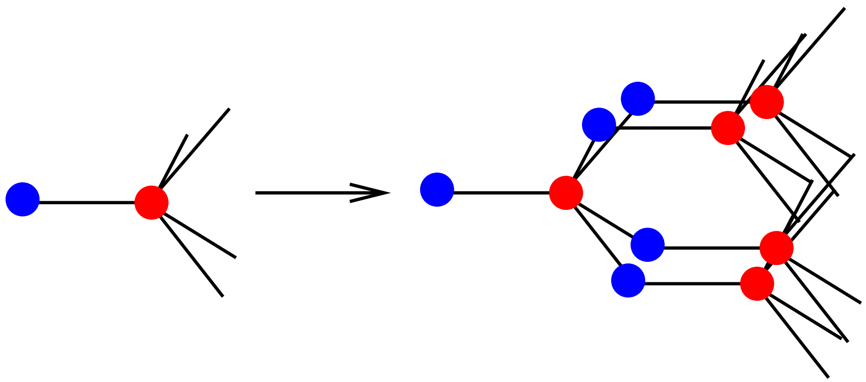


and iterate

smallest loops are cyclohexane “chairs”

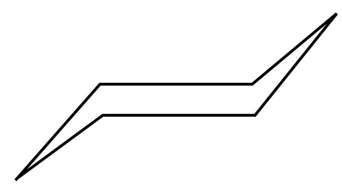
Minimally doubled fermions

Graphene (4 d): one bond splits into four



and iterate

smallest loops are hexagonal “chairs”



Minimally doubled fermions

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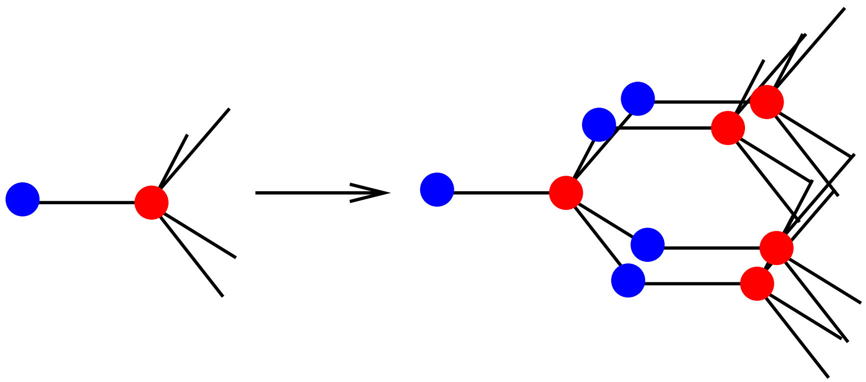

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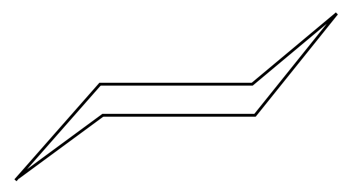
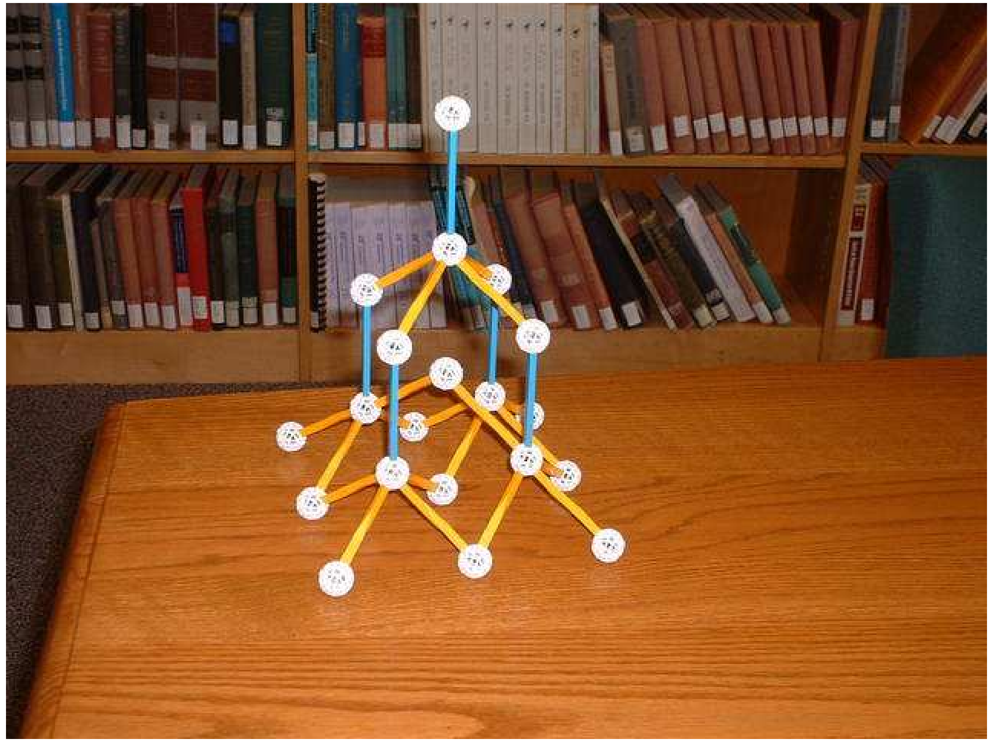


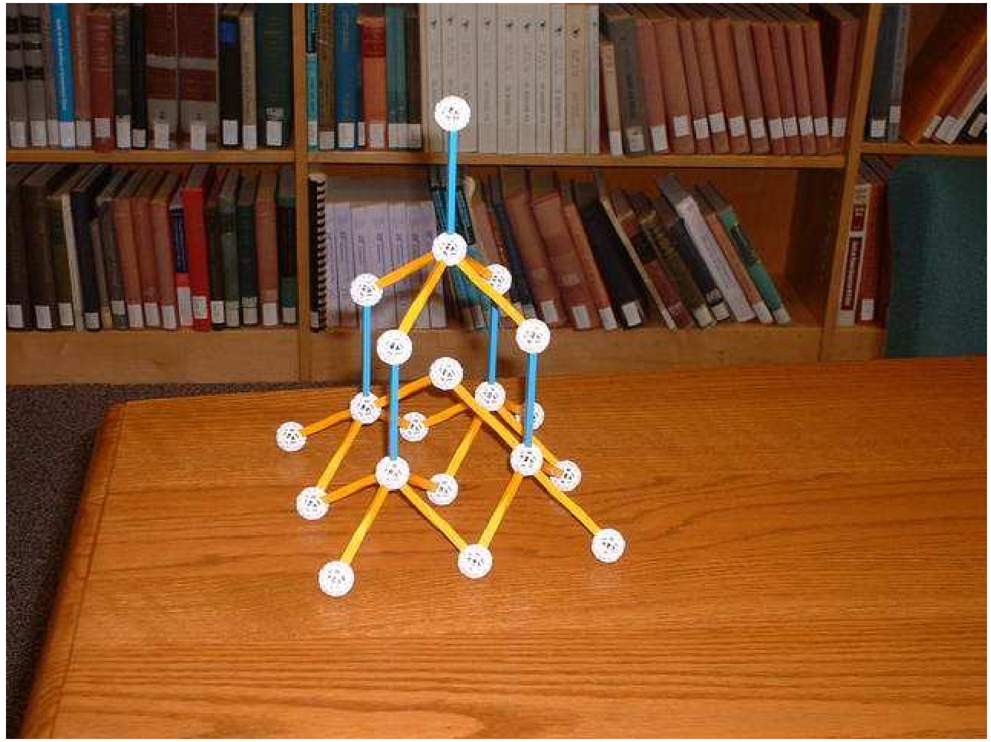
(thanks to Mike Creutz for providing many of these pictures)

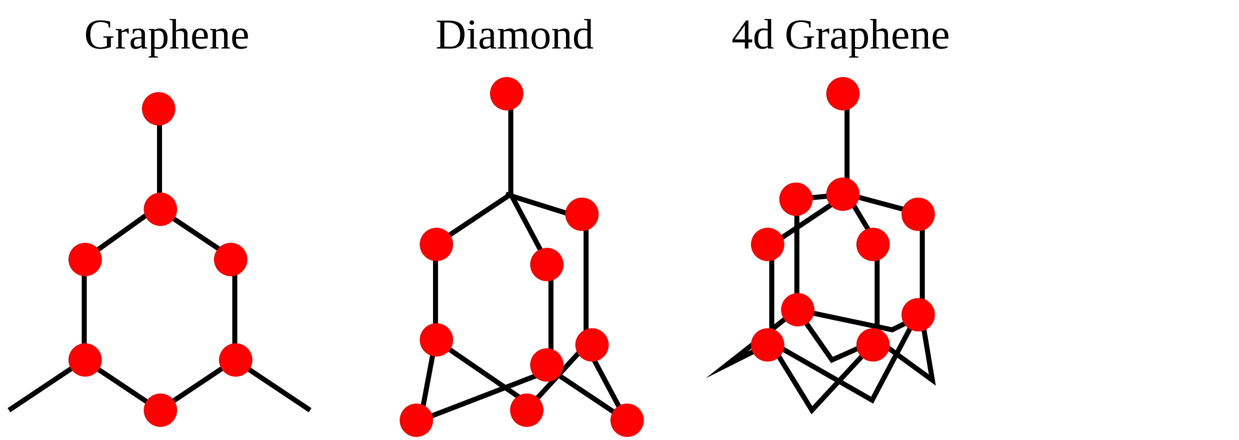
Minimally doubled fermions

4d graphene: 

Minimally doubled fermions

4d graphene:





Minimally doubled fermions

Boriçi : General family of (massless) actions on non-orthogonal lattices

$$D(p) = iB\gamma_4 \left(4C - \sum_\mu \cos p_\mu\right) + i\sum_{k=1}^{3} \gamma_k s_k(p)$$

where

$$\begin{aligned} s_1(p) &= \sin p_1 + \sin p_2 - \sin p_3 - \sin p_4 \\ s_2(p) &= \sin p_1 - \sin p_2 - \sin p_3 + \sin p_4 \\ s_3(p) &= \sin p_1 - \sin p_2 + \sin p_3 - \sin p_4 \end{aligned}$$

All these actions have two zeros, at $(\tilde{p}, \tilde{p}, \tilde{p}, \tilde{p})$ and $(-\tilde{p}, -\tilde{p}, -\tilde{p}, -\tilde{p})$, with

$$C = \cos\tilde{p}$$

Now we go on orthogonal lattices, where $B\sin\tilde{p} = C$

When we then put $B = 1$, after some translations of the momenta and rescalings we obtain the Boriçi-Creutz action

The Boriçi-Creutz action can be also constructed directly as a linear combination of two naive fermion formulations (*Creutz*)

Our focus here is on Boriçi-Creutz and Karsten-Wilczek fermions

Boriçi-Creutz fermions

The work of Boriçi and Creutz leads to a fermionic action whose free Dirac operator in momentum space reads

$$D(p) = i \sum_\mu (\gamma_\mu \sin p_\mu + \gamma'_\mu \cos p_\mu) - 2i\Gamma + m_0$$

where

$$\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \qquad (\Gamma^2 = 1)$$

and

$$\gamma'_\mu = \Gamma \gamma_\mu \Gamma = \Gamma - \gamma_\mu$$

Useful relations:

$$\sum_\mu \gamma_\mu = \sum_\mu \gamma'_\mu = 2\Gamma, \quad \{\Gamma, \gamma_\mu\} = 1, \quad \{\Gamma, \gamma'_\mu\} = 1$$

The action vanishes at $p_1 = (0,0,0,0)$ and $p_2 = (\pi/2, \pi/2, \pi/2, \pi/2)$

This ingenious construction represents a special linear combination of two (physically equivalent) naive fermions , corresponding to the first two terms in the action

Boriçi-Creutz fermions

Consider the massless case:

$$D(p) = i\sum_{\mu} \gamma_\mu \sin p_\mu + i\sum_{\mu} \gamma'_\mu \cos p_\mu - 2i\Gamma$$

Boriçi-Creutz fermions

Consider the massless case:

$$D(p) = \underbrace{i \sum_\mu \gamma_\mu \sin p_\mu}_{\text{16 doublers}} + i \sum_\mu \gamma'_\mu \cos p_\mu - 2i\Gamma$$

The first term, as widely known, has 16 zeros in the first Brillouin zone, that is when any component of the momentum is 0 or π

Boriçi-Creutz fermions

Consider the massless case:

$$D(p) = i \sum_\mu \gamma_\mu \sin p_\mu + \underbrace{i \sum_\mu \gamma'_\mu \cos p_\mu}_{\text{16 doublers}} - 2i\Gamma$$

The first term, as widely known, has 16 zeros in the first Brillouin zone, when any component of the momentum is 0 or π

The second term has also 16 doublers, but they are positioned at the momenta $(\pm\pi/2, \pm\pi/2, \pm\pi/2, \pm\pi/2)$

Boriçi-Creutz fermions

Consider the massless case:

$$D(p) = i\sum_\mu \gamma_\mu \sin p_\mu + \underbrace{i\sum_\mu \gamma'_\mu \cos p_\mu}_{\text{16 doublers}} - 2i\Gamma$$

The first term, as widely known, has 16 zeros in the first Brillouin zone, when any component of the momentum is 0 or π

The second term has also 16 doublers, but they are positioned at the momenta $(\pm\pi/2, \pm\pi/2, \pm\pi/2, \pm\pi/2)$

These are the maximally distant momenta from the zeros of the first term

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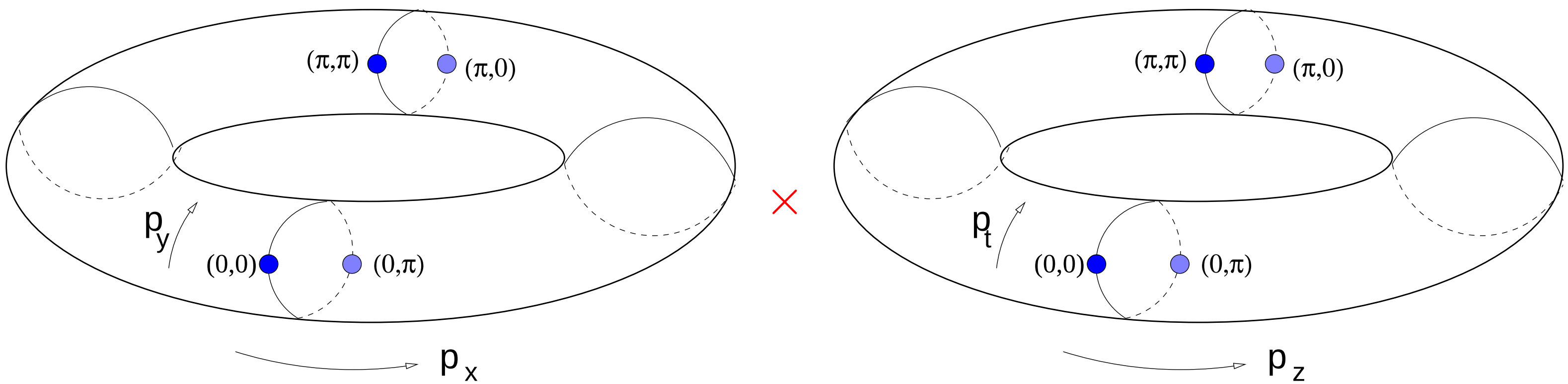
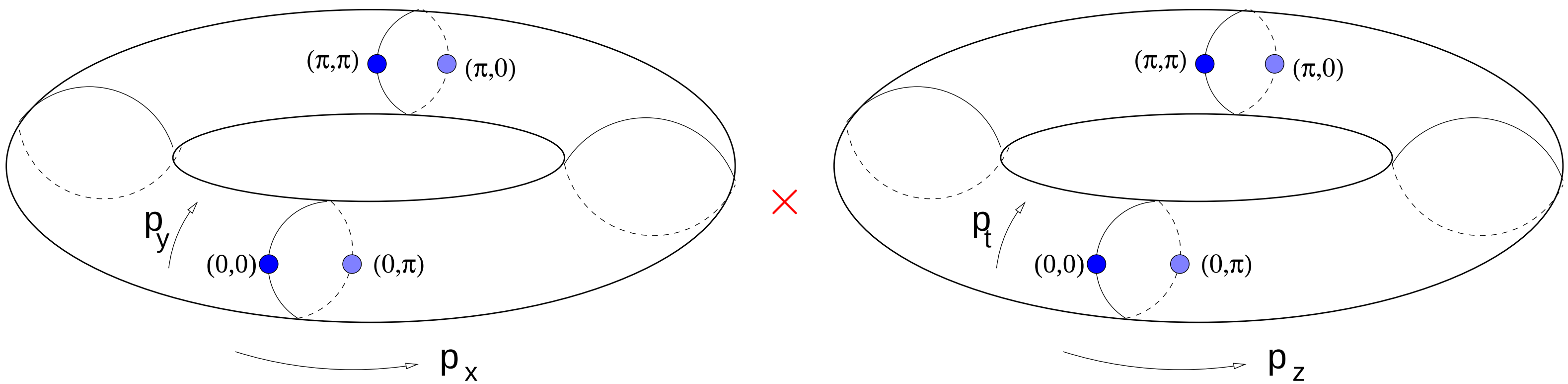
How does this come out?

Boriçi-Creutz fermions



The 16 doublers of the first naive fermion action, representing momentum space as a product of toroids

Boriçi-Creutz fermions



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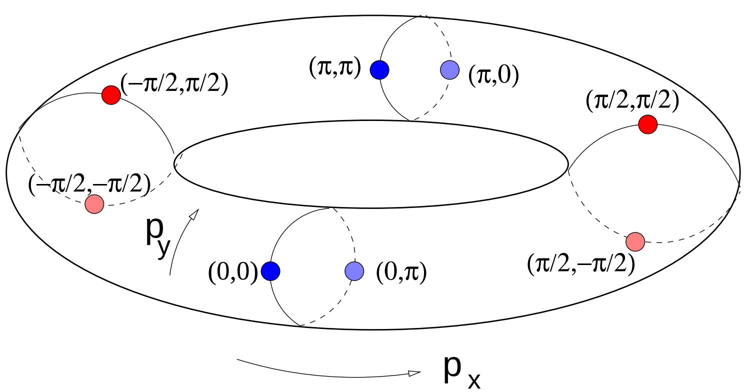

The 16 doublers of the second naive action are located at $p_\mu = \pm\pi/2$, furthest from the ones of the first naive action

(Michael Creutz, PoS LATTICE2008:080, 2008)

Boriçi-Creutz fermions

$$D(p) = \quad i\sum_{\mu}\gamma_\mu \sin p_\mu \quad + \quad i\sum_{\mu}\gamma'_\mu \cos p_\mu \qquad\qquad - 2i\Gamma$$

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$\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)$ selects a special direction $\rightarrow$ hypercubic breaking

Note: this Dirac operator is purely anti-hermitian

Karsten-Wilczek fermions

Already in the Eighties: Karsten (1981) and then Wilczek (1987) proposed some particular kind of minimally doubled fermions

Unitary equivalent to each other, after phase redefinitions

Wilczek [PRL 59, 2397 (1987)] proposed a special choice of the function $P_\mu(p)$ which minimizes the numbers of doublers

The Karsten-Wilczek Dirac operator

$$D(p) = i \sum_{\mu=1}^{4} \gamma_\mu \sin p_\mu + i\gamma_4 \sum_{k=1}^{\mathbf{3}} (1 - \cos p_k)$$

has zeros at $p_1 = (0,0,0,0)$ and $p_2 = (0,0,0,\pi)$

Drawback: it destroys the equivalence of the four directions under discrete permutations

Mixings with new operators then arise

So: now we cannot have $P_1(p) = P_2(p) = P_3(p) = P_4(p)$

Hypercubic breaking

The actions of minimally doubled fermions have two zeros

$\Rightarrow$ there is always a special direction in euclidean space

(given by the line that connects these two zeros)

Thus, these actions cannot maintain a full hypercubic symmetry

They are symmetric only under the **subgroup** of the hypercubic group which preserves (up to a sign) a **fixed direction**

For the Boriçi-Creutz action this is a major hypercube diagonal, while for other minimally doubled actions it may not be a diagonal – for example for the Karsten-Wilczek action is the x_4 axis

Although the distance between the 2 Fermi points is the same ($p_2^2 - p_1^2 = \pi^2$), these two realization of minimally doubled fermions are not equivalent

The breaking of the hypercubic symmetry implies the appearance of mixings with operators of different dimensionality, like $\overline{\psi}\Gamma\psi$ or $\overline{\psi}\Gamma D^2\psi$

For minimally doubled fermions a mixing with dimension-3 operators cannot be avoided (*Bedaque, Buchoff, Tiburzi and Walker-Loud*)

Quark propagator and vertices

Inverting the Boriçi-Creutz action we obtain the fermion propagator $S(p)$ as

$$a\,\frac{-i\sum_\mu \gamma_\mu(\sin ap_\mu - \cos ap_\mu) - i\Gamma\left(\sum_\mu \cos ap_\mu - 2\right) + am_0}{\sum_\mu(\sin ap_\mu \sum_\nu \cos ap_\nu - 2\sin ap_\mu(\cos ap_\mu + 1) - 2\cos ap_\mu) + 8 + (am_0)^2}$$

The denominator of this propagator cannot be cast (*as instead is conveniently done for many standard actions*) in a form which possesses a definite behavior under parity transformation of each single coordinate ($p_i \to -p_i$)

This is probably connected to the fact that there is an intrinsic special direction for these actions

By using $\{\gamma_\mu, \gamma_\nu\} = \{\gamma'_\mu, \gamma'_\nu\} = 2\delta_{\mu\nu}$ and $\{\gamma_\mu, \gamma'_\nu\} = 1 - 2\delta_{\mu\nu}$, the above quark propagator can also be written in the more convenient form

$$S(p) = a\,\frac{-i\sum_\mu \left[\gamma_\mu \sin ap_\mu - 2\,\gamma'_\mu\, \sin^2 ap_\mu/2\right] + am_0}{4\sum_\mu \left[\sin^2 ap_\mu/2 + \sin ap_\mu \left(\sin^2 ap_\mu/2 - \frac{1}{2}\sum_\nu \sin^2 ap_\nu/2\right)\right] + (am_0)^2}$$

where the limit of small p (continuum limit) is more transparent

The second pole at $ap = (\pi/2, \pi/2, \pi/2, \pi/2)$ describes (as expected) a particle of opposite chirality to the one at $ap = (0, 0, 0, 0)$

Quark propagator and vertices

Quark propagator for Karsten-Wilczek fermions (2nd pole at $ap = (0, 0, 0, \pi)$):

$$S(p) = a \, \frac{-i \sum_{\mu=1}^{4} \gamma_\mu \sin ap_\mu - 2i\,\gamma_4 \sum_{k=1}^{3} \sin^2 \frac{ap_k}{2} + am_0}{\sum_{\mu=1}^{4} \sin^2 ap_\mu + 4 \sin ap_4 \sum_{k=1}^{3} \sin^2 \frac{ap_k}{2} + 4 \left(\sum_{k=1}^{3} \sin^2 \frac{ap_k}{2} \right)^2 + (am_0)^2}$$

Quark-quark-gluon and quark-quark-gluon-gluon vertices (Boriçi-Creutz):

$$\begin{aligned} V_1(p_1, p_2) &= -ig_0 \left(\gamma_\mu \cos \frac{a(p_1 + p_2)_\mu}{2} - \gamma'_\mu \sin \frac{a(p_1 + p_2)_\mu}{2} \right) \\ V_2(p_1, p_2) &= \frac{1}{2} i a g_0^2 \left(\gamma_\mu \sin \frac{a(p_1 + p_2)_\mu}{2} + \gamma'_\mu \cos \frac{a(p_1 + p_2)_\mu}{2} \right) \end{aligned}$$

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(p_1 and p_2, momenta in and out of the vertex)

Counterterms

Each of these two bare actions does not contain all possible operators allowed by the respective symmetries

$\Rightarrow$ counterterms need to be added

Radiative corrections generate new contributions whose form is not matched by any term in the original bare actions

The counterterms are then necessary for a consistent renormalized theory

This consistency requirement will uniquely determine their coefficients

Our task: add to the bare actions all possible counterterms allowed by the remaining symmetries (after hypercubic symmetry has been broken)

These counterterms are not invariant under the hypercubic group – they are lattice artefacts peculiar to minimally doubled fermions

We consider operators of dimension four or lower, and we write them first in a continuum form

Afterwards, we look for convenient discretizations of these counterterms

Counterterms

In the following we will consider the massless case $m_0 = 0$

Chiral symmetry strongly restricts the number of possible counterterms

Since they have to anticommute with γ_5, we look only for operators in which a γ_μ matrix (or a sum of them) can be present – but not other matrices like 1, γ_5, $\gamma_\mu\gamma_5$ and $\sigma_{\mu\nu}$

For Boriçi-Creutz fermions, operators are allowed where summations over just single indices are present (in addition to the standard Einstein summation over two indices)

Then objects like $\sum_\mu \gamma_\mu = \Gamma$ appear

We find that there can be only one dimension-4 counterterm: $\overline{\psi}\,\Gamma \sum_\mu D_\mu \psi$

Possible discretization on the lattice: with a form similar to the hopping terms already present in the action

We then have the gauge invariant expression

$$c_4(g_0^2)\ \frac{1}{2a} \sum_\mu \Big(\overline{\psi}(x)\,\Gamma\, U_\mu(x)\, \psi(x+a\widehat{\mu}) - \overline{\psi}(x+a\widehat{\mu})\,\Gamma\, U_\mu^\dagger(x)\, \psi(x) \Big)$$

Counterterms

There is also one counterterm of dimension three: $\frac{ic_3(g_0^2)}{a}\,\overline{\psi}(x)\,\Gamma\,\psi(x)$

This is already present in the bare Boriçi-Creutz action, but with a fixed coefficient , $-2/a$

The appearance of this counterterm means that in the general renormalized action the coefficient of this operator must be kept general

Rather than a redefinition of some parameter, is the introduction of a new one

In perturbation theory it is convenient to use the convention that the operator $\overline{\psi}(x)\Gamma\psi(x)$ in the renormalized action has the coefficient $(-2+c_3)i/a$

For Monte Carlo simulations other choices might be more reasonable

This piece of the action determines among others the position of the poles of the propagators, which are moved by radiative corrections

Then, a possible renormalization condition is to choose the value of the coefficient which restores the poles to their original locations at p_1 and p_2

Counterterms

For Karsten-Wilczek fermions we find an analogous situation

Here objects are allowed in which Kronecker deltas can constrain any index to be equal to 4

It is easy to see that the only gauge-invariant counterterm of dimension four that can be added to the bare action is $\overline{\psi}\,\gamma_4 D_4\,\psi$

A suitable discretization:

$$d_4(g_0^2)\;\frac{1}{2a}\Big(\overline{\psi}(x)\,\gamma_4\,U_4(x)\,\psi(x+a\widehat{4}) - \overline{\psi}(x+a\widehat{4})\,\gamma_4\,U_4^\dagger(x)\,\psi(x)\Big)$$

There is also one counterterm of dimension three,

$$\frac{i d_3(g_0^2)}{a}\;\overline{\psi}(x)\,\gamma_4\,\psi(x)$$

which is already present in the bare Karsten-Wilczek action, with a fixed coefficient

The coefficient of the term $\overline{\psi}(x)\gamma_4\psi(x)$ in the renormalized action will be $(3+d_3)i/a$

Counterterms

In perturbation theory the coefficients of all these counterterms are functions of the coupling which start at order g_0^2

They give rise at one loop to additional contributions to fermion lines

These insertions need to be included in consistent one-loop calculations

The rules for the corrections to fermion propagators, needed for our one-loop calculations, can be easily derived

For external lines, they are given in momentum space respectively by

$$-ic_4(g_0^2)\;\Gamma\sum_\nu p_\nu, \qquad -\frac{ic_3(g_0^2)}{a}\,\Gamma$$

for Boriçi-Creutz fermions, and by

$$-id_4(g_0^2)\;\gamma_4\,p_4, \qquad -\frac{id_3(g_0^2)}{a}\,\gamma_4$$

for Karsten-Wilczek fermions

We will determine all these coefficients (at one loop) by requiring that the renormalized self-energy assumes its standard form

Counterterms

Counterterm interaction vertices are generated as well

These vertex insertions are at least of order g_0^3, and thus they cannot contribute to the one-loop amplitudes that we study here

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These vertex insertions are at least of order g_0^3, and thus they cannot contribute to the one-loop amplitudes that we study here

The form of the counterterms remains the same at all orders of perturbation theory

Only the values of their coefficients change according to the loop order

The same counterterms appear at the nonperturbative level, and will be required for a consistent numerical simulation of these fermions

We also want to emphasize that counterterms not only provide additional Feynman rules for the calculation of loop amplitudes

They can modify Ward identities and hence, in particular, contribute additional terms to the conserved currents

Self-energy

The tadpole of the self-energy can be easily computed from the vertex $V_2(p,p)$

The relevant expression for Boriçi-Creutz fermions is, in a general covariant gauge $\partial_\mu A_\mu = 0$,

$$\frac{1}{a^2}\cdot\frac{Z_0}{2}\Big(1-\frac{1}{4}(1-\alpha)\Big)\cdot iag_0^2 C_F\sum_\mu\Big(\gamma_\mu a p_\mu + (\Gamma-\gamma_\mu)(1+O(a^2)\Big)$$

which is equal to

$$g_0^2 C_F\,\frac{Z_0}{2}\Big(1-\frac{1}{4}(1-\alpha)\Big)\Big(i\not{p}+\frac{i}{a}\sum_\mu(\Gamma-\gamma_\mu)\Big)+O(a)$$

where

$$Z_0 = \int\frac{dp}{(2\pi)^4}\,\frac{1}{\widehat{p}^2} = 0.1549333...... = 24.466100\,\frac{1}{16\pi^2}$$

Terms of $O(a)$ and higher are not important here

Since $\sum_\mu \gamma_\mu = 2\Gamma$, the result of the one-loop tadpole is

$$g_0^2 C_F\,\frac{Z_0}{2}\Big(1-\frac{1}{4}(1-\alpha)\Big)\Big(i\not{p}+\frac{2i\Gamma}{a}\Big)$$

Self-energy

The $i\not{p}$ term is the same as for Wilson fermions, while the other term (as already noted by *Bedaque, Buchoff, Tiburzi and Walker-Loud*) would imply a power-divergent $1/a$ mixing with the dimension-3 operator $\overline{\psi}\,\Gamma\,\psi$

. . . if not compensated by an analogous term coming from the other diagram of the self-energy, the sunset diagram

In our work we have shown that there is no such compensation

The result of the sunset diagram is

$$i\not{p}\cdot\frac{g_0^2}{16\pi^2}\,C_F\left[\log a^2p^2-5.42642+(1-\alpha)\Big(-\log a^2p^2+7.850272\Big)\right]$$

$$+m_0\cdot\frac{g_0^2}{16\pi^2}\,C_F\left[4\,\log a^2p^2-29.48729+(1-\alpha)\Big(-\log a^2p^2+5.792010\Big)\right]$$

$$+1.52766\cdot\frac{g_0^2}{16\pi^2}\,C_F\cdot i\,\Gamma\sum_\mu p_\mu$$

$$+(5.07558+6.11653\,(1-\alpha))\cdot\frac{g_0^2}{16\pi^2}\,C_F\cdot i\,\frac{\Gamma}{a}$$

Self-energy

Note that gauge invariance forces the terms proportional to $1 - \alpha$ to be the same as (for example) Wilson or overlap fermions

This is an important check of the correctness of our calculations

The total self-energy (without counterterms) of a Boriçi-Creutz fermion is then given at this order by

$$\Sigma(p, m_0) = i\not{p}\,\Sigma_1(p) + m_0\,\Sigma_2(p) + c_1(g_0^2)\cdot i\,\Gamma\sum_\mu p_\mu + c_2(g_0^2)\cdot i\,\frac{\Gamma}{a}$$

with

$$\Sigma_1(p) = 1 + \frac{g_0^2}{16\pi^2}\,C_F\left[\log a^2p^2 + 6.80663 + (1-\alpha)\Big(-\log a^2p^2 + 4.792010\Big)\right] + O(g_0^4)$$

$$\Sigma_2(p) = 1 + \frac{g_0^2}{16\pi^2}\,C_F\left[4\,\log a^2p^2 - 29.48729 + (1-\alpha)\Big(-\log a^2p^2 + 5.792010\Big)\right] + O(g_0^4)$$

$$c_1(g_0^2) = 1.52766\cdot\frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4)$$

$$c_2(g_0^2) = 29.54170\cdot\frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4)$$

Self-energy

As expected, the two terms Γ/a coming from the tadpole and the half-circle diagrams do not cancel – in fact, they reinforce each other

Notice that the parts proportional to $1-\alpha$ instead exactly cancel, as required by gauge invariance

The full inverse propagator at one loop can be written (*without counterterms*) as

$$\Sigma^{-1}(p,m_0) = \Big(1-\Sigma_1\Big)\cdot\Big\{i\not{p}+m_0\Big(1-\Sigma_2+\Sigma_1\Big)-\frac{ic_1}{2}\sum_\mu \gamma_\mu \sum_\nu p_\nu - \frac{ic_2}{a}\,\Gamma\Big\}$$

We can only cast the renormalized propagator in the standard form

$$\Sigma(p,m_0) = \frac{Z_2}{i\not{p} + Z_m\, m_0}$$

with the wave-function and quark mass renormalization given by

$$Z_2 = \Big(1-\Sigma_1\Big)^{-1}, \qquad Z_m = 1 - \Big(\Sigma_2 - \Sigma_1\Big)$$

if we use the counterterms to cancel the Lorentz non-invariant factors (c_1 and c_2)

Self-energy

The term proportional to $c_1(g_0^2)$ can be eliminated by using the counterterm of the form $\overline{\psi}\,\sum_\mu \gamma_\mu\,\sum_\nu D_\nu\,\psi$ (permitted by the symmetries of the theory)

The term proportional to $c_2(g_0^2)$ can be eliminated using the counterterm

$$\frac{1}{a}\,\overline{\psi}\,\Gamma\,\psi$$

which is already present in the action:

$$S(x) = \;\cdots\; + a^4 \sum_x \overline{\psi}(x) \left(m_0 - \frac{2i\Gamma}{a} \right) \psi(x)$$

For Boriçi-Creutz fermions we then determine at one loop

$$\begin{aligned} c_3(g_0^2) &= 29.54170 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4) \\ c_4(g_0^2) &= 1.52766 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4) \end{aligned}$$

Self-energy

For Karsten-Wilczek fermions the result of the tadpole is

$$g_0^2 C_F \, \frac{Z_0}{2} \left(1 - \frac{1}{4}(1-\alpha)\right) \left(i\not{p} - \frac{3i\gamma_4}{a}\right)$$

The complete self-energy (without counterterms) comes out as

$$\Sigma(p, m_0) = i\not{p}\, \Sigma_1(p) + m_0\, \Sigma_2(p) + d_1(g_0^2) \cdot i\, \gamma_4 p_4 + d_2(g_0^2) \cdot i\, \frac{\gamma_4}{a}$$

where

$$\Sigma_1(p) = \frac{g_0^2}{16\pi^2}\, C_F \left[\log a^2 p^2 + 9.24089 + (1-\alpha)\Big(-\log a^2 p^2 + 4.792010\Big)\right]$$

$$\Sigma_2(p) = \frac{g_0^2}{16\pi^2}\, C_F \left[4\, \log a^2 p^2 - 24.36875 + (1-\alpha)\Big(-\log a^2 p^2 + 5.792010\Big)\right]$$

$$\begin{aligned} d_1(g_0^2) &= \; -0.12554 \cdot \frac{g_0^2}{16\pi^2}\, C_F \\ d_2(g_0^2) &= \; -29.53230 \cdot \frac{g_0^2}{16\pi^2}\, C_F \end{aligned}$$

Self-energy

The full inverse propagator at one loop can be written, without counterterms, as

$$\Sigma^{-1}(p,m_0) = \Big(1-\Sigma_1\Big)\cdot\Big(i\not{p} + m_0\Big(1-\Sigma_2+\Sigma_1\Big) - i d_1\,\gamma_4 p_4 - \frac{i d_2}{a}\,\gamma_4\Big)$$

Similarly to before, by adding to the Karsten-Wilczek action counterterms of the form

$$\overline{\psi}\,\gamma_4\,D_4\,\psi, \qquad \frac{1}{a}\,\overline{\psi}\,\gamma_4\,\psi$$

the terms which are not Lorentz invariant can be eliminated, and the renormalized propagator can be written in the standard form

$$\Sigma(p,m_0) = \frac{Z_2}{i\not{p} + Z_m\,m_0}$$

Then, at one loop

$$\begin{aligned} d_3(g_0^2) &= -29.53230\cdot\frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4) \\ d_4(g_0^2) &= -0.12554\cdot\frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4) \end{aligned}$$

Bilinears

We have also computed the renormalization of the (local) bilinears

Here we give the results of the vertex diagrams, to which the wave-function renormalization (Σ_1 *of the self-energy) must be added, to obtain the complete renormalization factors*

For the scalar density the vertex diagram gives, for Boriçi-Creutz fermions,

$$\frac{g_0^2}{16\pi^2}\, C_F \left[-4\, \log a^2p^2 + 29.48729 + (1-\alpha)\Big(\log a^2p^2 - 5.792010 \Big) \right]$$

There are no mixings here coming from the breaking of hypercubic invariance

For the tensor current we obtain the result of the vertex diagram as

$$\frac{g_0^2}{16\pi^2}\, C_F\, \sigma_{\mu\nu} \left[2.16548 + (1-\alpha)\Big(\log a^2p^2 - 3.792010 \Big) \right]$$

Also here there are no mixings coming from the breaking of hypercubic invariance

Bilinears

For the vector we obtain for the vertex diagram

$$\frac{g_0^2}{16\pi^2}\, C_F\, \gamma_\mu \left[-\log a^2p^2 + 9.54612 + (1-\alpha)\Big(\log a^2p^2 - 4.792010 \Big) \right] + c_1^v(g_0^2)\, \Gamma$$

with the coefficient of the mixing given by

$$c_1^v(g_0^2) \;=\; -0.10037 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4)$$

This is a mixing with an operator of the same dimensionality, which is not invariant under the hypercubic group

Notice that there can be no power-divergent mixing here (and in all the other bilinears), as one can see by simple dimensional counting

As a consequence of chiral symmetry, the same numbers appear in the renormalization of the vector and axial currents, and this also happens for the scalar and pseudo-scalar densities

We have verified this with our codes

Bilinears

For Karsten-Wilczek fermions, the vertex diagram of the scalar density gives

$$\frac{g_0^2}{16\pi^2}\, C_F \left[-4\, \log a^2p^2 + 24.36875 + (1-\alpha)\Big(\log a^2p^2 - 5.792010 \Big) \right]$$

For the vector current:

$$\frac{g_0^2}{16\pi^2}\, C_F\, \gamma_\mu \left[-\log a^2p^2 + 10.44610 - \delta_{\mu 4} \cdot 2.88914 + (1-\alpha)\Big(\log a^2p^2 - 4.792010 \Big) \right]$$

The spatial and temporal components of the vector (as well the axial) current receive different radiative corrections!

Cross-mixings between the spatial and temporal components appear to be absent – each of these components still renormalizes multiplicatively

For the tensor current:

$$\frac{g_0^2}{16\pi^2}\, C_F\, \sigma_{\mu\nu} \left[4.17551 + (1-\alpha)\Big(\log a^2p^2 - 3.792010 \Big) \right]$$

The tensor operator does not appear to show any preference for the temporal direction even after (one-loop) renormalization – all 6 independent components of the tensor operator renormalize with the same numbers

Renormalization of the mass

Chiral symmetry protects the quark mass m_0 from an additive renormalization

The relation between the bare and renormalized quark masses, m_0 and $m_{\rm R}$, is then

$$m_{\rm R} = Z_m \, m_0$$

The full expression for the renormalization factors of the scalar and pseudo-scalar densities at one loop is

$$Z_S = Z_P = 1 - \Big(\Lambda_S + \Sigma_1\Big)$$

where Λ_S is the result for the one-loop vertex diagram of the scalar density

Λ_S is exactly equal to the $O(g_0^2)$-contribution to the quark self-energy Σ_2, but comes with an opposite sign

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Then, the renormalization factors Z_S and Z_P satisfy

$$1/Z_m = Z_S = Z_P$$

The last equality is a consequence of chiral symmetry

The renormalization of the quark mass for minimally doubled fermions has the same form as (say) overlap fermions

Conserved vector and axial currents

We have seen that Z_V and Z_A are not equal to one

The local vector and axial currents are not conserved

We need to consider the chiral Ward identities in order to work with currents which are protected from renormalization

We have constructed the conserved vector and axial currents, and verified that at one loop their renormalization constants are equal to one

We act on the Boriçi-Creutz action in position space

$$S = a^4 \sum_x \Bigg[\frac{1}{2a} \sum_\mu \Big[\overline{\psi}(x)\,(\gamma_\mu + i\gamma'_\mu)\,U_\mu(x)\,\psi(x+a\widehat{\mu}) - \overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\gamma'_\mu)\,U^\dagger_\mu(x)\,\psi(x) \Big] + \overline{\psi}(x)\left(m_0 - \frac{2i\Gamma}{a}\right)\psi(x) \Bigg]$$

with the vector transformation

$$\delta_V \psi = i\alpha\,\psi, \quad \delta_V \overline{\psi} = -i\alpha\,\overline{\psi}$$

or the axial transformation

$$\delta_A \psi = i\alpha\,\gamma_5 \psi, \quad \delta_A \overline{\psi} = i\alpha\,\overline{\psi}\gamma_5$$

Conserved vector and axial currents

Take the Ward identity

$$\left\langle \frac{\delta O(x_1 \cdots x_n)}{\delta \alpha(x)} \right\rangle = \left\langle O(x_1 \cdots x_n)\, \frac{\delta S}{\delta \alpha(x)} \right\rangle$$

For an axial transformation we have

$$i \left\langle \frac{\delta S}{\delta \alpha(x)} \right\rangle = \nabla_x^{\mu} \left\langle O(x_1 \cdots x_n)\, A_\mu(x) \right\rangle$$

(similarly for a vector transformation)

For on-shell matrix elements, $O(x_1 \cdots x_n)$ is a product of the operators which generate the required initial and final states from the vacuum

Applying the axial transformation $\delta_A \psi$, we look for a current $A_\mu^{cons}(x)$ which satisfies

$$i \frac{\delta S}{\delta \alpha(x)} = \nabla^\star A_\mu^{cons}(x) = A_\mu^{cons}(x) - A_\mu^{cons}(x - a\widehat{\mu})$$

If the axial transformation is a symmetry of the action, then the current $A_\mu^{cons}(x)$ is conserved

(similarly for the vector current)

Conserved vector and axial currents

Using translational invariance, the vector transformation gives

$$\delta S = \frac{ia^3}{2}\sum_x \alpha(x) \sum_\mu \Big[\overline{\psi}(x-a\widehat{\mu})\,(\gamma_\mu + i\gamma'_\mu)\,U_\mu(x-a\widehat{\mu})\,\psi(x)$$

$$-\overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\gamma'_\mu)\,U^\dagger_\mu(x)\,\psi(x)$$

$$-\overline{\psi}(x)\,(\gamma_\mu + i\gamma'_\mu)\,U_\mu(x)\,\psi(x+a\widehat{\mu})$$

$$+\overline{\psi}(x)\,(\gamma_\mu - i\gamma'_\mu)\,U^\dagger_\mu(x-a\widehat{\mu})\,\psi(x-a\widehat{\mu})\Big]$$

The corresponding expression for the axial transformation is (for $m_0 = 0$)

$$\delta S = \frac{ia^3}{2}\sum_x \alpha(x) \sum_\mu \Big[\overline{\psi}(x-a\widehat{\mu})\,(\gamma_\mu + i\gamma'_\mu)\,\gamma_5\,U_\mu(x-a\widehat{\mu})\,\psi(x)$$

$$-\overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\gamma'_\mu)\,\gamma_5\,U^\dagger_\mu(x)\,\psi(x)$$

$$-\overline{\psi}(x)\,(\gamma_\mu + i\gamma'_\mu)\,\gamma_5\,U_\mu(x)\,\psi(x+a\widehat{\mu})$$

$$+\overline{\psi}(x)\,(\gamma_\mu - i\gamma'_\mu)\,\gamma_5\,U^\dagger_\mu(x-a\widehat{\mu})\,\psi(x-a\widehat{\mu})\Big]$$

Axial symmetry only works for $m_0 = 0$: $\quad \overline{\psi}(x)\psi(x) \to 2i\alpha(x)\,\overline{\psi}(x)\gamma_5\psi(x)$

Conserved vector and axial currents

We then obtain the conserved vector corrent for Boriçi-Creutz fermions as

$$V_\mu^{cons}(x) = \frac{1}{2}\left[\overline{\psi}(x)\,(\gamma_\mu + i\,\gamma'_\mu)\,U_\mu(x)\,\psi(x+a\widehat{\mu}) + \overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\,\gamma'_\mu)\,U_\mu^\dagger(x)\,\psi(x)\right]$$

while the axial current (conserved in the case $m_0 = 0$) is

$$A_\mu^{cons}(x) = \frac{1}{2}\left[\overline{\psi}(x)\,(\gamma_\mu + i\,\gamma'_\mu)\,\gamma_5\,U_\mu(x)\,\psi(x+a\widehat{\mu}) + \overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\,\gamma'_\mu)\,\gamma_5\,U_\mu^\dagger(x)\,\psi(x)\right]$$

We can only obtain isospin-singlet currents, since the action describes a degenerate doublet of fermions

We have computed the renormalization of these point-split currents

We give here the results for the individual diagrams of the conserved vector current

For the conserved axial current the numbers are the same, and one just needs to replace γ_μ with $\gamma_\mu\gamma_5$, and Γ with $\Gamma\gamma_5$

Conserved vector and axial currents

The vertex diagram gives the result

$$\frac{g_0^2}{16\pi^2}\, C_F\, \gamma_\mu \left[-\log a^2 p^2 + 0.61800 + (1-\alpha)\Big(\log a^2 p^2 - 1.73375 \Big) \right] + c_1^{vtx}(g_0^2)\, \Gamma$$

with the coefficient of the mixing given by

$$c_1^{vtx}(g_0^2) \;=\; -0.43749 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4)$$

The result of the sails is

$$\frac{g_0^2}{16\pi^2}\, C_F\, \gamma_\mu \left[4.80841 - 6.11653\,(1-\alpha) \right] + c_1^{sls}(g_0^2)\, \Gamma$$

with the coefficient of the mixing given by

$$c_1^{sls}(g_0^2) \;=\; -1.09017 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4)$$

Conserved vector and axial currents

Finally, the operator tadpole gives the same result as for Wilson fermions:

$$-g_0^2\, C_F\, \gamma_\mu\, \frac{Z_0}{2} \left(1 - \frac{1}{4}(1-\alpha)\right)$$

The sum of all these diagrams is

$$\frac{g_0^2}{16\pi^2}\, C_F\, \gamma_\mu \left[-\log a^2 p^2 - 6.80664 + (1-\alpha)\left(\log a^2 p^2 - 4.79202 \right) \right] + c_1^{cv}(g_0^2)\, \Gamma$$

with the coefficient of the mixing given by

$$c_1^{cv}(g_0^2) \;=\; -1.52766 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4)$$

The term proportional to γ_μ exactly compensates the contribution of $\Sigma_1(p)$ from the quark self-energy (wave-function renormalization)

But what about the mixing term, proportional to Γ ?

We should take into account the counterterms . . .

Conserved vector and axial currents

The counterterm $\overline{\psi}(x)\,\frac{i\Gamma}{a}\,\psi(x)$ does **not** modify the Ward identities

On the contrary, the counterterm

$$\frac{c_4(g_0^2)}{4}\sum_\mu\sum_\nu\Big(\overline{\psi}(x)\,\gamma_\nu\,U_\mu(x)\,\psi(x+a\widehat{\mu})+\overline{\psi}(x+a\widehat{\mu})\,\gamma_\nu\,U_\mu^\dagger(x)\,\psi(x)\Big)$$

generates new terms in the Ward identities and then in the conserved currents

The additional term in the conserved vector current so generated reads

$$\frac{c_4(g_0^2)}{4}\Big[\overline{\psi}(x)\Big(\sum_\nu\gamma_\nu\Big)U_\mu(x)\,\psi(x+a\widehat{\mu})+\overline{\psi}(x+a\widehat{\mu})\Big(\sum_\nu\gamma_\nu\Big)U_\mu^\dagger(x)\,\psi(x)\Big]$$

It is easy to compute its 1-loop contribution (*coefficient already of order* g_0^2 !)

The result is then $c_4(g_0^2)\,\Gamma\;=\;-c_1^{cv}(g_0^2)\,\Gamma$

This exactly cancels the Γ mixing term arising for the 1-loop conserved current without counterterms

Thus, we obtain that the renormalization constant of these point-split currents is one – which confirms that they are conserved currents

Conserved vector and axial currents

Let us now consider the Karsten-Wilczek action in position space:

$$S = a^4 \sum_x \left[\frac{1}{2a} \sum_{\mu=1}^{4} \left[\overline{\psi}(x) \left(\gamma_\mu - i\gamma_4 \left(1 - \delta_{\mu 4} \right) \right) U_\mu(x)\, \psi(x + a\widehat{\mu}) \right. \right.$$

$$\left. \left. - \overline{\psi}(x + a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(1 - \delta_{\mu 4} \right) \right) U_\mu^\dagger(x)\, \psi(x) \right] + \overline{\psi}(x) \left(m_0 + \frac{3i\gamma_4}{a} \right) \psi(x) \right]$$

Conserved vector and axial currents

Let us now consider the Karsten-Wilczek action in position space:

$$S = a^4 \sum_x \Bigg[\frac{1}{2a} \sum_{\mu=1}^{4} \Big[\overline{\psi}(x) \left(\gamma_\mu - i\gamma_4 \left(1-\delta_{\mu 4}\right)\right) U_\mu(x)\, \psi(x+a\widehat{\mu}) - \overline{\psi}(x+a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(1-\delta_{\mu 4}\right)\right) U_\mu^\dagger(x)\, \psi(x) \Big] + \overline{\psi}(x) \left(m_0 + \frac{3i\gamma_4}{a} \right) \psi(x) \Bigg]$$

For Karsten-Wilczek fermions, application of the chiral Ward identities gives for the conserved axial current

$$A_\mu^{\mathrm{c}}(x) = \frac{1}{2} \Big(\overline{\psi}(x) \left(\gamma_\mu - i\gamma_4 \left(1-\delta_{\mu 4}\right)\right) \gamma_5\, U_\mu(x)\, \psi(x+a\widehat{\mu}) + \overline{\psi}(x+a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(1-\delta_{\mu 4}\right)\right) \gamma_5\, U_\mu^\dagger(x)\, \psi(x) \Big) + \frac{d_4(g_0^2)}{2} \Big(\overline{\psi}(x)\, \gamma_4\gamma_5\, U_4(x)\, \psi(x+a\widehat{4}) + \overline{\psi}(x+a\widehat{4})\, \gamma_4\gamma_5\, U_4^\dagger(x)\, \psi(x) \Big)$$

Once more, is a simple expression which involve only nearest-neighbour points

We checked explicitly that its renormalization constant is one

Vacuum polarization

Our focus here: the radiative corrections to the gluon propagator due to fermion loops

Contributions to the vacuum polarization due to loops of gluons and ghosts: independent of the lattice fermionic action chosen (*at one loop*)

$\Rightarrow$ do not provide informations relevant for hypercubic breaking

Only the fermionic loops are able to generate hypercubic-breaking terms (as it in the end happens for both Karsten-Wilczek and Boriçi-Creutz fermions)

The fermionic contribution to the vacuum polarization for one flavor of Wilson fermions *(where neither breaking of hypercubic symmetry nor fermion doubling occur)* is

$$\Pi^{(f)}_{\mu\nu}(p) = \left(p_\mu p_\nu - \delta_{\mu\nu} p^2\right) \left[\frac{g_0^2}{16\pi^2} C_t \left(-\frac{4}{3} \log p^2 a^2 + 4.337002 \right)\right]$$

where $\mathrm{Tr}\,(\mathrm{t^a t^b}) = \mathrm{C_2}\,\delta^{\mathrm{ab}}$

We can see that this (gauge invariant) result satisfies the Ward identity $p^\mu \Pi^{(f)}_{\mu\nu}(p) = 0$, which expresses the conservation of the fermionic current

Vacuum polarization

For Boriçi-Creutz fermions (without the purely gluonic counterterm) :

$$\Pi^{(f)}_{\mu\nu}(p) = \left(p_\mu p_\nu - \delta_{\mu\nu}p^2\right)\left[\frac{g_0^2}{16\pi^2}C_2\left(-\frac{8}{3}\log p^2a^2 + 23.6793\right)\right]$$
$$-\left((p_\mu + p_\nu)\sum_\lambda p_\lambda - p^2 - \delta_{\mu\nu}\left(\sum_\lambda p_\lambda\right)^2\right)\frac{g_0^2}{16\pi^2}C_2 \cdot 0.9094$$

For Karsten-Wilczek fermions (without the purely gluonic counterterm) :

$$\Pi^{(f)}_{\mu\nu}(p) = \left(p_\mu p_\nu - \delta_{\mu\nu}p^2\right)\left[\frac{g_0^2}{16\pi^2}C_2\left(-\frac{8}{3}\log p^2a^2 + 19.99468\right)\right]$$
$$-\left(p_\mu p_\nu\left(\delta_{\mu4} + \delta_{\nu4}\right) - \delta_{\mu\nu}\left(p^2\,\delta_{\mu4}\delta_{\nu4} + p_4^2\right)\right)\frac{g_0^2}{16\pi^2}C_2 \cdot 12.69766$$

There are new terms, compared with a standard situation like Wilson fermions

Although each of these actions breaks hypercubic symmetry in its appropriate and peculiar way, these new terms still satisfy the Ward identity $p^\mu\Pi^{(f)}_{\mu\nu}(p) = 0$

Very important: there are no power-divergences ($1/a^2$ or $1/a$) in our results for the vacuum polarization!

Gluonic counterterms

We need counterterms also for the pure gauge part of the actions of minimally doubled fermions

Although at the bare level the breaking of hypercubic symmetry is a feature of the fermionic actions only, in the renormalized theory it propagates (*via the interactions between quarks and gluons*) also to the pure gauge sector

Effect: some terms in the purely gluonic parts of the action can renormalize with different factors – then, pure gauge counterterms need to be added in the renormalized actions to correct this imbalance

They must be of the (continuum) $\mathrm{tr}\, FF$ form, but with nonconventional choices of the indices, which reflect the breaking of the Lorentz (hypercubic) symmetry

Let us consider first the Boriçi-Creutz case

If we choose all indices of $\mathrm{tr}\, FF$ to appear only once in each summmation (as allowed by the hypercubic symmetry breaking), we can construct a counterterm which has the continuum form

$$\sum_{\lambda\rho\sigma\tau} \mathrm{tr}\, F_{\lambda\rho}(x)\, F_{\sigma\tau}(x)$$

However, this is identically zero

Gluonic counterterms

The next possibility is that one index is present twice in a summation (as in the usual Einstein convention), but there are other two indices, each appearing only once in a summation:

$$c_P(g_0^2) \sum_{\lambda\rho\tau} \mathrm{tr}\, F_{\lambda\rho}(x)\, F_{\rho\tau}(x)$$

This operator is the only purely gluonic counterterm possible: choosing two pairs of summed indices will reproduce the usual Lorentz invariant term $\sum_{\lambda\rho} \mathrm{tr}\, F_{\lambda\rho}(x)\, F_{\lambda\rho}(x)$, already present in the action

A possible lattice discretization: use the clover expression of the $F_{\mu\nu}$ tensor

It might be also convenient to employ $F_{\mu\nu}^{lat} = i\, \mathit{Im}\, (1 - P_{\mu\nu})$

At one loop this counterterm is relevant only for gluon propagators

Denoting the fixed external indices at both ends with μ and ν, all possible lattice discretizations of this counterterm give in momentum space the same Feynman rule:

$$-c_P(g_0^2) \left[(p_\mu + p_\nu) \sum_\lambda p_\lambda - p^2 - \delta_{\mu\nu} \left(\sum_\lambda p_\lambda \right)^2 \right]$$

The presence of this counterterm is essential for the correct renormalization of the vacuum polarization

Gluonic counterterms

It is not hard to imagine that in the case of Karsten-Wilczek fermions the temporal plaquettes will be renormalized differently from the other plaquettes

Indeed, the counterterm to be introduced contains an asymmetry between these two kinds of plaquettes, and can be written in continuum form as

$$d_P(g_0^2) \sum_{\rho\lambda} \mathrm{tr}\, F_{\rho\lambda}(x)\, F_{\rho\lambda}(x)\, \delta_{\rho 4}$$

This is the only purely gluonic counterterm needed for this action, since introducing also a $\delta_{\lambda 4}$ in the above expression will produce a vanishing object

It is immediate to write a lattice discretization for it, using the plaquette:

$$d_P(g_0^2)\, \frac{\beta}{2} \sum_{\rho\lambda} \Big(1 - \frac{1}{N_C}\, \mathrm{tr}\, P_{4\lambda}(x)\Big)$$

The Feynman rule for this counterterm reads

$$-d_P(g_0^2)\, \Big[p_\mu p_\nu\, (\delta_{\mu 4} + \delta_{\nu 4}) - \delta_{\mu\nu}\, \big(p^2\, \delta_{\mu 4}\delta_{\nu 4} + p_4^2\big)\Big]$$

and again is needed in the vacuum polarization

Counterterms

The cancellation of the hypercubic breaking terms of the vacuum polarization determines

$$
\begin{aligned}
c_P(g_0^2) &= -0.9094 \cdot \frac{g_0^2}{16\pi^2} C_2 + O(g_0^4) \\
d_P(g_0^2) &= -12.69766 \cdot \frac{g_0^2}{16\pi^2} C_2 + O(g_0^4)
\end{aligned}
$$

Counterterms

The cancellation of the hypercubic breaking terms of the vacuum polarization determines

$$
\begin{aligned}
c_P(g_0^2) &= -0.9094 \cdot \frac{g_0^2}{16\pi^2} C_2 + O(g_0^4) \\
d_P(g_0^2) &= -12.69766 \cdot \frac{g_0^2}{16\pi^2} C_2 + O(g_0^4)
\end{aligned}
$$

All counterterms remain of the same form at all orders of perturbation theory

Only the values of their coefficients depend on the number of loops

The same counterterms appear at the nonperturbative level, and will be required for a consistent simulation of these fermions

We would now like to see how the one-loop calculations presented so far can shed light on numerical simulations of minimally doubled fermions

These simulations will have to employ the <u>complete renormalized actions</u> (in position space), including the counterterms

We can write the renormalized actions as follows:

Simulations

For Boriçi-Creutz fermions

$$
\begin{aligned}
S^{f}_{BC} \quad = \quad & a^4 \sum_x \Bigg\{ \frac{1}{2a} \sum_{\mu=1}^{4} \Big[\overline{\psi}(x) \left(\gamma_\mu + c_4(\beta)\,\Gamma + i\gamma'_\mu\right) U_\mu(x)\, \psi(x + a\widehat{\mu}) \\
& -\overline{\psi}(x + a\widehat{\mu}) \left(\gamma_\mu - c_4(\beta)\,\Gamma - i\gamma'_\mu\right) U^\dagger_\mu(x)\, \psi(x) \Big] \\
& +\overline{\psi}(x) \left(m_0 + \widetilde{c}_3(\beta)\, \frac{i\,\Gamma}{a} \right) \psi(x) \\
& +\beta \sum_{\mu<\nu} \left(1 - \frac{1}{N_c} \mathrm{Re}\, \mathrm{tr}\, P_{\mu\nu} \right) + c_P(\beta) \sum_{\mu\nu\rho} \mathrm{tr}\, F^{lat}_{\mu\rho}(x)\, F^{lat}_{\rho\nu}(x) \Bigg\}
\end{aligned}
$$

We have redefined the coefficient of the dimension-3 counterterm, using $\widetilde{c}_3(\beta) = -2 + c_3(\beta)$ (which does not vanish at tree level)

F^{lat} is some lattice discretization of the field-strength tensor

Simulations

The renormalized action for Karsten-Wilczek fermions reads

$$S^f_{KW} \;=\; a^4 \sum_x \Bigg\{ \frac{1}{2a} \sum_{\mu=1}^{4} \Big[\overline{\psi}(x)\left(\gamma_\mu\left(1 + c_4(\beta)\,\delta_{\mu 4}\right) - i\gamma_4\left(1 - \delta_{\mu 4}\right)\right) U_\mu(x)\,\psi(x + a\widehat{\mu})$$

$$-\overline{\psi}(x + a\widehat{\mu})\left(\gamma_\mu\left(1 - d_4(\beta)\,\delta_{\mu 4}\right) + i\gamma_4\left(1 - \delta_{\mu 4}\right)\right) U^\dagger_\mu(x)\,\psi(x)\Big]$$

$$+\overline{\psi}(x)\left(m_0 + \widetilde{d}_3(\beta)\,\frac{i\,\gamma_4}{a}\right)\psi(x)$$

$$+\beta \sum_{\mu<\nu} \left(1 - \frac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\,P_{\mu\nu}\right)\left(1 + d_P(\beta)\,\delta_{\mu 4}\right)\Bigg\}$$

where $\widetilde{d}_3(\beta) = 3 + d_3(\beta)$ has a non-zero value at tree level

In perturbation theory the coefficients of the counterterms have the expansions

$$\begin{aligned}
\widetilde{c}_3(g_0^2) &= -2 + c_3^{(1)} g_0^2 + c_3^{(2)} g_0^4 + \ldots; & \widetilde{d}_3(g_0^2) &= 3 + d_3^{(1)} g_0^2 + d_3^{(2)} g_0^4 + \ldots \\
c_4(g_0^2) &= c_4^{(1)} g_0^2 + c_4^{(2)} g_0^4 + \ldots; & d_4(g_0^2) &= d_4^{(1)} g_0^2 + d_4^{(2)} g_0^4 + \ldots \\
c_P(g_0^2) &= c_P^{(1)} g_0^2 + c_P^{(2)} g_0^4 + \ldots; & d_P(g_0^2) &= d_P^{(1)} g_0^2 + d_P^{(2)} g_0^4 + \ldots
\end{aligned}$$

Simulations

In perturbation theory the four-dimensional counterterm to the fermionic action is necessary for the proper construction of the conserved currents

Its coefficient, as determined from the one-loop self-energy, has exactly the right value for which the conserved currents remain unrenormalized

One possible nonperturbative determination of c_4 (and d_4): require that the electric charge is one, using the (unrenormalized) conserved currents

Another effect of radiative corrections is to move the poles of the quark propagator away from their tree-level positions

It is the task of the dimension-3 counterterm, for the appropriate value of the coefficient c_3 (or d_3), to bring the two poles back to their original locations

An important point to remark here: only when the coefficients of $\overline{\psi}\Gamma\psi$ or $\overline{\psi}\gamma_4\psi$ are respectively $-2i/a$ or $3i/a$, are these actions real

Deviations from these values cause the appearance of oscillations due to the imaginary terms in hadronic correlators – for instance in 2-point functions

One possible way to determine the coefficients of the dimension-3 counterterms would then be to tune them in appropriate correlation functions until these oscillations are removed

Simulations

The purely gluonic counterterm for Boriçi-Creutz fermions introduces in the renormalized action operators of the kind $E \cdot B$, $E_1 E_2$, $B_2 B_3$ *(and similar)*

In a Lorentz invariant theory, instead, only the terms E^2 and B^2 are allowed

Fixing the coefficient c_P could then be done by measuring $\langle E \cdot B \rangle$, $\langle E_1 E_2 \rangle$, $\cdots$, and tuning c_P in such a way that one (or more) of these expectation values is restored to its proper value pertinent to a Lorentz invariant theory, i.e. zero

These effects could turn out to be rather small, given that in the tree-level action only the fermionic part breaks the hypercubic symmetry

It could also be that other derived quantities are more sensitive to this coefficient, and more suitable for its nonperturbative determination

In general one can look for Ward identities in which violations of the standard Lorentz invariant form, as functions of c_P, occur

For Karsten-Wilczek fermions the purely gluonic counterterm introduces an asymmetry between the plaquettes with a temporal index and the other ones

One can then fix d_P by computing a plaquette or Wilson loop lying entirely in two spatial directions, and then equating its result to an ordinary plaquette or Wilson loop which also extends in the time direction

Simulations

At the end only Monte Carlo simulations can reveal the actual amount of symmetry breaking

This could occur to be large or small according to the observable considered

One important such quantity is the mass splitting of the charged pions relative to the neutral pion

One must be a bit careful : there is only a $U(1) \otimes U(1)$ chiral symmetry

Consequence: π^0 is massless, as the unique Goldstone boson (for $m_0 \longrightarrow 0$), but π^+ and π^- are massive

Furthermore, the magnitude of these symmetry-breaking effects could turn out to be substantially different for Boriçi-Creutz compared to Karsten-Wilczek fermions

Thus, one of these two actions could in this way be raised to become the preferred one for numerical simulations

On the improvement

(tree level)

$$D^f_{\mathrm{Wilson}} = \frac{1}{2}\left\{\sum_{\mu=1}^{4}\gamma_\mu(\nabla_\mu+\nabla^*_\mu) - ar\sum_{\mu=1}^{4}\nabla^*_\mu\nabla_\mu\right\}$$

$$D^f_{\mathrm{BC}} = \frac{1}{2}\left\{\sum_{\mu=1}^{4}\gamma_\mu(\nabla_\mu+\nabla^*_\mu) + ia\sum_{\mu=1}^{4}\gamma'_\mu\,\nabla^*_\mu\nabla_\mu\right\}$$

$$D^f_{\mathrm{KW}} = \frac{1}{2}\left\{\sum_{\mu=1}^{4}\gamma_\mu(\nabla_\mu+\nabla^*_\mu) - ia\gamma_4\sum_{k=1}^{3}\nabla^*_k\nabla_k\right\}$$

where $\nabla_\mu\psi(x) = \frac{1}{a}\left[U_\mu(x)\,\psi\,(x+a\widehat{\mu}) - \psi(x)\right]$ is the nearest-neighbor forward covariant derivative, and ∇^*_μ the corresponding backward one

All these three formulations contain a dimension-5 operator in the bare action $\rightarrow$ we expect leading lattice artefacts to be of order a

Additional dimension-5 operators occur not only in the quark sector (*e.g.*, $\overline{\psi}\,\Gamma\sum_{\mu\nu}D_\mu D_\nu\psi$), but also in the pure gauge part (*e.g.*, $\sum_{\mu\nu\lambda}F_{\mu\nu}D_\lambda F_{\mu\nu}$)

When Lorentz invariance is broken, the statement that only operators with even dimension can appear in the pure gauge action is no longer true

Conclusions

- Boriçi-Creutz and Karsten-Wilczek fermions are described at one loop by a fully consistent quantum field theory
- Counterterms need to be added to the bare actions
- After these subtractions are consistently taken into account, the power divergence in the self-energy is eliminated
- No other power divergences occur for all quantities that we have calculated: *self-energy, Dirac bilinears and vacuum polarization*
- Scalar, pseudoscalar and tensor operators show no new mixings at all
- Local vector and axial currents mix with new operators which are not invariant under the hypercubic group
- The vacuum polarization does not present new divergences
- Leading lattice artefacts seem to be of order a

- Conserved vector and axial currents can be defined, and they involve only nearest-neighbors points
  - they do not present mixings, and their renormalization constant is one
  - the only case (apart from staggered fermions) where one can define a simple conserved axial current (also ultralocal)