Chiral symmetry breaking and the Banks–Casher relation on the lattice

Leonardo Giusti

CERN Theory Group and University of Milano-Bicocca

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Outline

- Introduction
- Banks-Casher relation
- Renormalization of the spectral density
- Spectral density in ChPT
- Computation of the mode number
- Exploratory numerical study
- Conclusions and outlook

- QCD action for $N_f = 3$, $M = \text{diag}(m_u, m_d, m_s)$

$$S = S_G + \int d^4x \left\{ \bar{\psi} D \psi + \bar{\psi} M \psi \right\} , \qquad D = \gamma_\mu (\partial_\mu + i A_\mu)$$

- For $M = 0$ chiral symmetry

$$\psi_{\rm R,L} \to V_{\rm R,L} \psi_{\rm R,L} \quad \psi_{\rm R,L} = \left(\frac{1 \pm \gamma_5}{2} \right) \psi$$

Chiral anomaly: measure not invariant

SSB: vacuum not symmetric

- Gauge symmetry

$$\psi(x) \to G(x) \psi(x)$$

Confinement: no isolated coloured charge

$$\mathrm{SU}(3)_c \times \mathrm{SU}(3)_{\rm L} \times \mathrm{SU}(3)_{\rm R} \times \mathrm{U}(1)_{\rm L} \times \mathrm{U}(1)_{\rm R} \times \mathcal{R}_{\rm scale}$$

$\downarrow$ (dim. transm., chiral anomaly)

$$\mathrm{SU}(3)_c \times \mathrm{SU}(3)_{\rm L} \times \mathrm{SU}(3)_{\rm R} \times \mathrm{U}(1)_{\rm B=L+R}$$

$\downarrow$ (Spont. Sym. Break.)

$$\mathrm{SU}(3)_c \times \mathrm{SU}(3)_{\rm L+R} \times \mathrm{U}(1)_{\rm B}$$

$\downarrow$ (Confinement)

$$\mathrm{SU}(3)_{\rm L+R} \times \mathrm{U}(1)_{\rm B}$$

QCD action and its (broken) symmetries [Fritzsch, Gell-Mann, Leutwyler 73; Gross, Wilczek 73; Weinberg 73]

- QCD action for $N_f = 3$, $M = \text{diag}(m_u, m_d, m_s)$

$$S = S_G + \int d^4x \left\{ \bar\psi D\psi + \bar\psi M\psi \right\}, \qquad D = \gamma_\mu(\partial_\mu + iA_\mu)$$

$$\mathrm{SU(3)}_c\times \mathrm{SU(3)}_\mathrm{L} \times \mathrm{SU(3)}_\mathrm{R} \times \mathrm{U(1)}_\mathrm{L} \times \mathrm{U(1)}_\mathrm{R} \times \mathcal{R}_\mathrm{scale}$$

$\downarrow$ (dim. transm., **chiral anomaly**)

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$\downarrow$ **(Spont. Sym. Break.)**

$$\mathrm{SU(3)}_c\times \mathrm{SU(3)}_\mathrm{L+R} \times \mathrm{U(1)}_\mathrm{B}$$

$\downarrow$ (Confinement)

$$\mathrm{SU(3)}_\mathrm{L+R} \times \mathrm{U(1)}_\mathrm{B}$$

- Confinement and SSB due to non-perturbative dynamics

- The mechanisms are still not know

- Today focus on SSB

- An axial Ward identity of the chiral group is [for simplicity $M = \text{diag}(m, m, m)$]

$$\langle \bar{\psi}_1 \psi_1 \rangle = m \int d^4x \, \langle P_{12}(x) P_{21}(0) \rangle \, , \qquad P_{ij} = \bar{\psi}_i \gamma_5 \psi_j$$

- In the limit $m \to 0$

$$\Sigma = - \lim_{m \to 0} \langle \bar{\psi}_1 \psi_1 \rangle \neq 0 \qquad \Longrightarrow \qquad M^2 = \frac{2m\Sigma}{F^2}$$

[Gell-Mann, Oakes, Renner 68]

where the decay constant is defined as

$$|\langle 0 | \hat{A}_{12,\mu} | \pi^-, p \rangle| = \sqrt{2} \, F_\pi \, p_\mu \, , \qquad F = \lim_{m \to 0} F_\pi$$

- Octet compatible with SSB pattern

$$\mathrm{SU}(3)_{\mathrm{L}} \times \mathrm{SU}(3)_{\mathrm{R}} \to \mathrm{SU}(3)_{\mathrm{L+R}}$$

and soft explicit symmetry breaking

$$m_u, m_d \ll m_s < \Lambda$$

- $m_u, m_d \ll m_s \Longrightarrow m_\pi \ll m_{\mathrm{K}}$

- A 9^{th} pseudoscalar with $m_{\eta'} \sim \mathcal{O}(\Lambda)$

I	I_3	S	Meson	Quark Content	Mass (GeV)
1	1	0	π^+	$u\bar{d}$	0.140
1	-1	0	π^-	$d\bar{u}$	0.140
1	0	0	π^0	$(d\bar{d} - u\bar{u})/\sqrt{2}$	0.135
$\frac{1}{2}$	$\frac{1}{2}$	+1	K^+	$u\bar{s}$	0.494
$\frac{1}{2}$	-$\frac{1}{2}$	+1	K^0	$d\bar{s}$	0.498
$\frac{1}{2}$	-$\frac{1}{2}$	-1	K^-	$s\bar{u}$	0.494
$\frac{1}{2}$	$\frac{1}{2}$	-1	$\overline{\mathrm{K}}^0$	$s\bar{d}$	0.498
0	0	0	η	$\cos\vartheta\eta_8 - \sin\vartheta\eta_0$	0.548
0	0	0	η'	$\sin\vartheta\eta_8 + \cos\vartheta\eta_0$	0.958

$$\eta_8 = (d\bar{d} + u\bar{u} - 2s\bar{s})/\sqrt{6}$$
$$\eta_0 = (d\bar{d} + u\bar{u} + s\bar{s})/\sqrt{3}$$
$$\vartheta \sim -10^\circ$$

- Chiral effective theory for pions

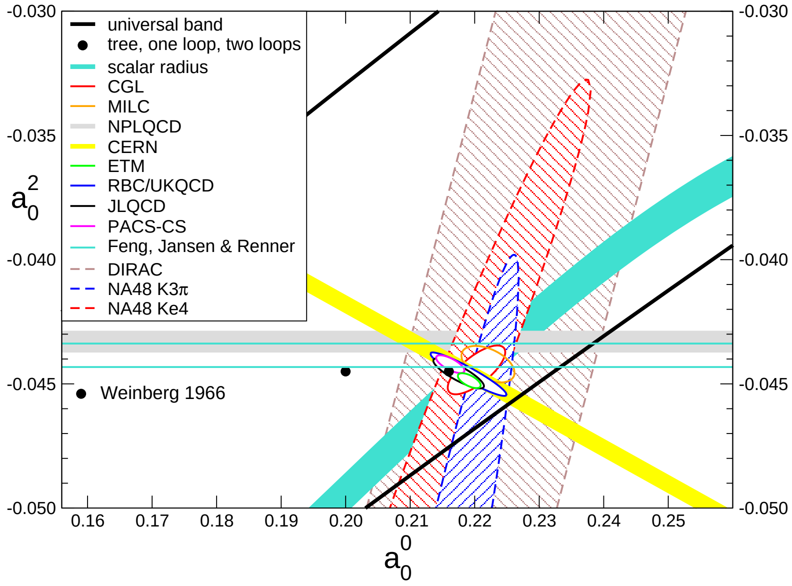
$$S_{\rm eff} = S^2_{\rm eff}(U; m, F, \Sigma) + S^4_{\rm eff}(U; m, F, \Sigma, \Lambda_i) + \cdots$$

encodes spontaneous symmetry breaking

- For $m = 0$ pions can interact only if they carry momentum. Expansion in p and m

- Chiral dynamics parameterized by effective low-energy coupling constants

[Colangelo, Gasser, Leutwyler 01; Leutwyler 09]



- For instance the pion mass and decay constant at $\mathcal{O}(p^4)$ are given by

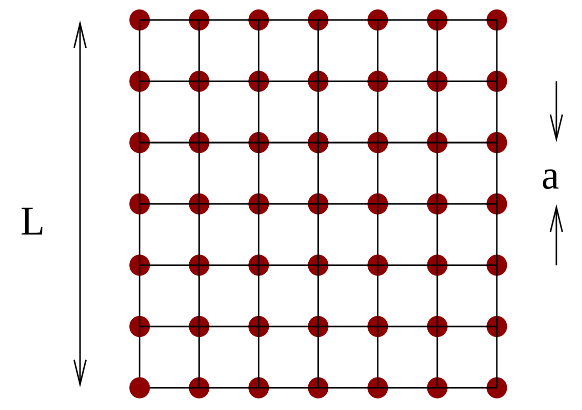
$$M_\pi^2 = M^2\left\{1 + \frac{M^2}{32\pi^2F^2}\ln\left(\frac{M^2}{\Lambda_3^2}\right)\right\}, \qquad F_\pi = F\left\{1 - \frac{M^2}{16\pi^2F^2}\ln\left(\frac{M^2}{\Lambda_4^2}\right)\right\}$$

Analogous expressions for other quantities such as S-wave $\pi\pi$ scattering lengths a_0^0 and a_0^2

Lattice QCD: action [Wilson 74]

- QCD can be defined on a discretized space-time so that gauge invariance is preserved

- Quark fields reside on a four-dimensional lattice, the gauge field $U_\mu \in \mathrm{SU}(3)$ resides on links

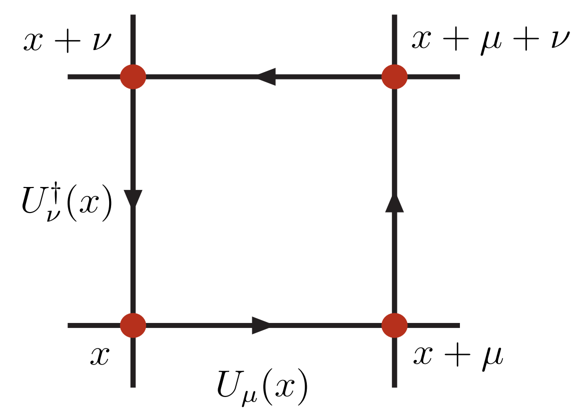


- The Wilson action for the gauge field is

$$S_G[U] = \frac{\beta}{2} \sum_x \sum_{\mu,\nu} \left[1 - \frac{1}{3}\mathrm{ReTr}\Big\{U_{\mu\nu}(x)\Big\}\right]$$

where $\beta = 6/g^2$ and the plaquette is defined as

$$U_{\mu\nu}(x) = U_\mu(x)\, U_\nu(x+\hat{\mu})\, U_\mu^\dagger(x+\hat{\nu})\, U_\nu^\dagger(x)$$



- Popular discretizations of fermion action: Wilson, Domain-Wall-Neuberger, perfect actions, tmQCD

Lattice QCD with two light dynamical quarks (I) [Del Debbio et al. 06-08]

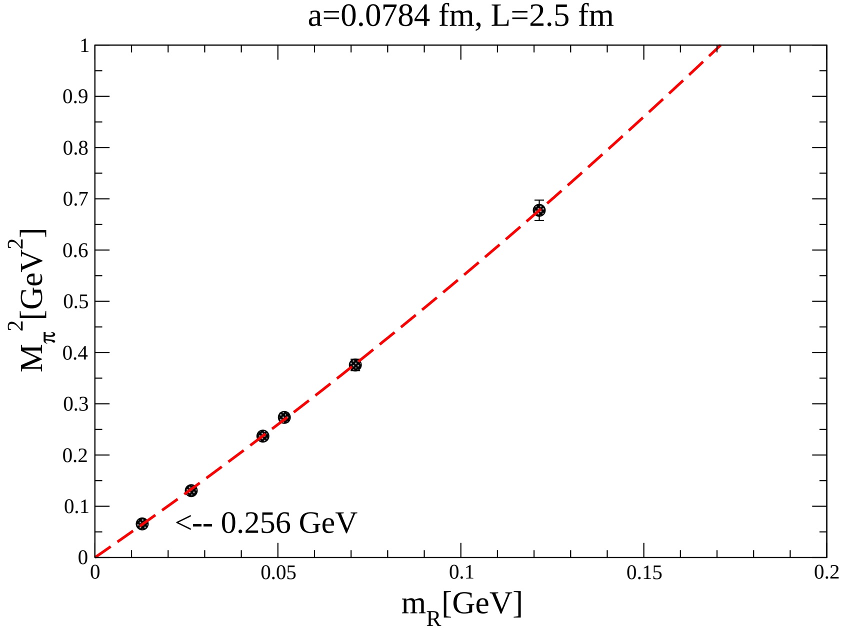
- Chiral regime is becoming accessible to lattice QCD simulations

- The pion mass squared is found to be a nearly linear function of quark mass up to $(0.5 \text{ GeV})^2$. At smallest masses non-linear correction is 1 - 3%

- Non-Abelian chiral symmetry spontaneously broken as expected

- Compatible with the fact that the bulk of the mass is given by the leading term in standard ChPT

- Relations dictated by SSB can be verified quantitatively. GMOR is maybe the simplest to start with



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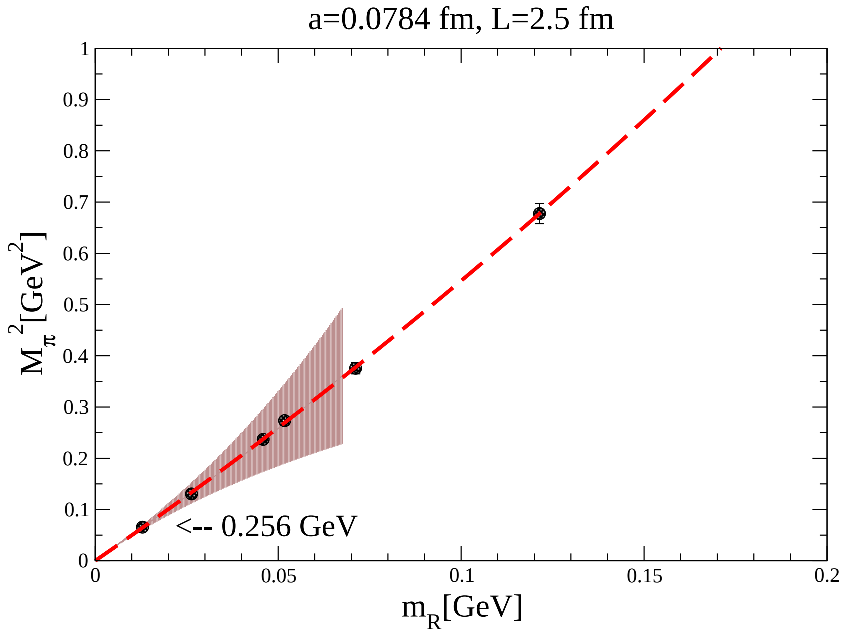
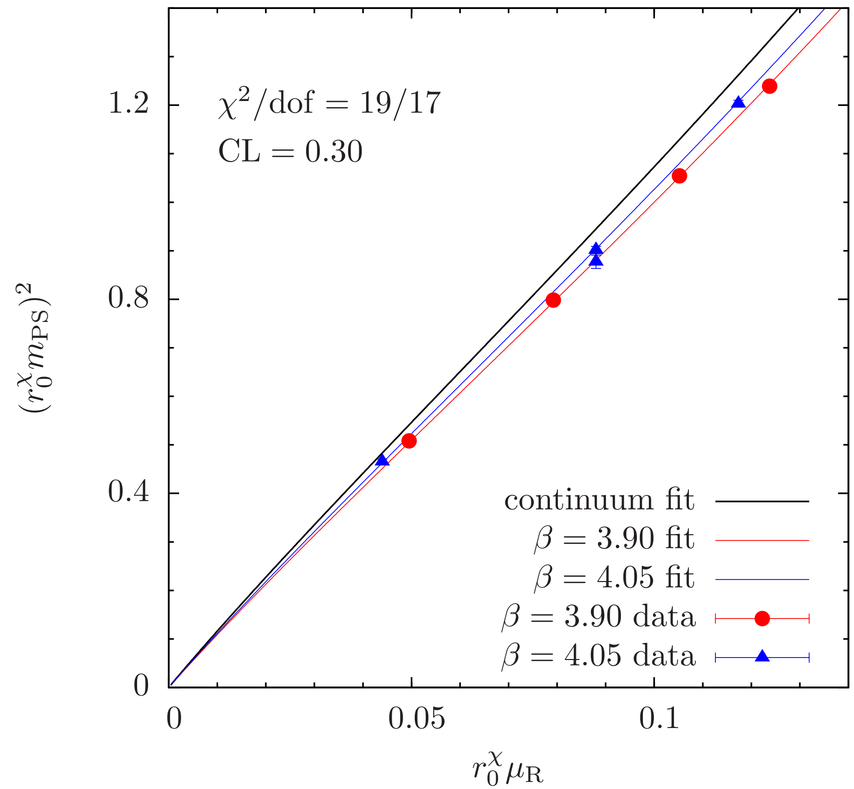


- Low-energy constants will finally be determined. As an example of the potentiality, from a fit to the curve

$0.47 \leq \Lambda_3 \leq 0.86$ GeV to be compared with $0.2 \leq \Lambda_3 \leq 2$ GeV [Gasser, Leutwyler 84]

Lattice QCD with two light dynamical quarks (II) [ETM Collaboration 07-09]



- A linear behaviour observed with different regularizations, different algorithms, etc.
- By assuming the standard ChPT formula, from an overall fit of the pion mass and decay constant the ETM collaboration obtains

$$\left[\Sigma_{\mathrm{R}}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV})\right]^{1/3}_{\mathrm{GMOR}} = 0.270(8)\ \mathrm{GeV} \qquad 0.69 \leq \Lambda_3 \leq 0.93\ \mathrm{GeV}$$

Banks–Casher relation [Banks, Casher 80]

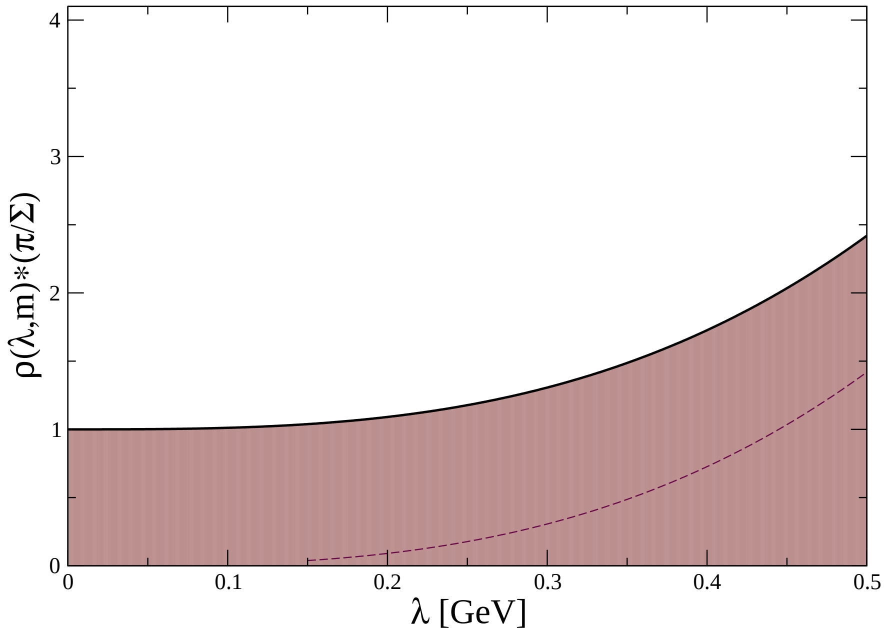
- For each gauge configuration

$$D_m \chi_k = (m + i\lambda_k)\chi_k$$

- The spectral density of D is

$$\rho(\lambda, m) = \frac{1}{V} \sum_k \langle \delta(\lambda - \lambda_k) \rangle$$

where $\langle \ldots \rangle$ indicates path-integral average



- The Banks–Casher relation

$$\lim_{\lambda \to 0} \lim_{m \to 0} \lim_{V \to \infty} \rho(\lambda, m) = \frac{\Sigma}{\pi}$$

provides a link between the condensate and the (non-zero) spectral density at the origin.
To be compared, for instance, with the free case $\rho(\lambda) \propto |\lambda^3|$

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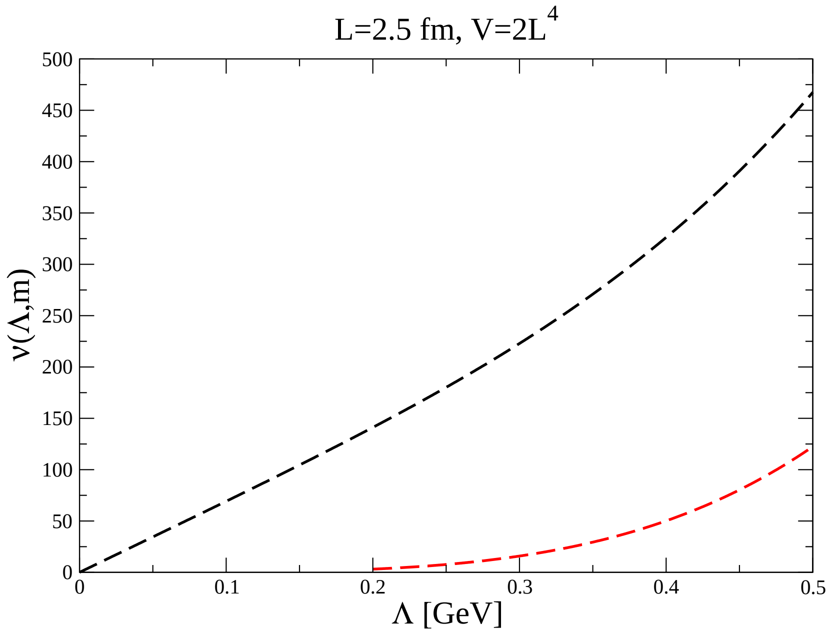

- The number of modes in a given energy interval

$$\nu(\Lambda, m) = V \int_{-\Lambda}^{\Lambda} d\lambda\ \rho(\lambda, m) \qquad \nu(\Lambda, m) = \frac{2}{\pi}\Lambda \Sigma V + \ldots$$

grows linearly with Λ, and they condense near the origin with values $\propto 1/V$

In the free case $\nu(\Lambda, m) \propto V\Lambda^4$

Renormalization and continuum limit [LG, Lüscher 09]

- Instead of the spectral density, consider the spectral sum

$$
\begin{aligned}
\sigma_k(m_v, m) &= V \int_{-\infty}^{\infty} d\lambda \frac{\rho(\lambda, m)}{(\lambda^2 + m_v^2)^k} \\
&= -a^{8k} \sum_{x_1 \ldots x_{2k}} \langle P_{12}(x_1) P_{23}(x_2) \ldots P_{2k1}(x_{2k}) \rangle
\end{aligned}
$$

  * Integral converges if $k \geq 3$

  * The relation between $\sigma_k(m_v, m)$ and $\rho(\lambda, m)$ invertible for every k

- Renormalization properties of $\rho(\lambda, m)$ can thus be inferred from those of σ_k

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$$= -a^{8k}\sum_{x_1\dots x_{2k}} \langle P_{12}(x_1)P_{23}(x_2)\dots P_{2k1}(x_{2k})\rangle$$

- Corr. functions of pseudoscalar densities at physical distance renormalized by $(1/Z_m)^{2k}$

- At short distance the flavour structure implies

$$P_{12}(x_1)P_{23}(x_2) \sim C(x_1-x_2)S_{13}(x_1) \qquad S_{13} = \bar\psi_1\psi_3$$

where $C(x)$ diverges like $|x|^{-3}$ and it is therefore integrable. Analogous argument for all other short-distance singularities. No extra contact terms needed to renormalize σ_k

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$$= -a^{8k} \sum_{x_1 \ldots x_{2k}} \langle P_{12}(x_1) P_{23}(x_2) \ldots P_{2k1}(x_{2k}) \rangle$$

- Once the gauge coupling and the mass(es) are renormalized, the spectral sum

$$\sigma_{k,\mathrm{R}}(m_{v\mathrm{R}}, m_{\mathrm{R}}) = Z_m^{-2k} \sigma_k \left(\frac{m_{v\mathrm{R}}}{Z_m}, \frac{m_{\mathrm{R}}}{Z_m} \right)$$

is ultraviolet finite. Continuum limit universal (if same renormalization conditions are used)

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$$= -a^{8k} \sum_{x_1 \ldots x_{2k}} \langle P_{12}(x_1) P_{23}(x_2) \ldots P_{2k1}(x_{2k}) \rangle$$

- The spectral density thus renormalizes as

$$\rho_{\rm R}(\lambda_{\rm R}, m_{\rm R}) = Z_m^{-1} \rho \left(\frac{\lambda_{\rm R}}{Z_m}, \frac{m_{\rm R}}{Z_m} \right)$$

Renormalization with $O(a)$-improved Wilson fermions [L.G., Lüscher 08]

- If we introduce k doublets of twisted-mass valence quarks with Dirac operator

$$D_{\rm tm} = D_m + i\mu\gamma_5\tau^3\ , \qquad P^{\pm}_{ij} = \bar\psi_i\gamma_5\tau^{\pm}\psi_j$$

the spectral sum is given by

$$\sigma_k(\sqrt{m^2+\mu^2}, m) = -a^{8k}\sum_{x_1,\dots,x_{2k}}\left\langle P^+_{12}(x_1)P^-_{23}(x_2)\dots P^+_{(2k-1)(2k)}(x_{2k-1})P^-_{(2k)1}(x_{2k})\right\rangle$$

thanks to the fact that for tm quarks the (modulo) square of the quark propagator is

$$(D_m^\dagger D_m + \mu^2)^{-1}$$

- As before the short-distance singularities are integrable ($x_1 \to x_2$)

$$P^+_{12}(x_1)P^-_{23}(x_2) \sim \mathcal{C}(x_1-x_2)S^{\pm}_{13}(x_1)\ , \qquad S^{\pm}_{13} = \bar\psi_1\tau^+\tau^-\psi_3$$

since by power counting $\mathcal{C}(x)$ diverges like $|x|^{-3}$

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thanks to the fact that for tm quarks the (modulo) square of the quark propagator is

$$(D_m^\dagger D_m + \mu^2)^{-1}$$

- In this case the spectral density is the one associated to

$$D_m^\dagger D_m \chi_k = \alpha_k \chi_k$$

and once the continuum limit is taken we can interpret $\alpha = \lambda^2 + m^2$

Extracting Σ from the spectral density

- The dynamical properties of the theory are encoded in $\rho(\lambda, m)$
- The power divergences in the chiral condensate

$$\Sigma(m_v, m) = \int_{-\infty}^{\infty} d\lambda \, \frac{\rho(\lambda, m)}{i\lambda + m_v} = 2m_v \int_0^{\infty} d\lambda \, \frac{\rho(\lambda, m)}{\lambda^2 + m_v^2}$$

do not originate from the spectral density itself but from the particular integral taken

- Note that if $\rho(\lambda, m)$ is extracted from

$$2\pi\rho(\lambda, m) = \lim_{\epsilon \to 0} \Big[\Sigma(i\lambda + \epsilon, m) + \Sigma(-i\lambda + \epsilon, m) \Big]$$

the divergences cancel out on the r.h.s.

- By choosing a different probe function, it is possible to extract Σ from integrals of the spectral density which are not plagued by power divergences [L.G., S. Necco 07]

Spectral projectors: the mode number

- The average number of eigenstates of D with $|\lambda| < \Lambda$

$$\nu(\Lambda, m) = V \int_{-\Lambda}^{\Lambda} d\lambda \, \rho(\lambda, m)$$

is maybe the simplest integral to consider

- From the renormalization pattern of the spectral density, it follows that the mode number is a renormalization-group invariant

$$\nu_{\mathrm{R}}(\Lambda_{\mathrm{R}}, m_{\mathrm{R}}) = \nu(\Lambda, m)$$

and its continuum limit is universal for any value of Λ and m

- $O(a)$ improvement (almost) automatic

Spectral density in the chiral effective theory (I) [Smilga, Stern 93; Osborn et al. 99; L.G., Lüscher 08]

- Standard chiral effective theory supplemented with a valence quark

$$SU(2)_L \otimes SU(2)_R \to SU(2)_V \quad \Longrightarrow \quad SU(3|1)_L \otimes SU(3|1)_R \to SU(3|1)_V$$

- The unitary fields are given by

$$U = \exp\left\{\frac{2i}{F}\Phi\right\}\,, \qquad\qquad \Phi = \sum_a \phi^a T^a$$

and the leading order Lagrangian is [$M = \text{diag}(m, m, m_v, m_v)$]

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\left\{\text{Str}\,\left[\partial_\mu U^\dagger \partial_\mu U\right] - \frac{2\Sigma}{F^2}\,\text{Str}\,\left[MU^\dagger + M^\dagger U\right]\right\}$$

- At the NLO the relevant term is given by

$$\mathcal{L}^{(4)} = -\frac{4\Sigma^2}{F^4} L_6\,\text{Str}\,\left[U^\dagger M + M^\dagger U\right]\text{Str}\,\left[U^\dagger M + M^\dagger U\right] + \ldots$$

Spectral density in the chiral effective theory (II) [Smilga, Stern 93; Osborn et al. 99; L.G., Lüscher 08]

- If the condensate is analytically continued in the valence quark mass, then the spectral density can be computed as

$$2\pi\rho(\lambda,m) = \lim_{\epsilon\to 0}\left[\Sigma^{\rm ChPT}(i\lambda+\epsilon,m)+\Sigma^{\rm ChPT}(-i\lambda+\epsilon,m)\right]$$

- In the infinite volume limit an at the NLO in ChPT

$$\nu^{\rm nlo}(\Lambda,m) = \frac{2\Lambda\Sigma V}{\pi}\left\{1-\frac{m\,\Sigma}{(4\pi)^2F^4}\left[3\ln\left(\frac{\Lambda\Sigma}{F^2\Lambda_6^2}\right)+\ln(2)+\frac{\pi}{2}\frac{m}{\Lambda}+O\left(\frac{m^2}{\Lambda^2}\right)\right]\right\}$$

- NLO corrections to leading behaviour (vanishing for $m\to 0$) expected rather small for $\Lambda = 0.05$–0.1 GeV and $m \leq 0.02$ GeV [$\overline{\rm MS}$ @ 2 GeV]. No chiral logs $\propto m\ln(m)$

- Finite volume effects: a fraction of a percent at NLO ChPT in the p-regime

- When $\Lambda\Sigma V$ is not large, threshold effects can be sizeable. They can be quantified in ChPT

Mode number: numerical algorithm

- The quantity $\nu(\Lambda, m)$ can be computed as

$$\nu(\Lambda, m) = \langle \mathcal{O} \rangle \;, \qquad \mathcal{O}[U] = \mathrm{Tr} P_M$$

$$P_M = \theta(M^2 - D_m^\dagger D_m) \;, \qquad M^2 = \Lambda^2 + m^2$$

- To avoid the (unnecessary) computation of $O(V)$ eigenvalues

$$\nu(\Lambda, m) = \langle \hat{\mathcal{O}} \rangle \;, \qquad \hat{\mathcal{O}}[U, \eta] = (\eta, P_M \eta)$$

  where η are Gaussian random sources

- The cost scales with V rather than V^2, and the relative stat. error scales like $V^{-1/2}$.
  Infinite volume and continuum limit extrapolation feasible

Coordinated Lattice Simulations (CLS) [https://twiki.cern.ch/twiki/bin/view/CLS/WebHome]

Several European groups coordinate their efforts to generate configuration ensembles with $N_{\rm f} = 2$ dynamical quarks degenerate in mass

Gluon action: Wilson
Quark action: $O(a)$-improved Wilson
Algorithm : DD-HMC

β	V	a[fm]	L[fm]	Masses	Status
5.3	48×24^3	0.08	1.9	6	Completed
5.5	64×32^3	0.06	1.9	6	Completed
5.7	96×48^3	0.04	1.9	3	
5.3	64×32^3	0.08	2.5	6	Completed
5.5	96×48^3	0.06	2.5	4	
5.7	128×64^3	0.04	2.5		
5.3	96×48^3	0.08	3.8		
5.5	128×64^3	0.06	3.8		
...	...	...	...	...	...

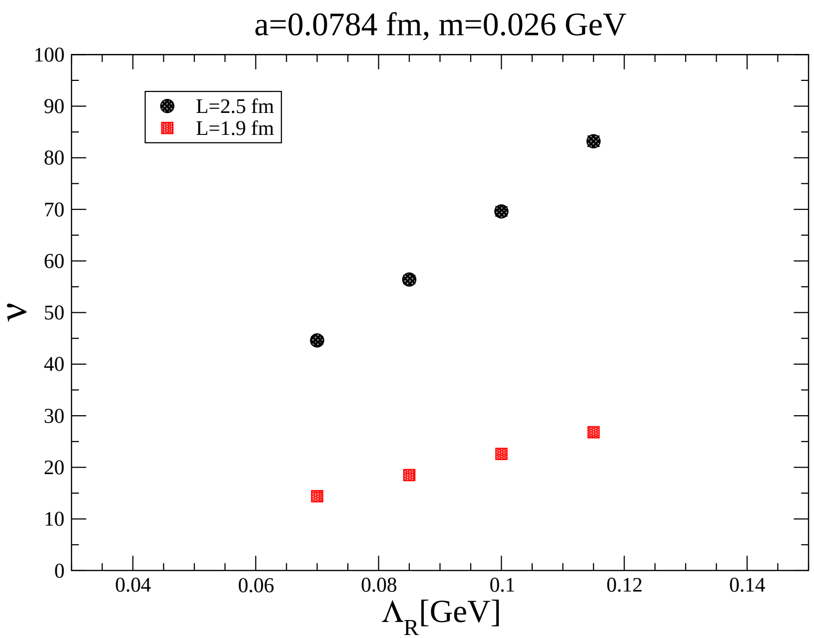
Groups:
Berlin (U. Wolff)
CERN (L. G., M. Lüscher)
DESY-Zeuthen (R. Sommer)
Madrid (C. Pena)
Mainz (H. Wittig)
Rome (R. Petronzio)
Valencia (P. Hernández)

Physics:
Spontaneous symmetry breaking
Fundamental parameters
Light-light mesons
Heavy-light mesons
Baryons
Weak matrix elements
.....

Numerical computation (I) [LG, Lüscher 09]

- Lattice details:

  * $N_f = 2$ degenerate quarks
  * Action: $O(a)$-improved Wilson
  * $a = 0.0784$ fm
  * $V = 2L \times L^3$, $L = 1.9, 2.5$ fm
  * $m_R^{\overline{\rm MS}}(2\,{\rm GeV}) = 0.013, 0.026, 0.046$ GeV
  * $\Lambda_R^{\overline{\rm MS}}(2\,{\rm GeV}) = 0.07, 0.085, 0.1, 0.115$ GeV



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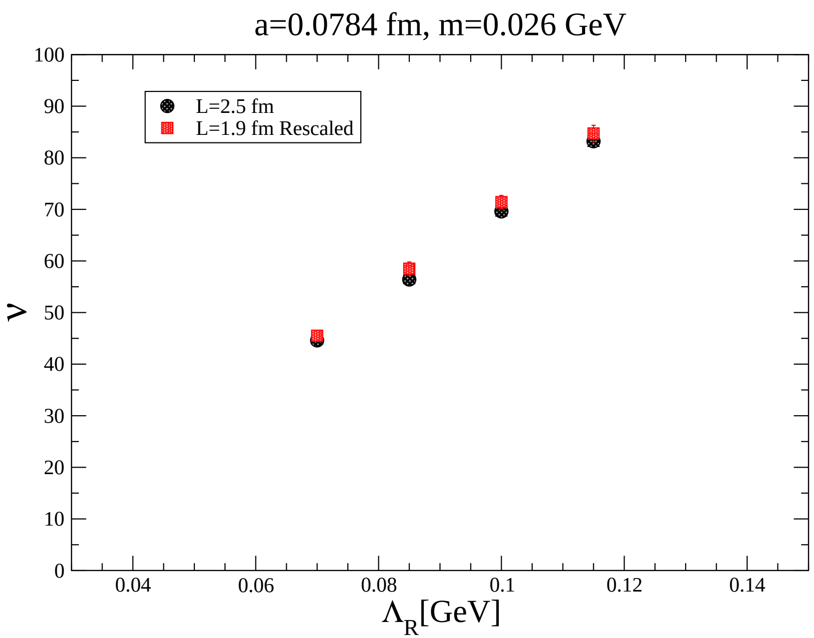


- Finite volume effects below stat. errors (1.5%). ChPT suggests a fraction of a percent

- A nearly linear function up to 0.1 Gev. Qualitative in line with ChPT, but the fact that the linear behaviour extends to such large values of $\Lambda_{\rm R}$ is rather striking and unexpected

- An effective condensate can be defined as

$$\overline{\Sigma}_{\rm R} = \frac{\pi}{2V} \frac{\partial}{\partial \Lambda_{\rm R}} \nu_{\rm R}(\Lambda_{\rm R}, m_{\rm R})$$

prefactor so that $\overline{\Sigma}_{\rm R}$ coincides with Σ at LO

$\Lambda_{\rm R}$	$m_{\rm R}$	ν_R	$\overline{\Sigma}_{\rm R}^{1/3}$
70	12.9(1)(5)	43.8(8)	266(2)(5)
	26.5(1)(10)	44.6(7)	267(1)(5)
	45.9(1)(18)	48.9(9)	276(2)(5)
85	12.9(1)(5)	54.9(9)	269(1)(5)
	26.5(1)(10)	56.4(8)	271(1)(5)
	45.9(1)(18)	63.4(10)	282(1)(5)
100	12.9(1)(5)	66.6(10)	271(1)(5)
	26.5(1)(10)	69.6(9)	275(1)(5)
	45.9(1)(18)	78.2(12)	286(1)(5)
115	12.9(1)(5)	78.7(11)	274(1)(5)
	26.5(1)(10)	83.2(10)	279(1)(5)
	45.9(1)(18)	93.9(12)	291(1)(5)

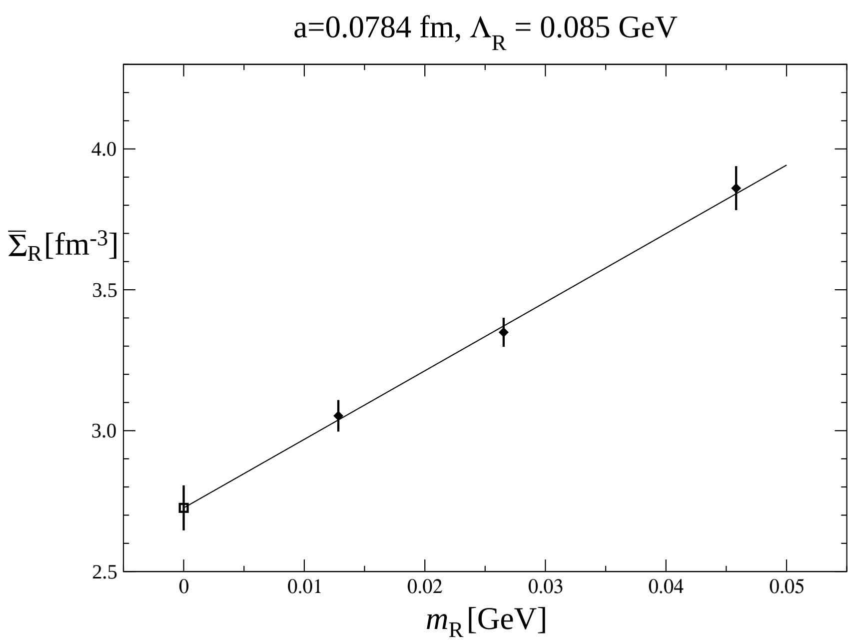
- An effective condensate can be defined as

$$\overline{\Sigma}_{\rm R} = \frac{\pi}{2V}\frac{\partial}{\partial \Lambda_{\rm R}}\nu_{\rm R}(\Lambda_{\rm R}, m_{\rm R})$$

prefactor so that $\overline{\Sigma}_{\rm R}$ coincides with Σ at LO

- A linear extrapolation to the chiral limit yields

$$\left[\Sigma_{\rm R}^{\overline{\rm MS}}(2\,{\rm GeV})\right]^{1/3} = 0.276(3)(4)(5)\ \text{GeV}$$



- To be compared, for instance, with the value extracted from the GMOR by the ETMC

$$\left[\Sigma_{\rm R}^{\overline{\rm MS}}(2\,{\rm GeV})\right]^{1/3} = 0.270(8)\ {\rm GeV}$$ [R. Baron et al. ETM Coll. 09]

or from fixed topology simulations by JLQCD

$$\left[\Sigma_{\rm R}^{\overline{\rm MS}}(2\,{\rm GeV})\right]^{1/3} = 0.243(4)(0)\ \text{GeV}$$ [Fukaya et al. 09]

Numerical computation (II) [LG, Lüscher 09]

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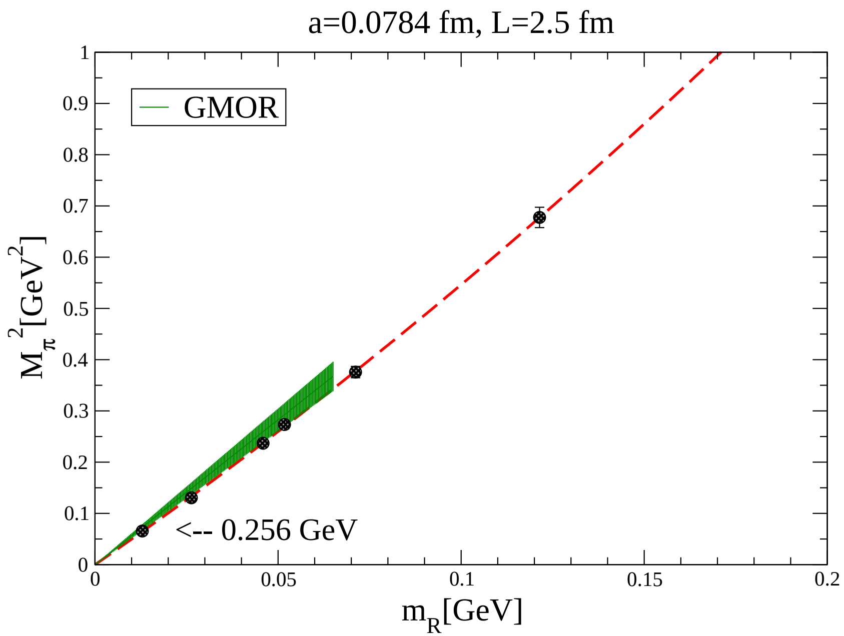


- A clear and consistent picture is emerging. For $m_R \leq 0.05$ GeV the GMOR formula accounts for the bulk of the pion mass. But discretization errors not quantified yet

Why is the symmetry spontaneously broken?

- Dynamical process not yet known. Studies of low modes can provide important clues

- The Banks–Casher mechanism is:

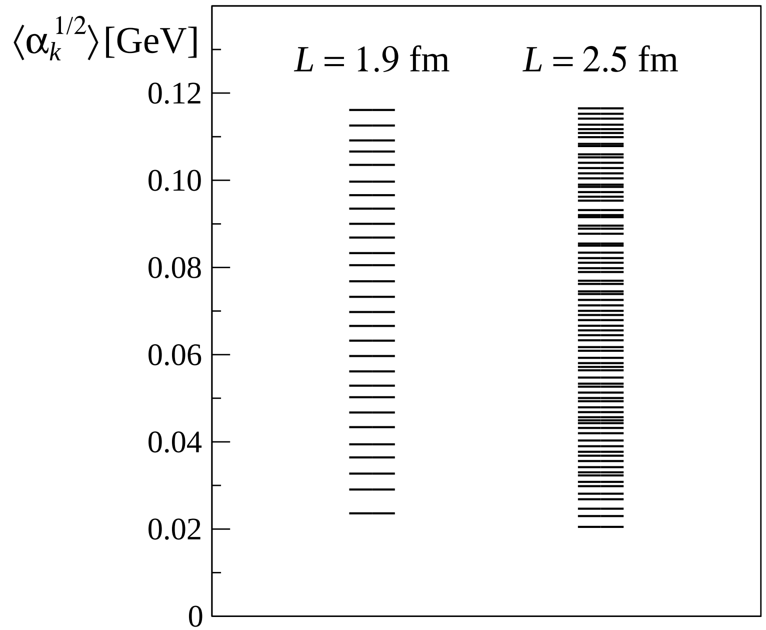
  ∗ insensitive to lattice details (universality)

  ∗ largely insensitive to dynamical quark effects

  ∗ present also in quenched QCD

- It is tempting to read the relation in the other direction, i.e. chiral symmetry is broken because the low-modes of the Dirac operator condense

$$\frac{\Sigma}{\pi} = \lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty} \rho(\lambda, m)$$



Eigenvalues of $D_m^\dagger D_m$

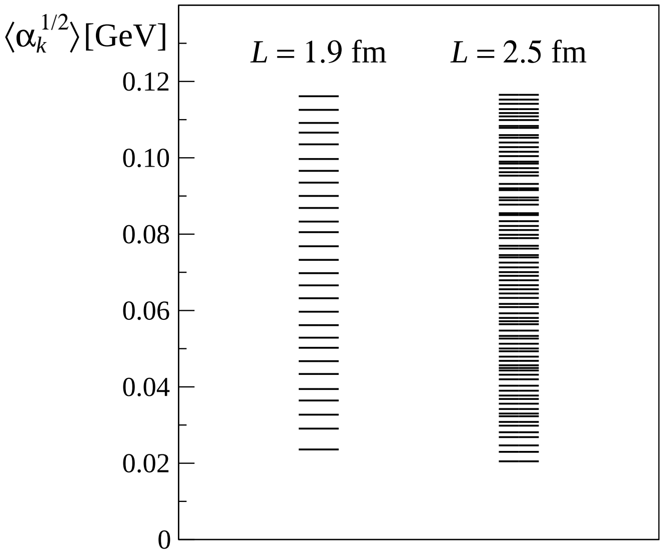
Conclusions

- Thanks to the recent extraordinary conceptual, technical and algorithmic advances the chiral regime of (lattice) QCD is becoming accessible to numerical simulations

- The Banks-Casher relation offers an extremely powerful tool to study the SSB of chiral symmetry, thanks to the renormalizability and universality of the spectral density

- Condensation of low-modes of the Dirac operator most direct piece of theoretical evidence for SSB

- The rate of condensation explains the bulk of the pion mass up to 0.5 GeV



- The moderate computational cost allows for the infinite volume and continuum limits

- More data needed to estimate the systematics on Σ with confidence

Conclusions

- Thanks to the recent extraordinary conceptual, technical and algorithmic advances the chiral regime of (lattice) QCD is becoming accessible to numerical simulations

- Spectral projectors open a new perspective to study the chiral regime of QCD:

  * Chiral condensate without power divergences

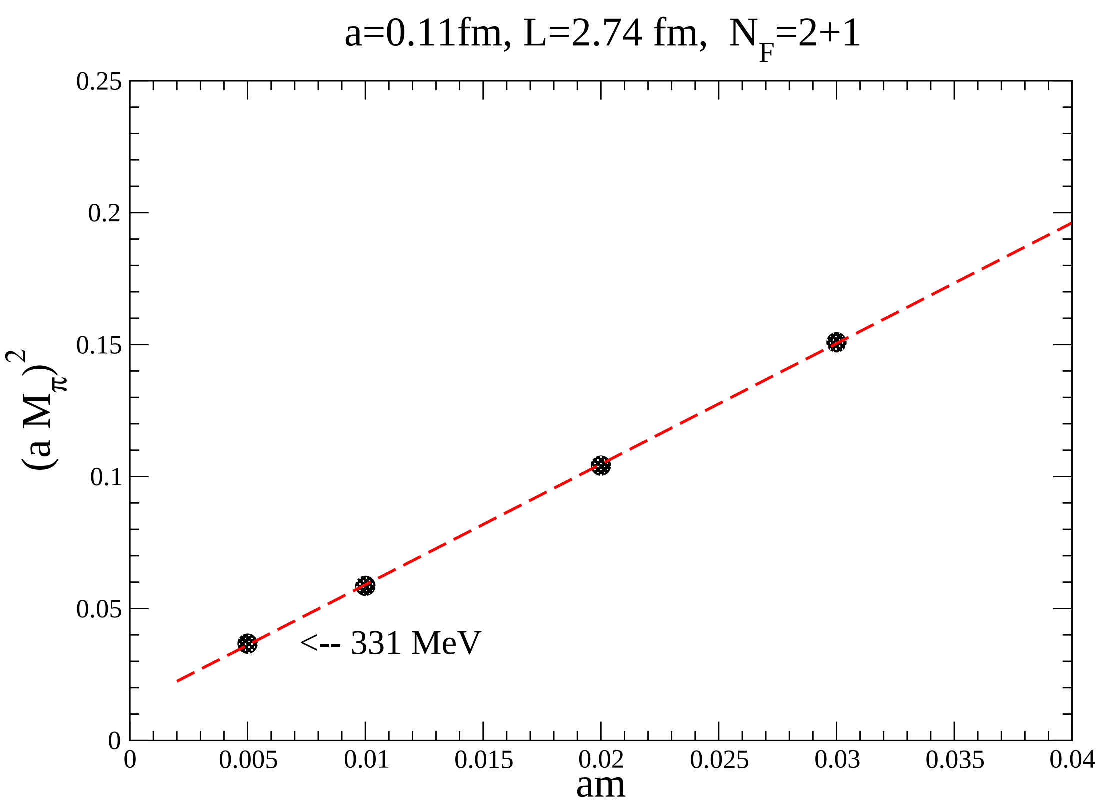
  * Topological susceptibility without power divergences

  * Ward Identities . . .

- The moderate computational cost allows for the infinite volume and continuum limits

- Application to finite temperature or other QCD-like theories straightforward

Pion Mass with Domain-Wall fermions ($N_F = 2 + 1$) [C. Allton et al. RBC and UKQCD Coll. 07-08]



- ... and striking linearity observed in this case too
- .....