Heavy quark lattice calculations

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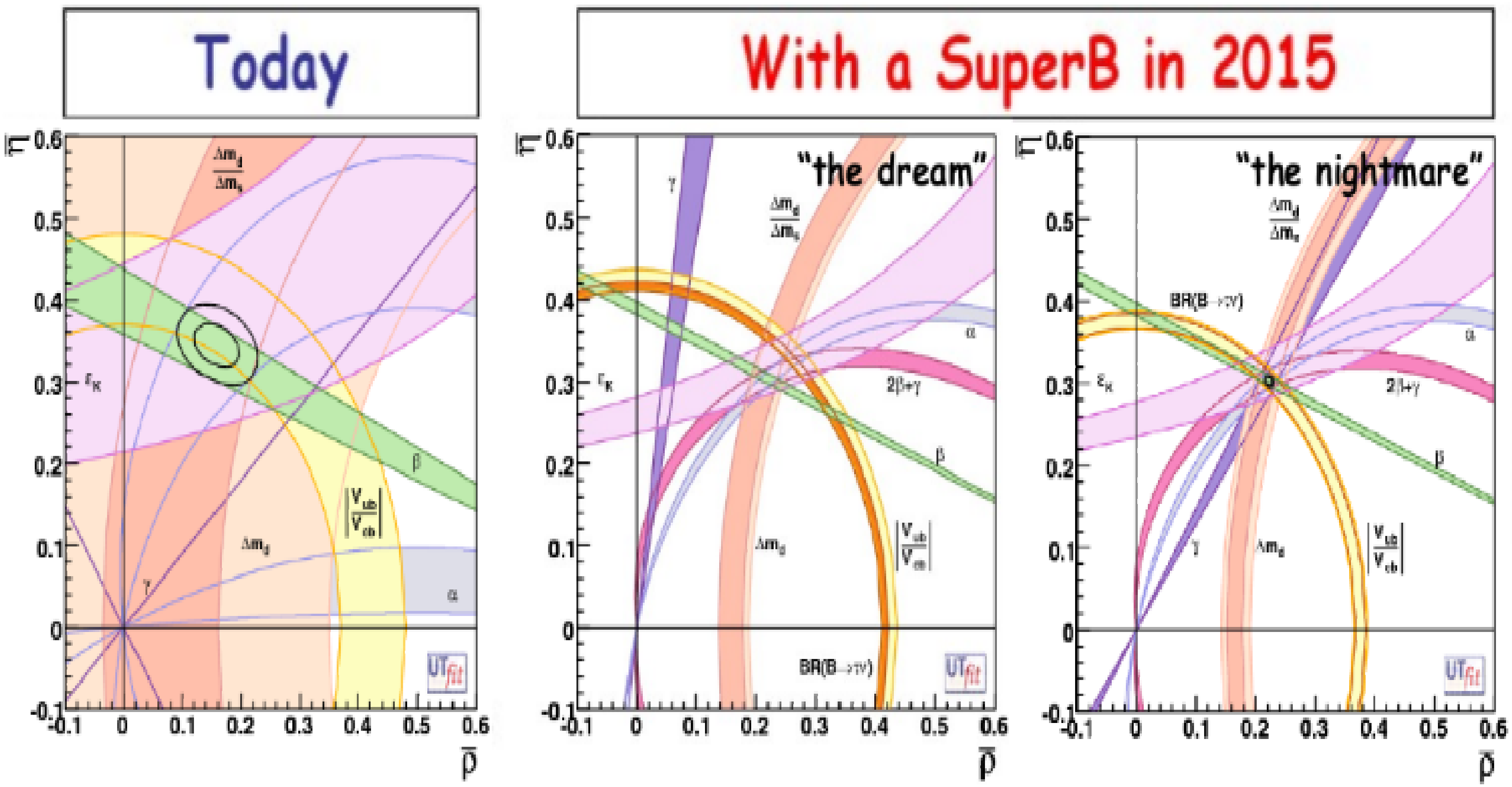
- Lattice QCD and heavy quarks
- Reduce systematic uncertainties
- Outlook

We are entering the area of high precision tests of the Standard Model. It is claimed that from LHCb and a SuperFlavour factory the CKM paradigm might be tested at the level of 1-2 %.

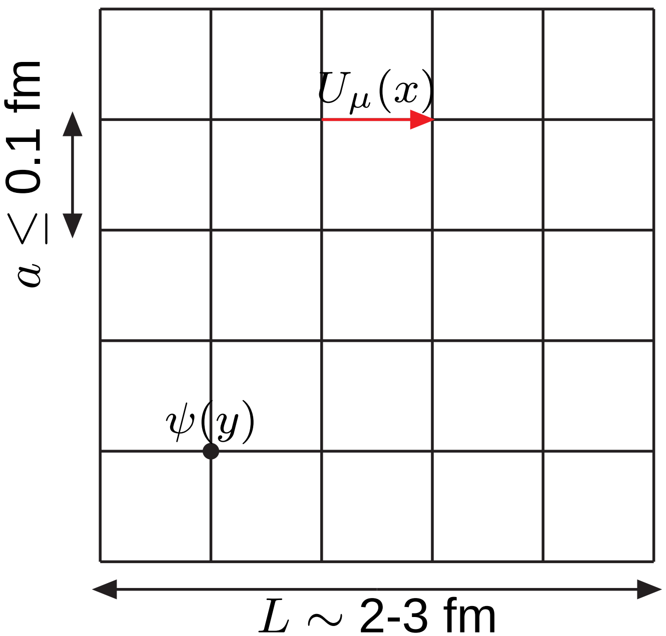
[M. Ciuchini, SuperB workshop ’09]



The achievement of that goal needs the halving at least of the theoretical uncertainty on the hadronic quantities encoding the long-distance dynamics of QCD.

In the following we will discuss recent lattice techniques to deal with B physics and illustrate them in exclusive $B \to X_c l\nu$, B spetrum and decay constants.

Lattice QCD and heavy quarks



Computation of Green functions of the theory from first principle of QFT:

$$\langle O(U,\psi,\bar\psi)\rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}U\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\, O(U,\psi,\bar\psi) e^{-S(U,\psi,\bar\psi)}\,,$$

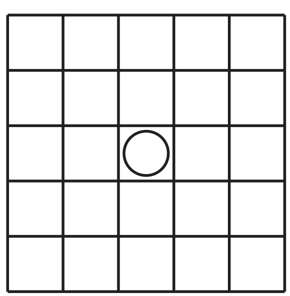
$$\mathcal{Z} = \int \mathcal{D}U\,\mathcal{D}\psi\,\mathcal{D}\bar\psi e^{-S(U,\psi,\bar\psi)}\,,$$

$$S(U,\psi,\bar\psi) = S^{\rm YM}(U) + \bar\psi^i_x M^{ij}_{xy}(U)\psi^j_y\,,$$

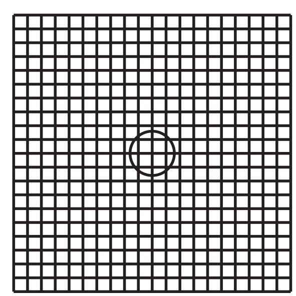
$$\mathcal{Z} = \int \mathcal{D}U\, \mathrm{Det}[\mathrm{M(U)}] e^{-S^{\rm YM}(U)} \equiv \int \mathcal{D}U e^{-S_{\rm eff}(U)}\,.$$

Monte Carlo simulation: $\langle O\rangle \sim \frac{1}{\mathrm{N_{conf}}} \sum_i O(\{U\}_i)$: sampling of $\{U\}_i$ with Boltzmann weight $e^{-S_{\rm eff}}$.

An important issue is how to take under control discretisation effects ($\Lambda_{\rm Compt} \sim 1/m_Q$).



Cut-off Effects



cut-off effects

Several strategies are proposed in the literature to deal with those cut-off effects.

Relativistic heavy quarks [N. Christ et al, '06]

Purpose: define a lattice action for heavy quarks such that the improvement of the hadron spectrum is realised at $\mathcal{O}(a)$, $\mathcal{O}(a\vec{p})$ and at all orders of (am_0), $am_0 \sim 1$.

Kinematics: $|\vec{p}| \sim \Lambda_{QCD}$ (hl mesons), $|\vec{p}| \sim \alpha_s m_Q$ (hh mesons).

Necessity to break the axis symmetry because $p_0 \gg \Lambda_{QCD}$.

Statement: only 3 parameters are required in the effective action.

However the improvement of matrix elements needs also the introduction of counter-terms to the operators and interpolating fields:

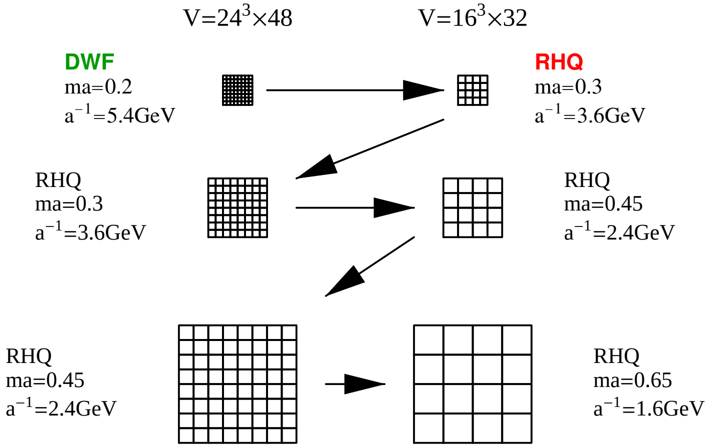
$$S_{\text{lat}} = \sum_{n',n} \overline{\psi}_{n'} \Big(\gamma^0 D^0 + \zeta \vec{\gamma} \cdot \vec{D} + m_0 - \frac{r_t}{2} (D^0)^2 - \frac{r_s}{2} \vec{D}^2 + \sum_{i,j} \frac{i}{4} c_B \sigma_{ij} F_{ij} + \sum_i \frac{i}{2} c_E \sigma_{0i} F_{0i} \Big)_{n',n} \psi_n$$

$$\Psi = z_q^{-1/2} (1 + \delta a \vec{\gamma} \cdot \vec{\partial}) \psi$$

Symanzik programm: the effective action S_{eff} describes a continuum theory approximating the lattice theory up to a given order in a.

RHQ power counting: expansion in a where it is not compensated by a factor m_0 or D_0 $a \sim \mathcal{O}(a)$ but am_0 is $\mathcal{O}(1)$ and D_0 is $\mathcal{O}(a^{-1})$.

It has been shown that one can fix $r_t = 0$, $r_s = 0$ and $c_E = c_B \equiv c_P$.



Several applications of step scaling and matching are performed.

The matching at the smallest lattice spacing is done with DWF

$\Longrightarrow$ discretisation effects are $\mathcal{O}(am)^2$

One performs the matching between hadronic quantities involving heavy-light and heavy-heavy system:

- Spin-average: $m^{hh}_{\rm sa} = \frac{1}{4}\left(m^{hh}_{\rm PS} + 3m^{hh}_{\rm V}\right) \quad m^{hl}_{\rm sa} = \frac{1}{4}\left(m^{hl}_{\rm PS} + 3m^{hl}_{\rm V}\right)$
- Hyperfine splitting: $m^{hh}_{\rm hs} = m^{hh}_{\rm V} - m^{hh}_{\rm PS} \quad m^{hl}_{\rm hs} = m^{hl}_{\rm V} - m^{hl}_{\rm PS}$
- Spin-orbit average and splitting: $m^{hh}_{\rm soa} = \frac{1}{4}\left(m^{hh}_{\rm S} + 3m^{hh}_{\rm AV}\right) \quad m^{hh}_{\rm sos} = m^{hh}_{\rm AV} - m^{hh}_{\rm S}$
- Mass ratio: m_1/m_2 where $E^2 = m_1^2 + \frac{m_1}{m_2}p^2$

What about this procedure beyond the quenched approximation? Usually short-distance renormalisation conditions are imposed independently of the masses and volumes.

At very small lattice spacing one can use small volumes and relatively heavy seas quarks.

☹ The hadronic spectrum is distorted.

☺ It is taken into account by the parameters that are tuned at each step of the matching.

For larger a: bigger physical volume and lighter seas quarks, until the lattice size where the improved action is used to compute non perturbative quantities.

Fermilab action

Formulation is similar to RHQ. Tree level improvement: $E^2 = M_1^2 + \frac{M_1}{M_2}\mathbf{p}^2 + \cdots$

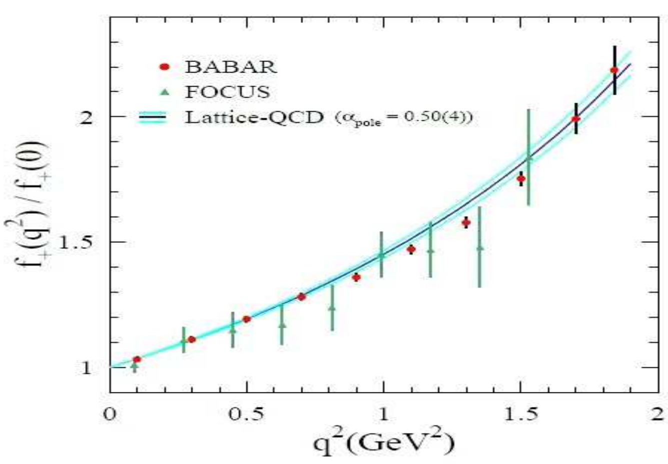
$M_1 = E(0)$ is the "rest" mass; $M_2^{-1} = \left.\frac{\partial^2 E}{\partial p_i^2}\right|_{\vec{p}=0}$ is the kinetic mass.

Imposing $M_1 = M_2$ fixes ζ and m_0; c_P is fixed by imposing that the term $\delta\mathcal{S}$ coming from a transformation *à la* Foldy-Wouthuysen is absent.

To improve the currents one does formally an intermediate matching with HQET, regularised both on the lattice and in a continuum-like scheme:

$$\begin{aligned}
\langle Q^{\rm QCD}\rangle_{\rm lat} &= C_{\rm lat}\langle Q^{\rm HQET}\rangle_{\rm lat} + B^i_{\rm lat}\langle \mathcal{Q}^i_{\rm HQET}\rangle_{\rm lat} \\
\langle Q^{\rm QCD}\rangle_{\rm cont} &= C_{\rm cont}\langle Q^{\rm HQET}\rangle_{\rm cont} + B^i_{\rm cont}\langle \mathcal{Q}^i_{\rm HQET}\rangle_{\rm cont} \\
Z &= \frac{C_{\rm cont}}{C_{\rm lat}}
\end{aligned}$$

Application to heavy-light semileptonic decay form factors [C. Aubin et al, '04]



$f_+^{D\to\pi}(0) = 0.64(3)(6) \quad f_+^{D\to K}(0) = 0.73(3)(7)$

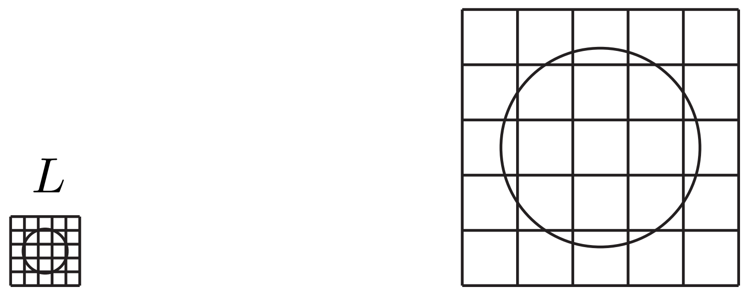
$\delta V_{cd} \sim 14$ % $\quad \delta V_{cs} \sim 11$ %

Step Scaling method

The Step Scaling Functions integrate out the various degrees of freedom between m_b and $\Lambda_{\rm QCD}$ by doing the calculation in different physical volumes and taking for each of them the continuum limit: $A = \underbrace{\sigma_1(L)}_{a\to 0} \times \underbrace{\sigma_2(2L)}_{a\to 0} \times \cdots \times \underbrace{\sigma_n(nL)}_{a\to 0}$.

$4L$

L



$\Lambda_{\rm Compt} \sim 1/m_Q$ $\qquad$ $\Gamma_{\rm bind} \sim 1/\Lambda_{\rm QCD}$

How to fix the volume along the trajectory in the renormalisation flow? We define the Schödinger Functional with the partition function $\mathcal{Z}[C, C'] = \langle C'|e^{-H\,T}|C\rangle$ [K. Symanzik, '81]

$C(x_0 = 0)$ and $C'(x_0 = T)$ are 2 particular configurations.

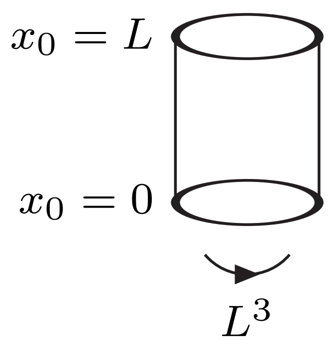
The Schrödinger functional is renormalisable with Yang-Mills theories. [M. Lüscher et al, '92]

One associates a finite volume renormalisation scheme which is independent of any regularisation.

$$\Gamma(B) \equiv -\ln \mathcal{Z}[C, C'] = g_0^{-2}\Gamma_0[B] + \Gamma_1[B] + g_0^2\Gamma_2[B] + \ldots \qquad \left.\frac{\delta S}{\delta \Phi}\right|_{\Phi=B} = 0$$

$$C^{(\prime)} \equiv C^{(\prime)}(\eta) \qquad \bar{g}^2(L) = \left[\frac{\partial \Gamma_0(B)}{\partial \eta}\right] \Big/ \left[\frac{\partial \Gamma(B)}{\partial \eta}\right] \qquad \bar{g}^2(L) = \left\langle \frac{\partial S}{\partial \eta} \right\rangle$$

The SF is renormalisable with QCD. [S. Sint, '93]



$$P_+\psi(x)|_{x_0=0} = \rho(\vec{x}) \qquad P_-\psi(x)|_{x_0=T} = \rho'(\vec{x})$$

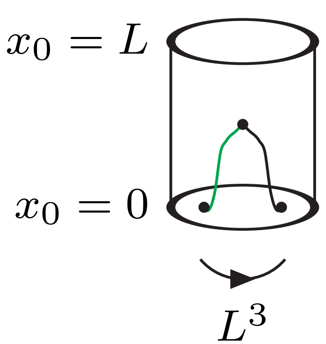
$\psi(x + L\hat{k}) = e^{i\theta_k}\psi(x)$ (allow computation at $w \neq 1$)

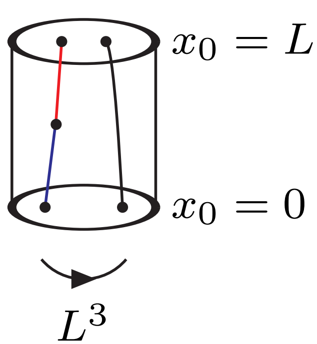
$$\langle O \rangle = \left.\left(\tfrac{1}{\mathcal{Z}} \int [\mathcal{D}U][\mathcal{D}\psi][\mathcal{D}\bar{\psi}] O e^{-S(U,\psi,\bar{\psi})}\right)\right|_{\rho=\bar{\rho}=\rho'=\bar{\rho}'=0}$$

The Dirac operator has no zero modes in the chiral limit.

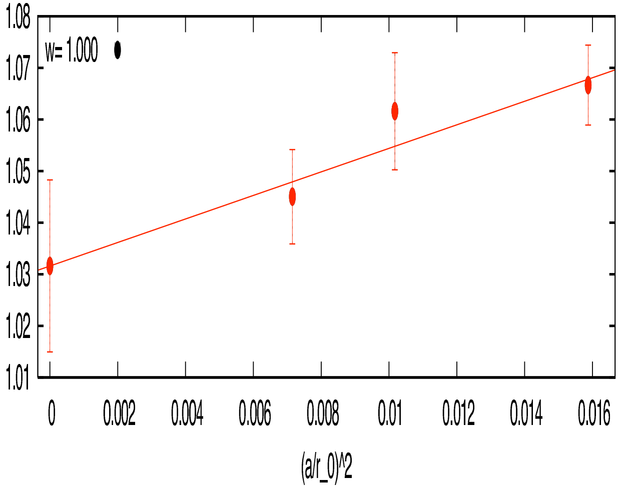
Application to the $B \to Dl\nu$ form factor [G. M. de Divitiis et al, '07]

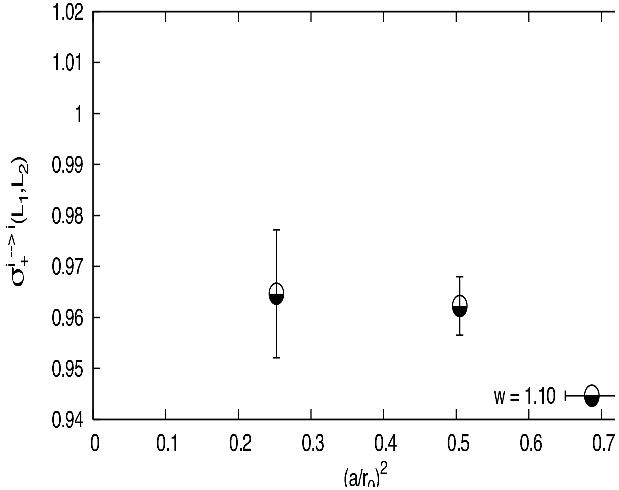
$$\frac{d\Gamma}{dw}(\bar{B} \to D\ell\bar{\nu}_\ell) = \frac{G_F^2 |V_{cb}|^2 m_B^5}{192\pi^3} K_V(w) G(w)^2 \left(1 - \frac{m_\ell^2}{m_B^2}\left|1 + \frac{t(w)}{(m_b - m_c)m_\ell} C_{NP}^\ell\right|^2 K_s(w) \Delta_S(w)^2\right)$$

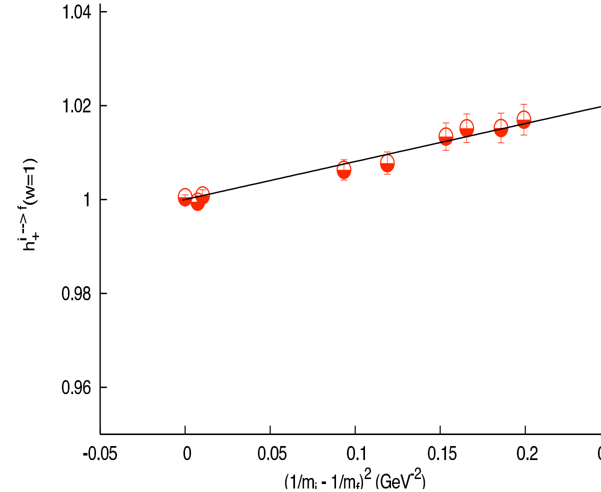




Computation of boundary to bulk and boundary to boundary correlators to extract $\langle B|V_\mu|D\rangle$.



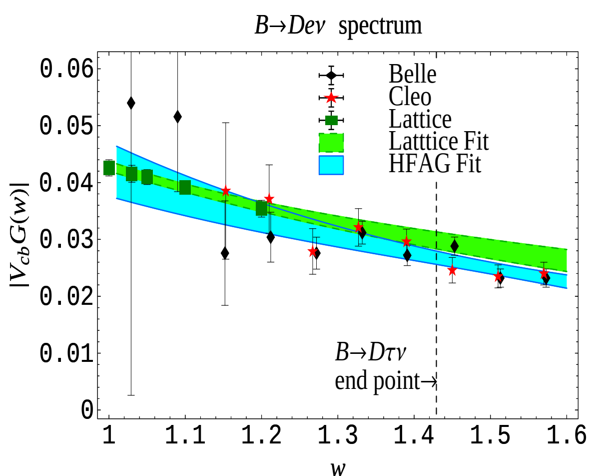


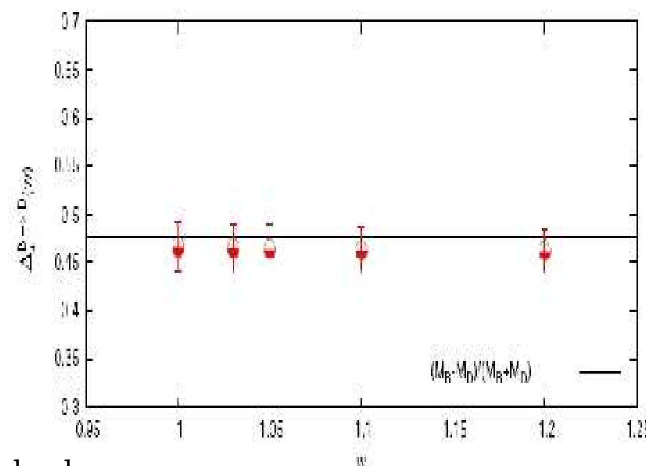


Lattice data on G cover a kinematical region hardly reachable by the experiment.

The function $\Delta^{B \to D}$ can be measured experimentally from $\frac{d\Gamma^{B \to D \tau \nu_\tau}}{d\Gamma^{B \to D(e,\mu)\nu_{e,\mu}}}$.

Lepton-flavour universality checks on the extraction of V_{cb}.





$$\delta[\Delta(w)]^{\text{quenched}} \sim 2\%$$

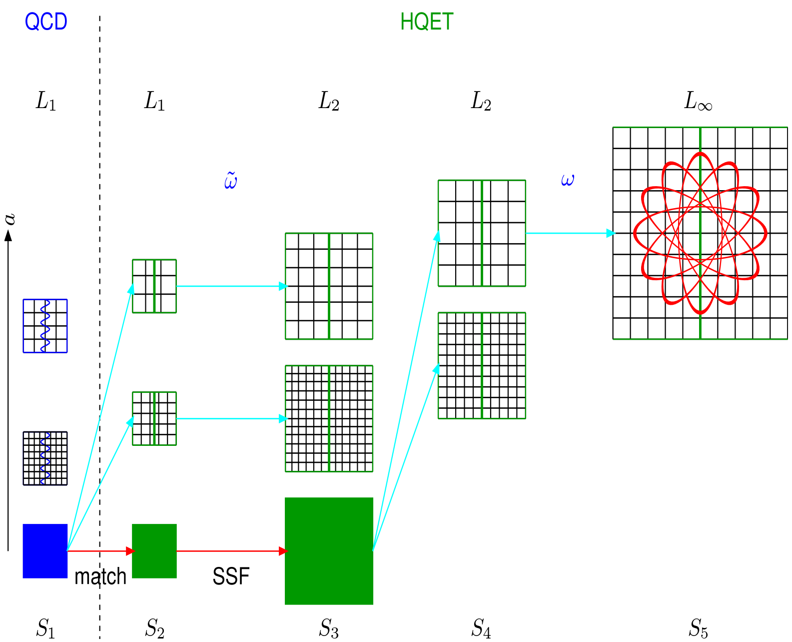
☺ Extension to $N_f = 2$ simulations is in principle straightforward.

☹ However the simulations in large volumes seemed to be delicate to analyse (multi-pion states created at the boundaries).

Heavy Quark Effective Theory

The purpose is to integrate out degrees of freedom $\sim m_b$ to keep more easily cut-off effects under control.

It allows to interpolate at the b quark mass results extracted from computations performed in the charm sector.



Bare parameters of the HQET Lagrangian and currents need to be tuned.

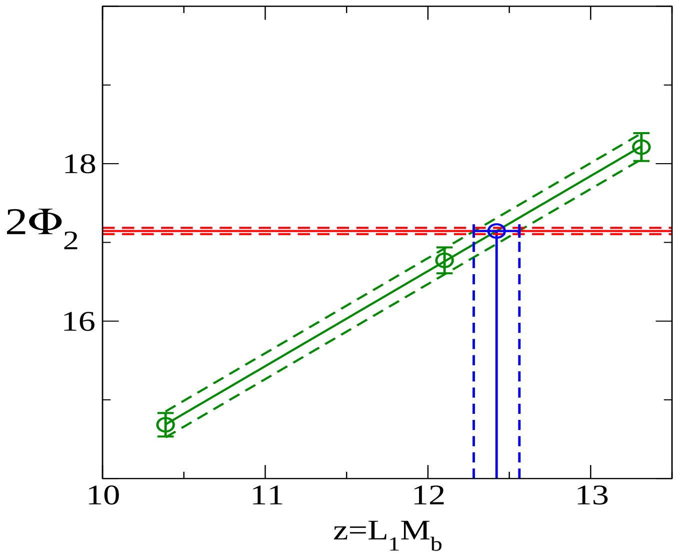
It is done non perturbatively by imposing in a small volume $L_1 \sim$ 0.5 fm several matching conditions between correlators defined in QCD and their HQET counterpart:

$$\underbrace{\Phi_i^{\mathrm{QCD}}}_{\text{cont lim}} = \underbrace{f_{ij}(\omega_k)\Phi_j^{\mathrm{HQET}}}_{\text{finite } a}$$

Ultraviolet divergences of HQET have been absorbed in the ω_k coefficients.

One uses once again Step Scaling functions: $\Phi_i(sL/a) = \Sigma_{ij}(L/a)\Phi_j(L/a)$ in order to run observables from the volume L_1 up to a volume $L_{\mathrm{inf}} = s^k L_1$ where long-distance physics dominates and where one extracts hadronic quantities.

Application to the b quark mass measurement



[M. Della Morte *et al*, '06]

$\bar{m}_b^{\mathrm{Nf}=0}(\bar{m}_b) = 4.347(48)$ GeV

SF framework is not used anymore to extract hadronic quantities.

Applications to the B spectrum study and f_{B^n} are underway [B. B. *et al*, '10].

☺ The extension to a computation with dynamical fermions is straightforward.

☺ A control of contribution to correlators from radial excitations is possible.

☹ However that approach does not seem so great for $B \to D$ semileptonic decays (need to go to $\mathcal{O}(1/m_{b,c}^2)$ to extract something non trivial).

Step Scaling in mass

A recent proposal has been tested to extrapolate in the m_b region results obtained around m_c, using scaling laws in the heavy quark limit [B. B. *et al* for ETMC, '09].

$$q(x,\lambda,\hat{m}_l) = \lambda^\alpha \frac{A_{hl}(1/x,\hat{m}_l)}{A_{hl}(1/\lambda x,\hat{m}_l)} \frac{\mathcal{Z}(\ln x\lambda)}{\mathcal{Z}(\ln x)} \left[\frac{\rho(\ln x)}{\rho(\ln \lambda x)}\right]^\alpha$$

$\lambda = \frac{x^{(n-1)}}{x^n} > 1$ is the heavy mass step, $x = 1/\hat{m}_h$, $\hat{m}_{l(h)}$ are renormalised light (heavy) quark masses, $\rho(\ln \hat{m}_h)\hat{m}_h = m_h^{\mathrm{pole}}$, $A^{\mathrm{QCD}} = \mathcal{Z} A^{\mathrm{HQET}}$

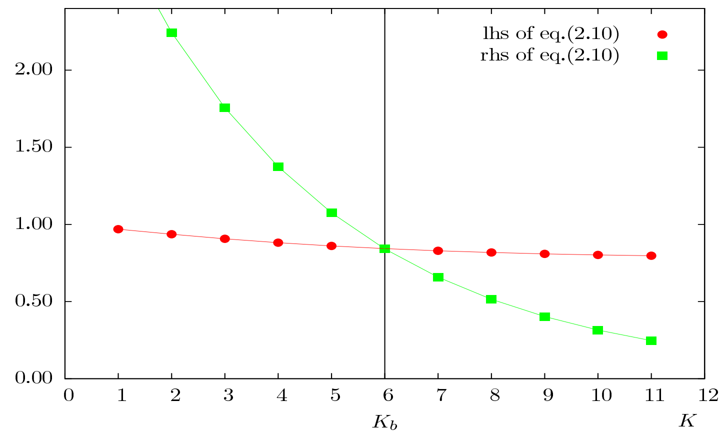
$$\lim|_{x\to 0}[\rho(\ln x)/x]^\alpha A_{hl}(1/x)/\mathcal{Z}(\ln x) = \mathrm{C_{ste}} \quad q(\Phi) = \lim|_{a\to 0} q^L(\Phi,a) \quad \hat{m}_b \sim \lambda^K \hat{m}_c$$

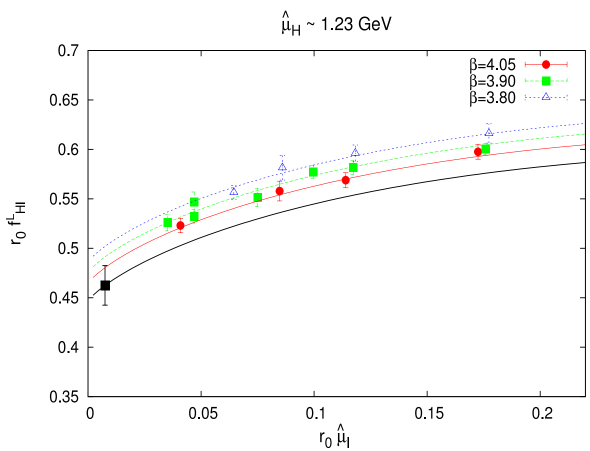
$$q_p^{(2)} q_p^{(3)} \cdots q_p^{(K+1)} = \lambda^{K\alpha} \frac{A_{hl}(\hat{m}_h^{(1)})}{A_{hl}(\hat{m}_h^{(K+1)})} \left\{ \frac{\mathcal{Z}(\ln \hat{m}_h^{(1)})}{\mathcal{Z}(\ln \hat{m}_h^{(K+1)})} \left[\frac{\rho(\ln \hat{m}_h^{(K+1)})}{\rho(\ln \hat{m}_h^{(1)})}\right]^\alpha \right\}_p$$

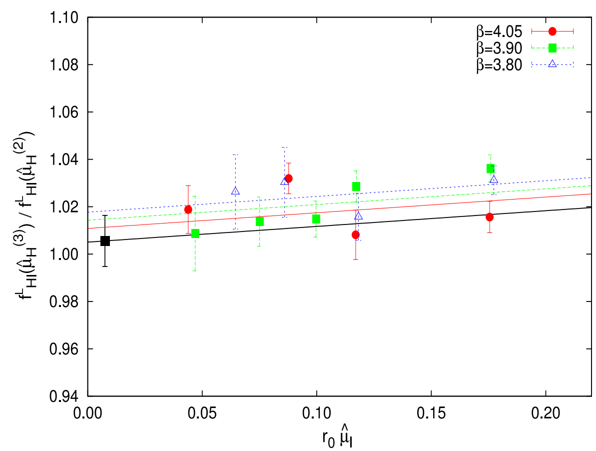
One has to determine K, λ and interpolate lattice data q^L to a sequence of "reference masses" $\hat{m}_h^i = \lambda^i \hat{m}_h^{(1)}$ by a smooth function, then perform a fit of $q^L(\hat{m}_h^{(i)})$ obtained at different lattice spacings to extrapolate to the continuum limit;
$A_{hl}(\hat{m}_b) = A_{hl}(\hat{m}_h^{(1)}) \times \Pi_i q(\hat{m}_h^{(i)})$.

Application to the measurement of f_B

$$y = \lambda^{-1} \frac{M_{hl}(1/\lambda x, \hat{m}_l)}{M_{hl}(1/x, \hat{m}_l)} \left[\frac{\rho(\ln x)}{\rho(\ln \lambda x)} \right]^{-1} \qquad z = \lambda^{1/2} \frac{f_{hl}(1/x, \hat{m}_l)}{f_{hl}(1/\lambda x, \hat{m}_l)} \frac{Z_{\rm stat}(\ln x\lambda)}{Z_{\rm stat}(\ln x)} \left[\frac{\rho(\ln x)}{\rho(\ln \lambda x)} \right]^{1/2}$$



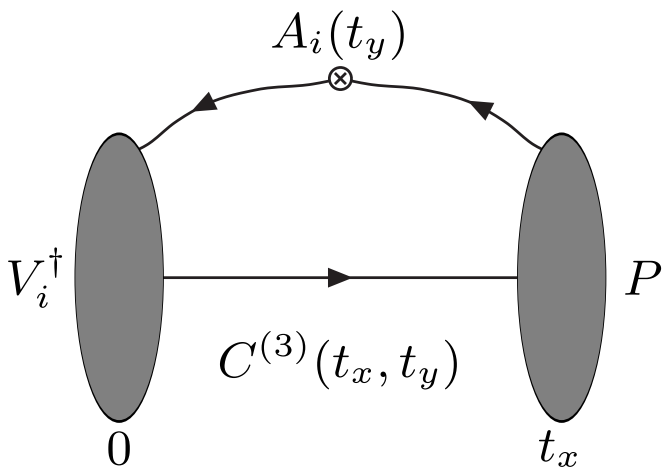




Idea of "error budget": 50% comes from the extraction of $f_{hl}(\hat{m}^{(1)})$ and 50% comes from the product of ratios $z's$. The truncation in p of $\rho^{1/2}/Z_{\rm stat}$ has a negligible impact on f_B

Some investigation wether one can apply the method to extract B_B parameters or $B \to D$ form factors is suitable.

Reduce systematic uncertainties



A lot of work has been done on 3-pts functions to reduce as much as possible systematic errors (for example from renormalisation constants) or to improve signal over noise ratios.

We will present few techniques developed over 10 last years by the lattice community.

Double ratios and normalisation point

The "double ratio" technique $R = \frac{\langle A(p)|O_\Gamma|B(p')\rangle\langle B(p')|O'_\Gamma|A(p)\rangle}{\langle A(p)|O^1_\Gamma|A(p)\rangle\langle B(p')|O^2_\Gamma|B(p')\rangle}$ has been widely used to reduce the error on form factors [FNAL, '99; SPQcdR, '04; RBC/UKQCD, '07]. No renormalisation constant, knowledge of normalisation points $\langle A(p)|O^1_\Gamma|A(p)\rangle$.

Application to $B \to D^*$ [C. Bernard et al, '08]

$$\frac{d\Gamma}{dw}(\bar{B} \to D^* \ell \bar{\nu}_\ell) = \frac{G_F^2}{4\pi^2}|V_{cb}|^2 m_{D^*}^3 (m_B - m_{D^*})^2 \sqrt{w^2-1}\, G'(w)|\mathcal{F}_{B\to D^*}(w)|^2$$

$G'(1) = 1$ $\quad \mathcal{F}_{B\to D^*}(1)$ depends on the single form factor $h_{A_1}(1)$ $\quad h_{A_1}^{N_f=2+1}(1) = 0.921(13)(20)$

$$\langle D^*(v,\epsilon')|A^\mu|\overline{B}(v)\rangle = \sqrt{2m_B 2m_{D^*}}\;\bar{\epsilon}'^\mu\, h_{A_1}(1) \quad h_{A_1}(1) = \frac{\langle D^*|\bar{c}\gamma_j\gamma_5 b|\overline{B}\rangle\langle\overline{B}|\bar{b}\gamma_i\gamma_5 c|D^*\rangle}{\langle D^*|\bar{c}\gamma_0 c|D^*\rangle\langle\overline{B}|\bar{b}\gamma_4 b|\overline{B}\rangle}$$

Stochastic propagators

Until few years ago we were computing a single column of the quark propagator $S(0,y)$ (point to all propagator) by solving $D_{x,y}\psi_y = R_x$, $R_x = \delta_{x,0}$.
Techniques to compute all to all propagators have been massively used recently.

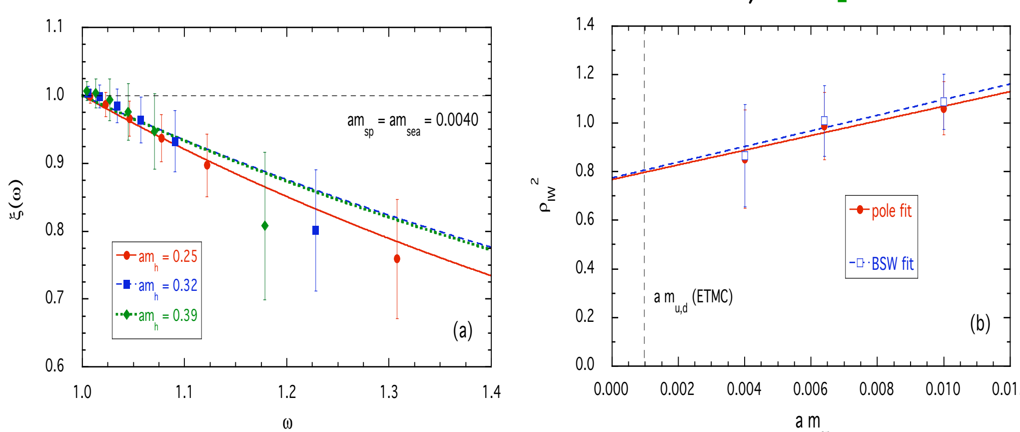
☺ The whole information contained in $\mathrm{N_f} > 0$ gauge configurations, costly to be produced, is taken into account.

☺ It opens perspectives to calculate interesting quantities involving disconnected diagrams.

Stochastic propagator: $S(y,x) = \langle\langle \psi(y)\eta^\dagger(x)\rangle\rangle \quad \langle\langle \eta(y)\eta^\dagger(x)\rangle\rangle = \delta_{xy} \quad D\psi = \eta$.

"One-end trick" [UKQCD, '06]: a single random noise used in the computation of propagators entering in 2-pts and 3-pts correlators, so that one obtains variance reduction with the signal $\mathcal{O}(V)$ and the noise $\mathcal{O}(V/\sqrt{N})$.

Application to $B \to D$ form factor at $w \neq 1$ [S. Simula, '07]



$\rho^2_{\mathrm{N_f}=0} = 0.89 \pm 0.17 \qquad \rho^2_{\mathrm{N_f}=2} = 0.77 \pm 0.28$

Computation of form factors at non zero recoil

Momenta quantified on the lattice because of boundary conditions:

$\psi(x + L\hat{k}) = \psi(x) \Longrightarrow p_k = 2\pi n/L$.

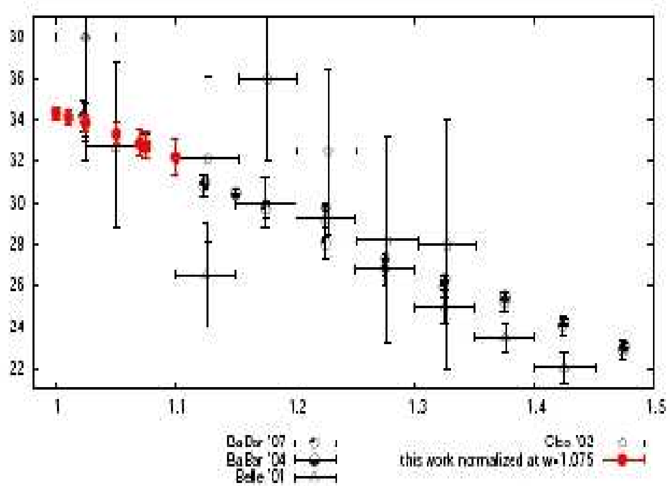
It is possible to shift p_k by a continuous term $2\pi\theta_k/L$ imposing $\tilde{\psi}(x + L\hat{k}) = e^{i\theta_k}\tilde{\psi}(x)$,

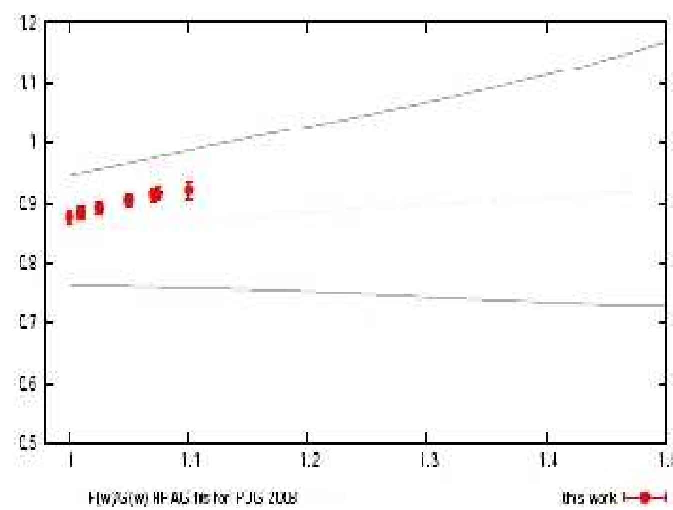
$\tilde{\psi}(x) = e^{2i\pi\theta_k x_k/L}\psi(x)$ [F. Bedaque, '04; R. Petronzio et al, '04]

$$D(U)\tilde{\psi} = R \leftrightarrow D(U^{\theta_k})\psi = R \quad U_k^{\theta_k}(x) = e^{2i\pi\theta_k/L}U_k(x)$$

Boundary conditions of valence quark fields have been properly modified. What about sea quarks? It has been shown that finite volume effects induced by partially θ-boundary conditions (i.e. only for valence quarks) are exponentially small in quantities without FSI [C. Sachrajda and G. Villadoro, '04].

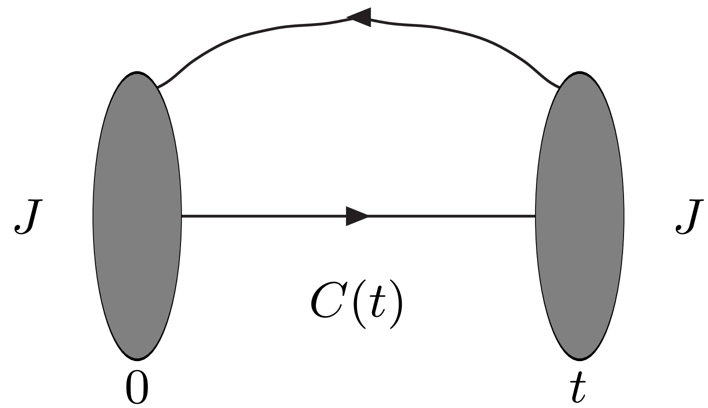
Application to $\mathcal{F}_{B\to D^*}(w \neq 1)$ [G.M. de Divitiis et al, '08]





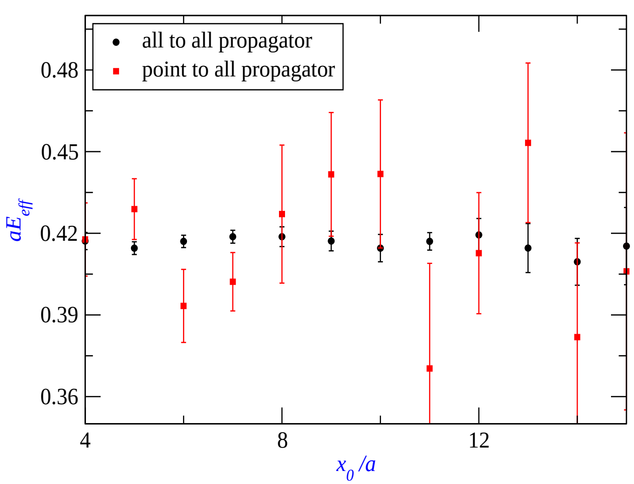
$$\mathcal{F}_{B\to D^*}^{\mathrm{N_f}=0}(1) = 0.917(8)(5)$$

The spectrum of pseudoscalar heavy-light systems is more narrow than the one of their lighter counterparts. It means that one has to take care of the contribution of excited states to the correlators analysed to extract for instance f_{B_s} from lattice simulations.
A well-established method to measure masses has been further investigated by Alpha Collaboration to deal with excited states in HQET [B. B. *et al*, '09].



$$C_{JJ}(t) = \sum_{\vec{x}} \langle \Omega | \mathcal{T}[J(\vec{x},t) J^\dagger(0)] | \Omega \rangle$$
$$= \sum_n \frac{\mathcal{Z}_n^2 e^{-E_n t}}{2E_n}$$
$$\mathcal{Z}_n = \langle \Omega | J | n \rangle \qquad \langle n | m \rangle = 2E_n \delta_{mn}$$
$$C_{JJ}(t) \xrightarrow{(E_1 - E_0)t \gg 1} \frac{\mathcal{Z}_0^2 e^{-E_0 t}}{2E_0}$$

Issue: at $t \gtrsim 1$ fm, the statistical noise enters severely in competition with the usable signal.



All to all propagators increase dramatically the statistical efficiency [C. Michael and J. Peisa, '98] [J. Foley et al, '05]

Example of a B_s meson 2pts correlator
$\Gamma^{\rm stat}(x_0) = -\ln[C_{PP}(x_0 + a)/C_{PP}(x_0)]$

$\# = 50 \quad N_f = 0 \quad a \sim 0.1$ fm $\quad L \sim 1.5$ fm $\quad m_q \sim m_s$

We are now in a good position to study the systematic effects induced by excited states.

$m_{B'} - m_B \sim 500$ MeV $\quad m_{B''} - m_{B'} \sim 200$ MeV

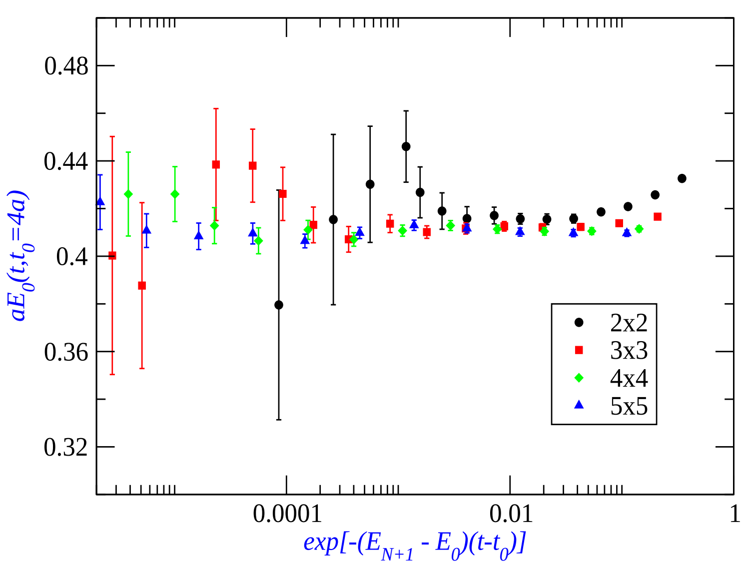
Highlights of the strategy

The Variational method is an appealing approach to define an operator $O^n_{J^P}$ weakly coupled to other states than $|n\rangle$ [C. Michael, '85] [M. Lüscher and U. Wolff, '90].

– Compute an $N \times N$ matrix of correlators $C^{ij}_{PP}(t) = \sum_{\vec{x},\vec{y}} \langle \Omega | O^i_{J^P}(\vec{x},t) O^j_{J^P}(\vec{y},0) | \Omega \rangle$ with $O^i_{J^P}(\vec{x},t) = \sum_{\vec{z}} \bar{q}(\vec{x},t) [\Gamma \times \Phi(|\vec{x}-\vec{z}|)]^i_{J^P} q(\vec{z},t)$

– Solve the generalised eigenvalue problem $C^{ij}(t)\, v_n(t,t_0) = \lambda_n(t,t_0)\, C^{ij}(t_0)\, v_n(t,t_0)$

– $\lambda_n(t,t_0) \sim e^{-E_n\,(t-t_0)}$



$\# = 100 \quad N_f = 0 \quad a \sim 0.1$ fm $\quad L \sim 1.5$ fm $\quad m_q \sim m_s$

The effects of excited states on the effective mass of the ground state is clearly visible

We are not sure to keep them under control within 1% unless incorporating in our system the $3^{\rm rd}$ excited state ($E_3 - E_0 \sim 850$ MeV).

Impossible to do it by a multi-exponential fit without imposing some priors.

It has been proved in the literature that $aE^{\rm eff}_n(t,t_0) \equiv -\ln\left(\frac{\lambda_n(t+a,t_0)}{\lambda_n(t,t_0)}\right) = aE_n + \mathcal{O}(e^{-\delta E_n\, t})$

$\delta E_n = \min_m |E_n - E_m|$

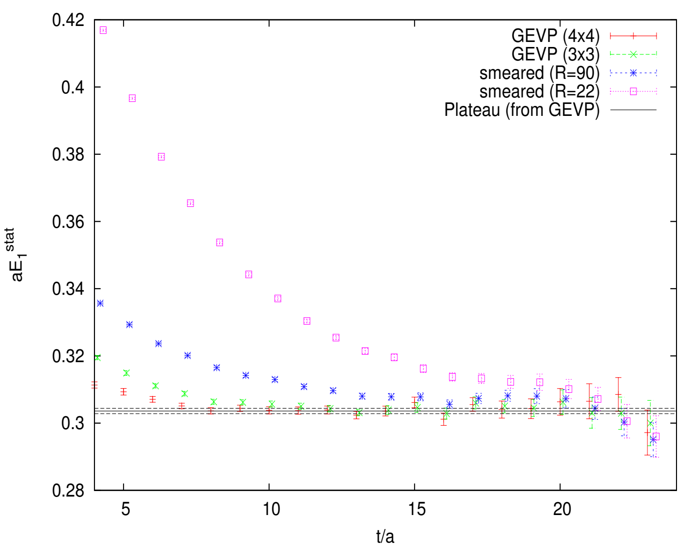
Issue if $\delta E_n \lesssim 500$ MeV (Example: $E_{X+\pi+\pi} - E_X$)

New statements

Our claim: actually the rate of convergence is even faster than $e^{-\delta E_n t}$ under the condition that t_0 is large enough ($t_0 \geq t/2$):

$$\begin{aligned} aE_n^{\rm eff}(t,t_0) &= aE_n + \mathcal{O}(e^{-\Delta E_{N,n}\, t}) \\ \langle\Omega|\hat{P}e^{-Ht}[Q_n^{\rm eff}]^\dagger(t,t_0)|\Omega\rangle &= \langle n|P|0\rangle + \mathcal{O}(e^{-\Delta E_{N,n}\, t_0}) \\ \langle\Omega|Q_n^{\rm eff}e^{-H\,t}\hat{P}e^{-H\,t}[Q_{n'}^{\rm eff}]^\dagger(t,t_0)|\Omega\rangle &= \langle n|P|n'\rangle + \mathcal{O}(e^{-\Delta E_{N,n}\, t_0}) \end{aligned}$$

$$\Delta E_{n,n'} = |E_n - E_{n'}| \quad Q_n^{\rm eff}(t,t_0) = \frac{O^i(t)v_{n\,i}(t,t_0)}{\sqrt{v_{n\,i}(t,t_0)C^{ij}(t)v_{n\,j}(t,t_0)}}\left(\frac{\lambda_n(t_0+1,t_0)}{\lambda_n(t_0+2,t_0)}\right)^{t/2}$$



Application to B meson spectroscopy: by considering a 4×4 matrix of correlators, one has $\Delta E_{0,4} \sim 1$ GeV.

= 100 $\quad L \sim 1.5$ fm $\quad a \sim 0.07$ fm $\quad m_q \sim m_s$

Thanks to the GEVP analysis one can quantify the systematic error coming from excited states even in a region where the statistical error starts to increase strongly.

$$C_{ij}(t) = \langle\Omega|O_i(t)O_j^\dagger(0)|\Omega\rangle = \sum_{n=0}^{\infty} \psi_i^n \psi_j^n e^{-E_n\,t},\ i,j = 1,\dots,N$$

$$\psi_i^n = \langle\Omega|O_i|n\rangle \quad E_n < E_{n+1}$$

Truncation: $C_{ij}(t) = C_{ij}^{(0)}(t) + C_{ij}^{(1)}(t) \quad C_{ij}^{(0)}(t) = \sum_{n=0}^{N-1} \psi_i^n \psi_j^n e^{-E_n\,t}$

Assuming that $A^{(0)} v_n^{(0)} = \lambda_n^{(0)} B^{(0)} v_n^{(0)}$ with $A^{(0)} = C^{(0)}(t)$ and $B^{(0)} = C^{(0)}(t_0)$ we have

$$A\,v_n = \lambda_n\,B\,v_n \quad A = A^{(0)} + \epsilon A^{(1)} \quad B = B^{(0)} + \epsilon B^{(1)} \quad \epsilon = e^{-(E_N - E_{N-1})\,t_0}$$

$$\begin{aligned} \lambda_n &= \lambda_n^{(0)} + \epsilon\lambda_n^{(1)} + \epsilon^2\lambda_n^{(2)} + \dots \\ v_n &= v_n^{(0)} + \epsilon v_n^{(1)} + \epsilon^2 v_n^{(2)} + \dots \end{aligned}$$

$$\rho_n(t) = e^{-E_n\,t} \quad \Delta_n = A^{(0)} - \lambda_n^{(0)} B^{(1)} \quad \alpha_{mn}^{(0)} = \delta_{mn} \quad \lambda_n^{(0)}(t,t_0) = e^{-E_n(t-t_0)}$$

$$\lambda_n^{(k)}\,\rho_n = (v_n^{(0)}\,,\,\Delta_n v_n^{(k-1)}) - \sum_{l=1}^{k-1} \lambda_n^{(l)}\,(v_n^{(0)}\,,\,B^{(1)} v_n^{(k-1-l)}) \quad v_n^{(k)} = \sum_{m\neq n} \alpha_{mn}^{(k)} v_m^{(0)}$$

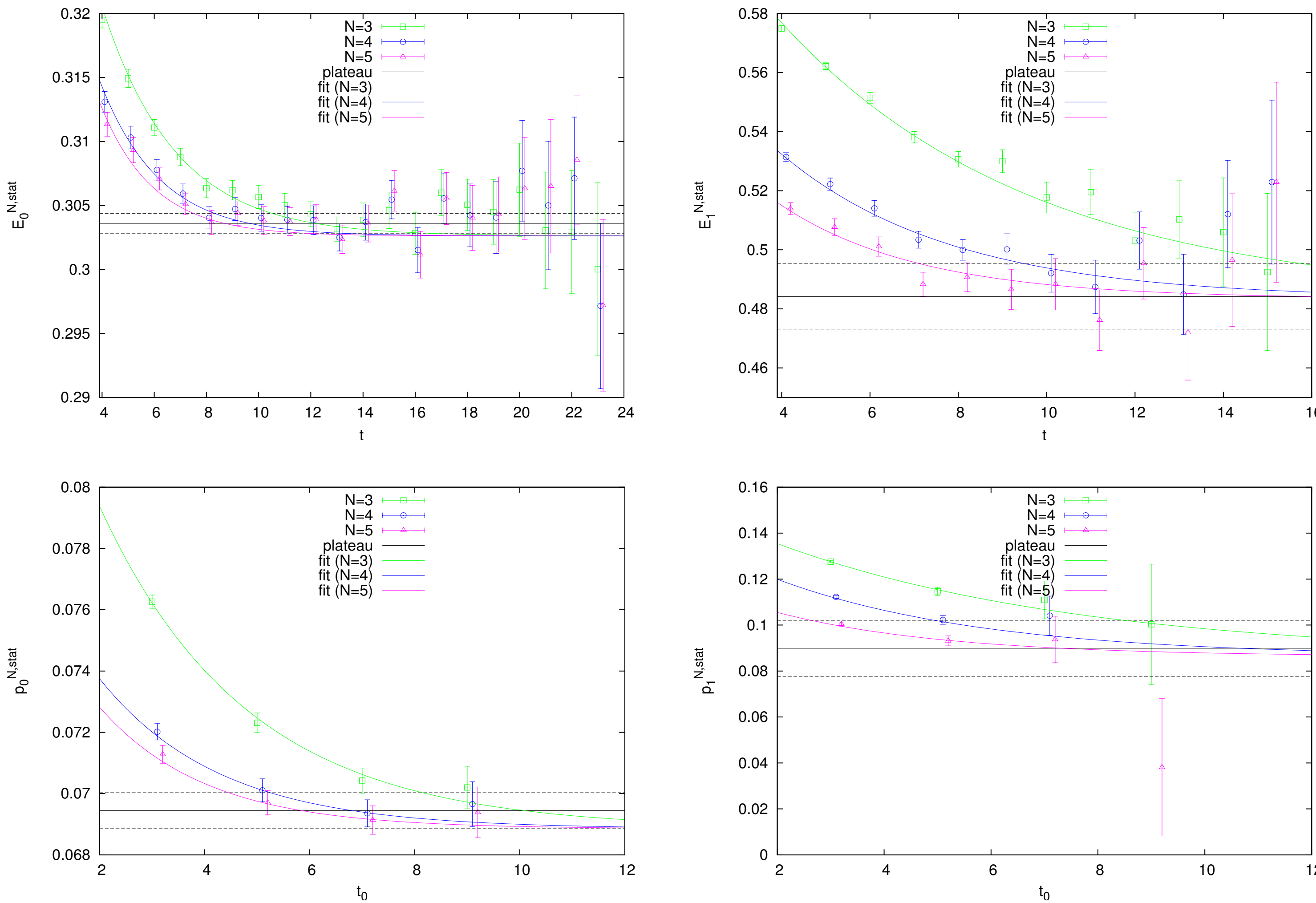
$$\begin{aligned} \alpha_{mn}^{(k)}\,\rho_m &= \frac{1}{\lambda_n^{(0)} - \lambda_m^{(0)}} \Bigg\{ (v_m^{(0)}, \Delta_n v_n^{(k-1)}) - \sum_{l=1}^{k-1} \lambda_n^{(l)}\,(v_m^{(0)}\,,\,B^{(1)} v_n^{(k-1-l)}) \\ &\quad - \sum_{l=1}^{k-1} \lambda_n^{(l)}\,(v_m^{(0)}\,,\,B^{(0)} v_n^{(k-l)}) \Bigg\} \end{aligned}$$

After looking at the dependence on t and t_0 of $\lambda_n^{(k)}$ and $\alpha_{mn}^{(k)}$ we arrive at the conclusions that if $t_0 \geq t/2$

$$aE_n^{\mathrm{eff}}(t,t_0) = aE_n + \epsilon_n(t,t_0) \quad \epsilon_n(t,t_0) = \mathcal{O}(e^{-\Delta E_{N,n}\, t})$$

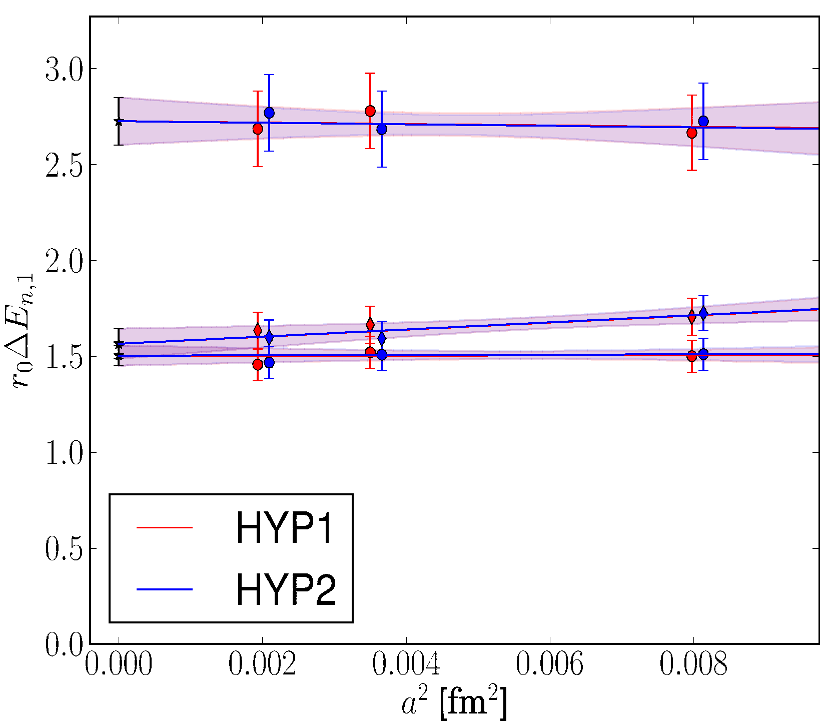
$$\langle \Omega | \hat{P} e^{-Ht} [Q_n^{\mathrm{eff}}]^\dagger (t,t_0) | \Omega \rangle = \langle n | P | 0 \rangle + \pi_{n,n'} \quad \pi_{n,n'} = \mathcal{O}(e^{-\Delta E_{N,n}\, t_0})$$

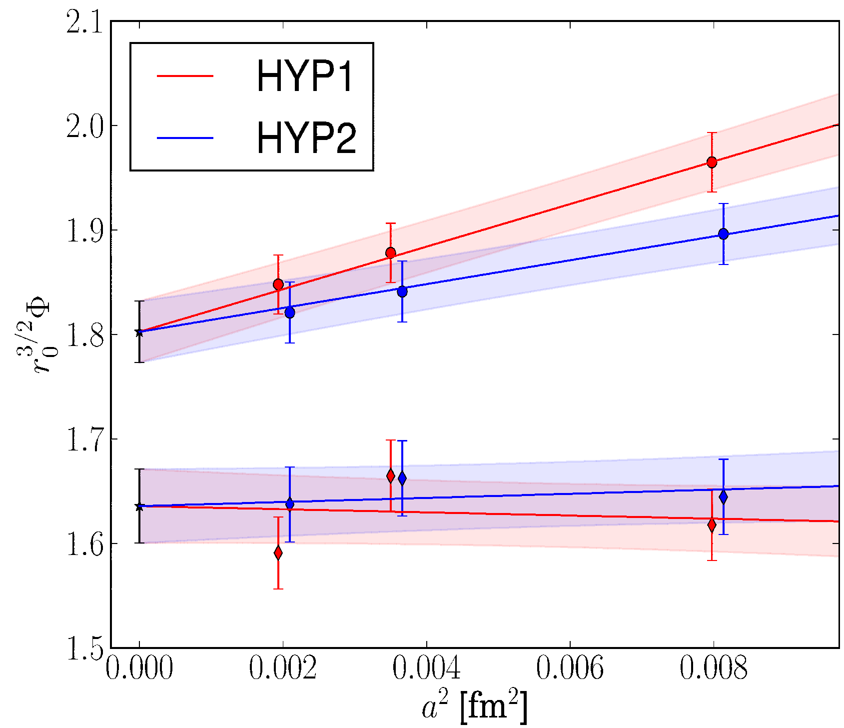
The study of the B_s meson spectroscopy and the extraction of the decay constants performed at $N_f = 0$ ($a \sim 0.07$ fm, $m_q \sim m_s$) seem to confirm our analytical proof.



Application to the quenched B_s spectrum in HQET at $\mathcal{O}(1/m)$

Once having determined the parameters of the matching between QCD and HQET at $\mathcal{O}(1/m)$ and extracted the hadronic matrix elements from a GEVP analysis, we can perform a continuum limit extrapolation of $r_0 \Delta E_{n,1}$ and $r_0^{3/2} f_B \sqrt{m_B}$.





Outlook

- Recently several novel techniques have been developed by the lattice community in order to extract from first principle computations hadronic quantities with a competitive accuracy with respect to experimental measurements.

- Appropriate descriptions of heavy quark are crucial to keep under control discretisation effects, reduce as much as possible the number of sources of systematic uncertainties (e.g. renormalisation constants) helps. Close to zero recoil lattice data are nicely complementary to experiment for form factors.

- Reach 1% level of accuracy is (very) ambitious; of course quenching effects (essentially) disappear by including $2+1$ sea flavours in the theory but other sources of uncertainties enter the game: contribution of radial excitations to the correlators on the theoretical side, fake events with soft photons on the experimental side.

- Exploratory study of static-light meson spectrum and decay constants involving a non perturbative matching between QCD and HQET and a GEVP analysis gives promising results as far as continuum limit is concerned.

- Its extension to 3-pts correlation functions might be a nice challenge, for instance to eliminate a source of uncertainty in the determination of B bag parameters or establish the sign of $g_{B^* B' \pi}$.

- A combination of GEVP and Step Scaling in Masses might help to measure form factors of $B \to D^{**} l \nu$.