With Wilson quarks towards the epsilon regime
O(a^2) corrections in ChPT

Oliver Bär
Humboldt Universität zu Berlin

in collaboration with
S. Necco, S. Schaefer and A. Shindler

Outline

- The epsilon regime: What and Why
- ...
- ...

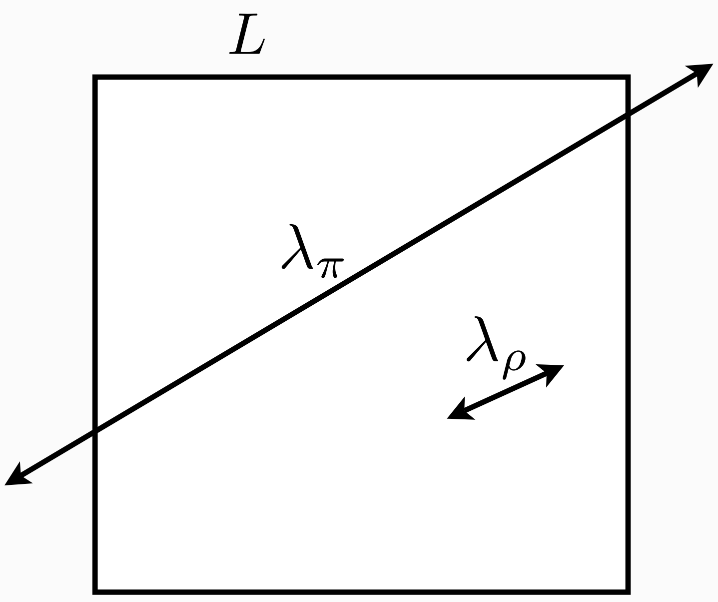
Berlin
May 3 2010

QCD in the epsilon regime

- Epsilon regime: QCD in a finitel volume such that

  a) $L \ll 1/M_\pi = \lambda_\pi$

  b) $L \gg 1/M_\rho = \lambda_\rho$



- Finite size effects are large instead of exponentially suppressed and prop to $e^{-M_\pi L}$

J.Gasser, H. Leutwyler 1987
F. Hansen 1990

- ChPT analysis of the epsilon regime shows
  - NLO results depend on two low-energy couplings (LECs) only: F, Σ
  - Gasser-Leutwyler coefficients L_i enter at NNLO and higher

  ➥ Use lattice simulations in the epsilon regime to get F, Σ

epsilon regime on the lattice

- Simulations in the epsilon regime need
  - large volumes (expansion parameter is $\dfrac{1}{(FL)^2}$)
  - small pion (quark) masses
- Widespread belief: Ginsparg-Wilson fermions are necessary
  - many quenched simulations in the epsilon regime have been done
  - bottleneck: unquenched. So far only one simulation by JLQCD

H. Fukaya et.al. 2008

- Recently: Simulation with (tree-level improved) Wilson fermions

A.Hasenfratz, R. Hoffmann, S. Schaefer 2008

  - small quark masses reached using reweighting
  - lattice spacing $a \approx 0.115$ fm
- Surprise: data very well described by continuum ChPT predictions
  no sign of explicit chiral symmetry breaking by the Wilson term

Can we understand this ?

Answer: Yes *

- An analysis in ChPT for nonzero lattice spacing a shows

  a, a^2 corrections are highly suppressed, typically to NNLO

* but ... (will be explained later on)

References:

OB, S. Necco, S. Schaefer, JHEP 0903 (2009) 006

essentially same conclusion in

A. Shindler, Phys. Lett. B 672 (2009) 82

Outline

- Introduction
- Some basics on Chiral Perturbation Theory (ChPT) for Lattice QCD
- ChPT in finite volume
  - Continuum ChPT: p-regime and epsilon regime
  - Extension to nonzero lattice spacings
  - Natural suppression of cut-off effects
- Computation of the leading lattice spacing corrections to correlation functions
- Numerics: Reanalysis of Stefan Schaefer's data
- Comment on twisted mass Wilson fermions
- Conclusions and outlook

ChPT for Lattice QCD: strategy

Two-step matching to effective theories

Lee, Sharpe 1998
Sharpe, Singleton 1998

Lattice QCD — Fundamental theory

↓

1. Symanzik effective theory — = Continuum QCD plus a-corrections

Symanzik 1983

↓

2. Chiral perturbation theory (ChPT) — = Continuum ChPT plus a-corrections

(for Wilson fermions → Wilson ChPT)

For an introduction and references → Les Houches lecture notes by M. Golterman, arXiv: 912.4042 [hep-lat]

Wilson ChPT
Chiral Lagrangian

- Continuum part for $N_f = 2$, degenerate quark mass m, $\hat{m} = 2Bm$

$$\mathcal{L}_2 = \frac{F^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial_\mu U^\dagger\right) - \frac{F^2}{4}\hat{m}\mathrm{Tr}\left(U + U^\dagger\right)$$

$$U(x) = \exp\left(\frac{2i}{F}\pi^a(x)T^a\right)$$

F, B: low-energy couplings (LECs)

- Leading lattice spacing corrections $\hat{a} = 2W_0 a$

$$\mathcal{L}_a = \hat{a}W_{45}\mathrm{Tr}\left(\partial_\mu U \partial_\mu U^\dagger\right)\mathrm{Tr}\left(U + U^\dagger\right) - \hat{a}\hat{m}W_{68}(\mathrm{Tr}\left(U + U^\dagger\right))^2$$

$$\mathcal{L}_{a^2} = \frac{F^2}{16}c_2 a^2(\mathrm{Tr}\left(U + U^\dagger\right))^2$$

W_{45}, W_{68}, c_2: additional LECs

Subtlety: quark mass in cont ChPT and WChPT are not the same !

Wilson ChPT
Currents and densities

- Continuum part

$$A^a_{\mu,\mathrm{ct}} = i\frac{F^2}{2}\mathrm{Tr}\left(T^a(U^\dagger\partial_\mu U - U\partial_\mu U^\dagger)\right)$$

$$P^a_{\mathrm{ct}} = i\frac{F^2 B}{2}\mathrm{Tr}\left(T^a(U - U^\dagger)\right)$$

- Leading lattice spacing corrections

W_{10}: additional LEC

$$A^a_{\mu,\mathrm{WChPT}} = A^a_{\mu,\mathrm{ct}}\left(1 + \frac{4}{F^2}\hat{a}W_{45}\mathrm{Tr}\left(U + U^\dagger\right)\right) + 2\hat{a}W_{10}\partial_\mu\mathrm{Tr}\left(T^a(U - U^\dagger)\right)$$

$$P^a_{\mathrm{WChPT}} = P^a_{\mathrm{ct}}\left(1 + \frac{4}{F^2}\hat{a}W_{68}\mathrm{Tr}\left(U^\dagger + U\right)\right)$$

Wilson ChPT

Power counting schemes in infinite volume

- Two parameters that explicitly break chiral symmetry: m and a

  ➥ power counting determined by their relative size

- Relevant ``regimes" for today's simulations

GSM regime	$m \sim a\Lambda_{\rm QCD}^2$	$\mathcal{L}_{\rm LO} = \mathcal{L}_2$
Aoki regime	$m \sim a^2\Lambda_{\rm QCD}^3$	$\mathcal{L}_{\rm LO} = \mathcal{L}_2 + \mathcal{L}_{a^2}$

GSM: generically small quark masses   S. Sharpe, J. Wu 2006

Wilson ChPT

Pion and PCAC quark mass at LO

- Pion mass

GSM regime $M_\pi^2 = 2Bm$

Aoki regime $M_\pi^2 = 2Bm - 2c_2 a^2$

Aoki regime: Different phase diagram depending on the sign of c_2

S. Sharpe, R. Singleton 1998

- PCAC mass

GSM regime $m_{\rm PCAC} = m\big(1 + a[d_A - d_P]\big)$

Aoki regime $m_{\rm PCAC} = \Big(m - \frac{c_2}{B}a^2\Big)\big(1 + a[d_A - d_P]\big)$

d_A, d_P : combinations of previously defined LECs

- Combine results ➞ M_π^2 as a function of $m_{\rm PCAC}$

Outline

- Introduction
- Some basics on Chiral perturbation theory (ChPT) for Lattice QCD
- ChPT in finite volume
  - Continuum ChPT: p-regime and epsilon regime
  - Extension for nonzero lattice spacing
  - Natural suppression of cut-off effects
- Computation of the leading lattice spacing corrections to correlation functions
- Numerics: Reanalysis of Stefan Schaefer's data
- Comment on twisted mass Wilson fermions
- Conclusion and outlook

Continuum ChPT in finite volume

- p-expansion for $M_\pi L \gg 1$

$$\partial_\mu \sim p \qquad m \sim p^2 \qquad 1/L \sim p$$

- p-expansion breaks down for $M_\pi L \approx 1$ or smaller

collective zero-mode U_0 has to be treated exactly:

$$U(x) = \exp\left(\frac{2i}{F}\xi^a(x)T^a\right)U_0$$

$$\int \prod_{x,a} [d\xi^a(x)] \qquad\qquad \int [dU_0]$$

perturbative
Saddle-point expansion

exact

Continuum ChPT in the epsilon regime
Why exact integration over the constant mode ?

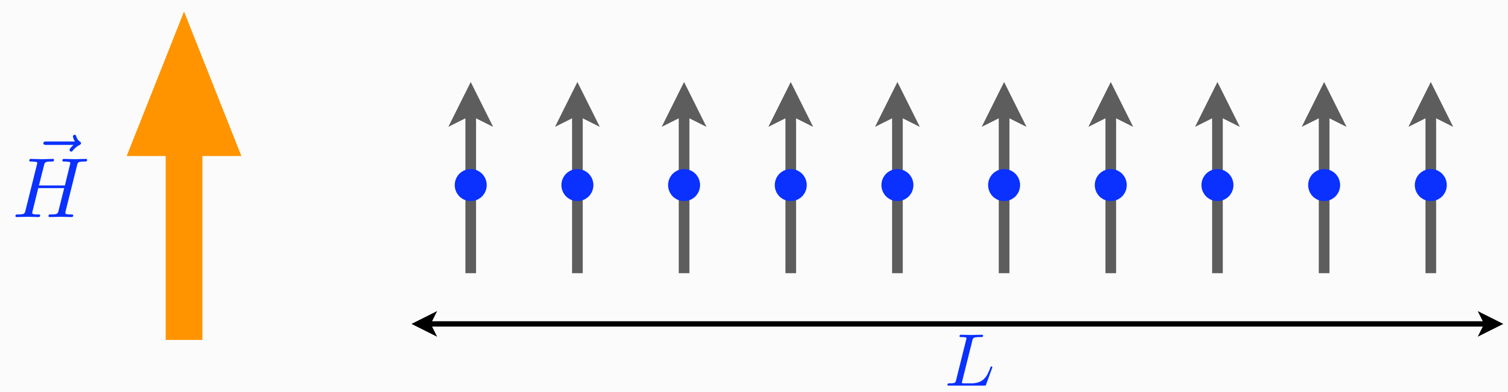
- Switch to *magnetic* language: Spin system in an external magnetic field

$$\mathcal{L}_{\rm eff} = \frac{F^2}{2}\partial_\mu \vec{S}\cdot\partial_\mu \vec{S} - \Sigma\, \vec{H}\cdot\vec{S} \qquad \vec{S}\in S^n$$

effective Lagrangian for the linear O(N) sigma model with SSB O(N) → O(N-1)

P. Hasenfratz, H. Leutwyler 1990

- Alignment by the magnetic field → spins point into direction of $\vec{H}$



Continuum ChPT in the epsilon regime
Why exact integration over the constant mode ?

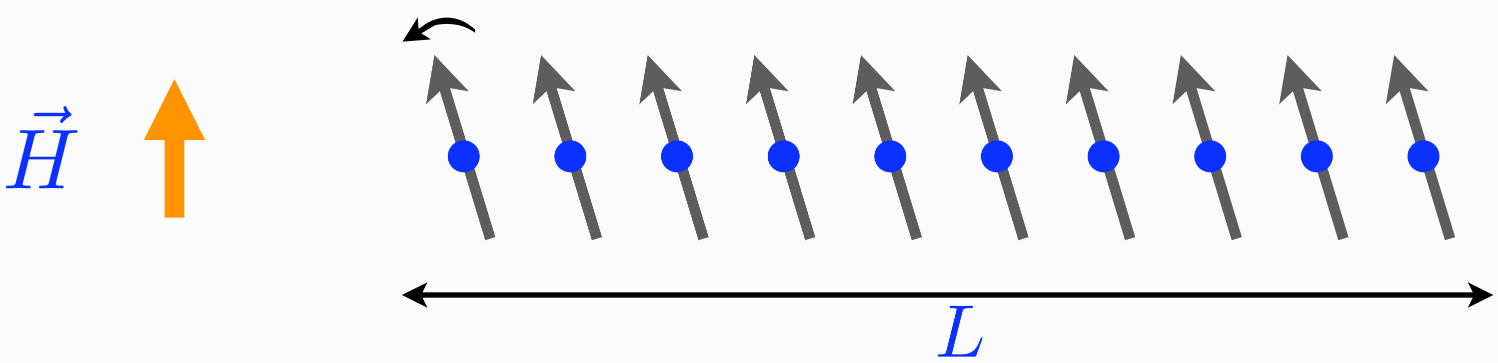
- Switch to *magnetic* language: Spin system in an external magnetic field

$$\mathcal{L}_{\rm eff} = \frac{F^2}{2}\partial_\mu \vec{S}\cdot\partial_\mu \vec{S} - \Sigma\, \vec{H}\cdot\vec{S} \qquad\qquad \vec{S}\in S^n$$

effective Lagrangian for the linear O(N) sigma model with SSB O(N) → O(N-1)

P. Hasenfratz, H. Leutwyler 1990

- Alignment by weaker magnetic field → spins are allowed to move collectively



constant collective mode is no longer suppressed

Continuum ChPT in the epsilon regime

Why exact integration over the constant mode ?

- Same physics in QCD with replacement
  - Spin(field) $\rightarrow$ Pion(field)
  - Magnetic field $\rightarrow$ Quark mass

Continuum ChPT in the epsilon regime
Modified power counting

- Pion propagator

$$G(x-y) = \frac{1}{V}\sum_k \frac{e^{-ik(x-y)}}{E_k^2} = \frac{1}{M_\pi^2 V} + \ldots$$

diverges in the chiral limit

- Reordering of the perturbative expansion necessary J. Gasser, H. Leutwyler 1987

Achieved by the so-called epsilon-expansion

$$\partial_\mu \sim \epsilon \qquad 1/L \sim \epsilon \qquad M_\pi \sim \epsilon^2 \quad \Rightarrow \quad m \sim \epsilon^4$$

➥ $M_\pi^2 V \sim \epsilon^0$

Zero-mode propagator is unsuppressed in the epsilon expansion
equivalent to treating the zero-mode exactly

Wilson ChPT in the epsilon regime

How to count a ?

- Counting determined
  1. as in continuum $1/L \sim \epsilon$ and $M_\pi^2 L^4 \sim \epsilon^0$
  2. by the relative size of m and a

- GSM regime: $M_\pi^2 = 2Bm \quad \Rightarrow \quad m \sim \epsilon^4 \quad \Rightarrow \quad a \sim \epsilon^4$

- Aoki regime: $M_\pi^2 = 2Bm - 2c_2 a^2$

a) $c_2 < 0$ $\quad \Rightarrow \quad m \sim a^2 \sim \epsilon^4 \quad \Rightarrow \quad a \sim \epsilon^2$

b) $c_2 > 0$ $\quad a^2 \sim \epsilon^4 \quad \Rightarrow \quad m \sim \epsilon^4 \quad \Rightarrow \quad a \sim \epsilon^2$

assumption !

In practice: a < 0.1 fm

Wilson ChPT in the epsilon regime

How to count a ?

- Summary

GSM regime	$a \sim \epsilon^4$
GSM* regime	$a \sim \epsilon^3$
Aoki regime	$a \sim \epsilon^2$

Intermediate regime between GSM and Aoki: GSM*

Epsilon expansion of correlators

- Split into continuum part + correction

$$O(x) = O_{\rm ct}(x) + \delta O(x)$$

$$S = S_{\rm ct} + \delta S$$

- Two-point correlator

$$\langle O_1 O_2 \rangle_W = \langle O_{1,\rm ct} O_{2,\rm ct} \rangle + \delta \langle O_1 O_2 \rangle$$

$$\delta \langle O_1 O_2 \rangle = \langle O_{1,\rm ct} \delta O_2 + \delta O_1 O_{2,\rm ct} \rangle$$
$$- \langle O_{1,\rm ct} O_{2,\rm ct} \delta S \rangle + \langle O_{1,\rm ct} O_{2,\rm ct} \rangle \langle \delta S \rangle$$

$\langle \ldots \rangle_W$ functional integral with e^{-S}

$\langle \ldots \rangle$ $e^{-S_{\rm ct}}$

Epsilon expansion of correlators

Example: Pseudo scalar correlator <PP>

- Pseudo scalar density $P^c = P^c_{\rm ct} + \delta P^c$ $\qquad \delta P^c \propto a\, P^c_{\rm ct}$

  ($a \leftarrow \epsilon^{n_a}$, $P^c_{\rm ct} \leftarrow \epsilon^0$)

  $$\Rightarrow \quad \delta P^c \sim \epsilon^{n_a} \qquad n_a = 2, 3, 4$$

- Action $\delta S = \delta S_a + \delta S_{a^2}$ $\qquad \delta S_a \propto a\, m \int d^4x \ldots$

  ($a \leftarrow \epsilon^4$, $d^4x \leftarrow \epsilon^{-4}$)

  $$\Rightarrow \quad \delta S_a \sim \epsilon^{n_a}$$

  $$\Rightarrow \quad \delta S_{a^2} \sim \epsilon^{2n_a - 4}$$

Epsilon expansion of correlators

Example: Pseudo scalar correlator <PP>

- PP correlator in the GSM regime $\Rightarrow$ $n_a = 4$

$\Rightarrow$ LO continuum part $\langle P^c_{\rm ct} P^c_{\rm ct} \rangle \sim \epsilon^0$

correction $\delta \langle P^c P^c \rangle \sim \epsilon^4$

- Strong suppression by four powers of epsilon $\Rightarrow$ NNLO correction

- possible explanation* for success of continuum formulae in describing the lattice data

* but: why is data in the GSM regime ?

Epsilon expansion of correlators
Example: Pseudo scalar correlator <PP>

- PP correlator in the GSM* regime $\Rightarrow$ $n_a = 3$

$\Rightarrow$ LO continuum part $\langle P^c_{\mathrm{ct}} P^c_{\mathrm{ct}} \rangle \sim \epsilon^0$

correction $\delta \langle P^c P^c \rangle \sim \epsilon^2$

- Still suppression by two powers of epsilon $\Rightarrow$ NLO correction
- Interesting: caused only by a^2 term in the action $\delta S_{a^2} \sim \epsilon^2$

  corrections linear in a are $\sim \epsilon^3$

  ➥ Symanzik improvement affects sub-leading corrections only

Epsilon expansion of correlators
Example: Pseudo scalar correlator <PP>

- PP correlator in the Aoki regime $\Rightarrow$ $n_a = 2$

$\Rightarrow$ LO continuum part $\langle P^c_{\rm ct} P^c_{\rm ct} \rangle \sim \epsilon^0$

``correction” $\delta \langle P^c P^c \rangle \sim \epsilon^0$

- δS_{a^2} gives zero-mode contribution $\sim \epsilon^0$
  - ➥ cannot be expanded
  - ➥ modified integrals over the constant mode U_0
  - ➥ non-trivial change to continuum results

Epsilon expansion of correlators
Other correlators

- Analogous results for currents
  - vector-vector
  - axial vector-axial vector
  - left-left
- Lattice spacing corrections are NNLO in the GSM regime
- Bottom line:

  Chiral symmetry breaking effects due to the Wilson term
  are highly suppressed (and less severe than previously thought)

Outline

- Introduction
- Some basics on Chiral perturbation theory (ChPT) for Lattice QCD
- ChPT in finite volume
  - Continuum ChPT: p-regime and epsilon regime
  - Extension for nonzero lattice spacing
  - Natural suppression of cut-off effects
- Computation of the leading lattice spacing corrections to correlation functions
- Numerics: Reanalysis of Stefan Schaefer's data
- Comment on twisted mass Wilson fermions
- Conclusion and outlook

Leading correction in the GSM* regime

Motivation

- Q: How do we know in which regime we are ?
- A: Compare data with formulae for each regime
  and use formulae that the data favors
- In practice: restrict yourself to NLO

  Reason: Only two LECs — decay constant in the chiral limit F

  chiral condensate $\Sigma = F^2 B = -\langle \overline{\psi}\psi \rangle$
- Relevant NLO formulae
  - GSM regime → continuum results F.C. Hansen 1990
  - GSM* regime → continuum results + a^2 correction — OB, Necco, Schaefer 2008; Shindler 2008

    advantage: Only one more LEC: c_2

Leading correction in the GSM* regime

- Leading correction in the GSM* regime:

$$\delta\langle PP\rangle\Big|_{\text{leading}} = -\langle P_{\text{ct}} P_{\text{ct}} \delta S_{a^2}\rangle + \langle P_{\text{ct}} P_{\text{ct}}\rangle\langle\delta S_{a^2}\rangle$$

↳ leading order contimuum expression

$$\delta S_{a^2} = \frac{\rho}{16}(\text{Tr}(U_0 + U_0^\dagger))^2 \qquad \rho = F^2 c_2 a^2 V$$

- Expectation values → SU(N_f) integrals, here $N_f = 2$

$$\langle \ldots \rangle = \frac{1}{Z_0}\int_{SU(2)} [dU_0] \ldots e^{\frac{\mu}{2}\text{Tr}(U_0 + U_0^\dagger)} \qquad \mu = m\Sigma V \qquad Z_0 = \langle 1 \rangle$$

➥ $\langle\delta S_{a^2}\rangle = \rho\left(1 - \frac{3}{2\mu}\frac{I_2(2\mu)}{I_1(2\mu)}\right)$ $\qquad I_n(z)$: modified Bessel functions

Leading correction in the GSM* regime
PCAC mass

- The PCAC mass

$$m_{\rm PCAC} = \frac{\langle \partial_\mu A^a_\mu(x) P^a(0)\rangle}{2\langle P^a(x) P^a(0)\rangle}$$

- PCAC mass as a function of m
  - Continuum ChPT $\quad m_{\rm PCAC} = m$
  - WChPT incl. a^2 $\quad m_{\rm PCAC} = m\left[1+\rho\left(\frac{2}{\mu^2} - \frac{I_1(2\mu)}{\mu I_2(2\mu)}\right)\right]$

➥ $m = m(m_{\rm PCAC})$

express correlators as functions of the PCAC mass

Leading correction in the GSM* regime

Results for PP correlator

- Used in practice: integrated correlator

$$C_{PP}(t) = \int d^3\vec{x}\, C_{PP}(x-y)\Big|_{y=0} = C_{PP,\mathrm{ct}}(t) + C_{PP,a^2}(t)$$

$$x_0 = t$$

- Continuum part (schematic) $C_{PP,\mathrm{ct}}(t) = \alpha + \beta t^2$

  α, β : functions of m_{PCAC}

  F.C. Hansen 1990

- a^2 part $C_{PP,a^2}(t) = \rho L^3 \Sigma^2 \Delta_{a^2}$ $\rho = F^2 c_2 a^2 V$

  Δ_{a^2} is a complicated function of the PCAC mass

$$\Delta_{a^2} = \frac{4\mu^2 I_1^3(2\mu) - 11\mu I_1^2(2\mu) I_2(2\mu) + 2(3-2\mu^2) I_1(2\mu) I_2^2(2\mu) + 5\mu I_2^3(2\mu)}{2\mu^3 I_1^2(2\mu) I_2(2\mu)}$$

Leading correction in the GSM* regime

Results for other correlators

- Results for other correlators

vector-vector $$C_{VV,a^2}(t) = -\rho \frac{F^2}{T} \Delta_{a^2}$$

axial vector-axial vector $$C_{AA,a^2}(t) = +\rho \frac{F^2}{T} \Delta_{a^2}$$

left-left $$C_{LL,a^2}(t) = 0$$

- All correlators involve the same Δ_{a^2}
- All correlators depend on 3 LECs only: F, Σ, c_2

➥ 3 fit parameter in practice

Outline

- Introduction
- Some basics on Chiral perturbation theory (ChPT) for Lattice QCD
- ChPT in finite volume
  - Continuum ChPT: p-regime and epsilon regime
  - Extension for nonzero lattice spacing
  - Natural suppression of cut-off effects
- Computation of the leading lattice spacing corrections to correlation functions
- Numerics: Reanalysis of Stefan Schaefer's data
- Comment on twisted mass Wilson fermions
- Conclusion and outlook

How big are the a^2 corrections?

- Consider the ratios

$$R_{\rm PP} = \left| \frac{C_{\rm PP,a^2}}{C_{\rm PP,ct}} \right| \quad \text{at} \quad t = \frac{T}{2}$$

$$R_{\rm AA} = \text{ same but } P \to A$$

- Typical parameter values

$$F = 90\,\mathrm{MeV}$$

$$a = 0.08\,\mathrm{fm}$$

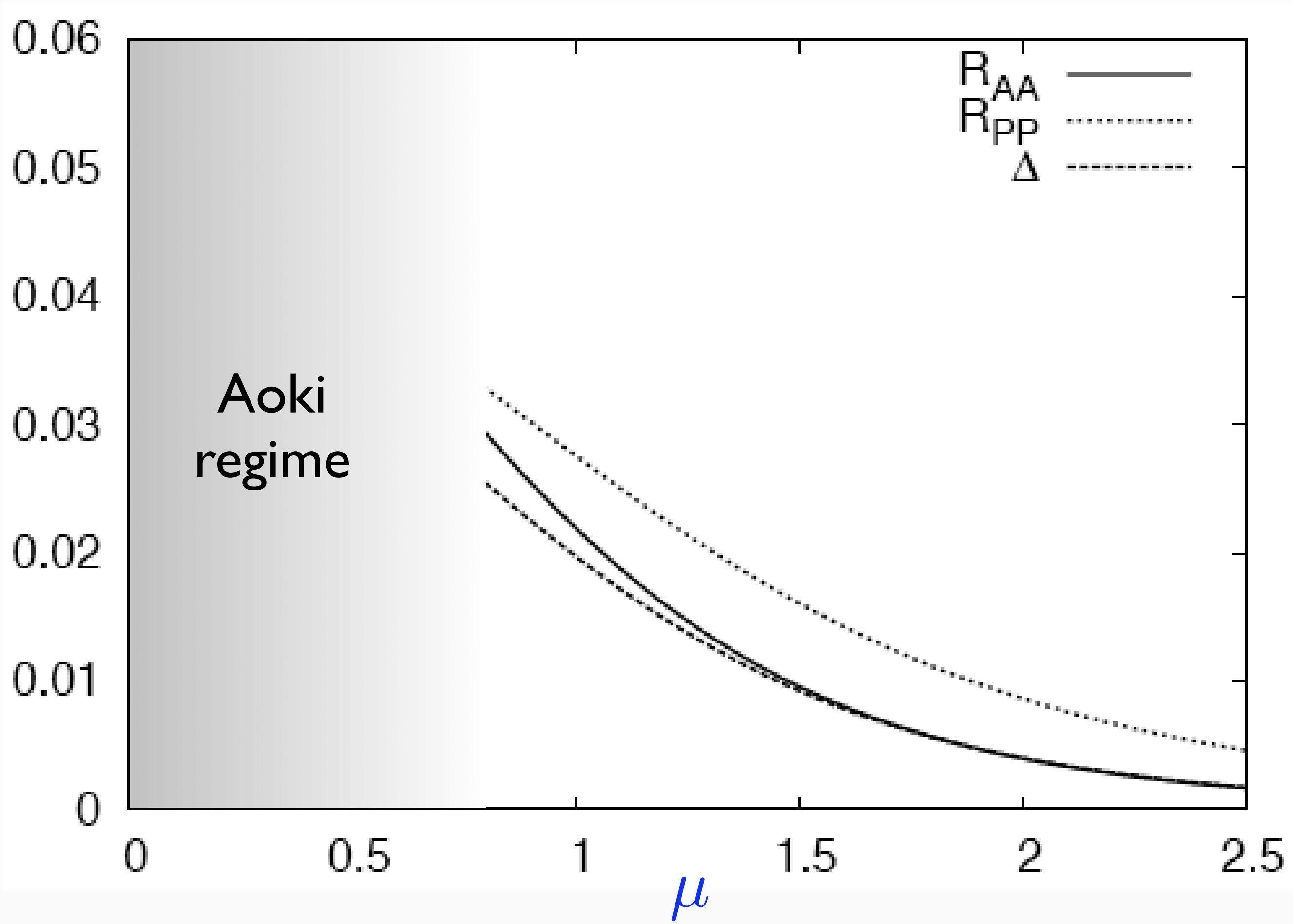
$$L = 1.92\,\mathrm{fm} \qquad N = 24 \quad L = T$$

$$c_2 = (500\,\mathrm{MeV})^4 \; *$$

* from pion mass splitting in twisted mass QCD (ETM collaboration)

$$M_{\pi^0}^2 - M_{\pi^\pm}^2 = 2c_2a^2 \quad \Rightarrow \quad c_2 \approx (550\,\mathrm{MeV})^4$$

How big are the a^2 corrections?



$\mu = m_{\mathrm{PCAC}} \Sigma V$

Conclusion: small correction ≈ few %

Reanalysis of lattice data

- Lattice data of A. Hasenfratz, R. Hoffmann and S. Schaefer
  Phys. Rev. D78 (2008) 054511

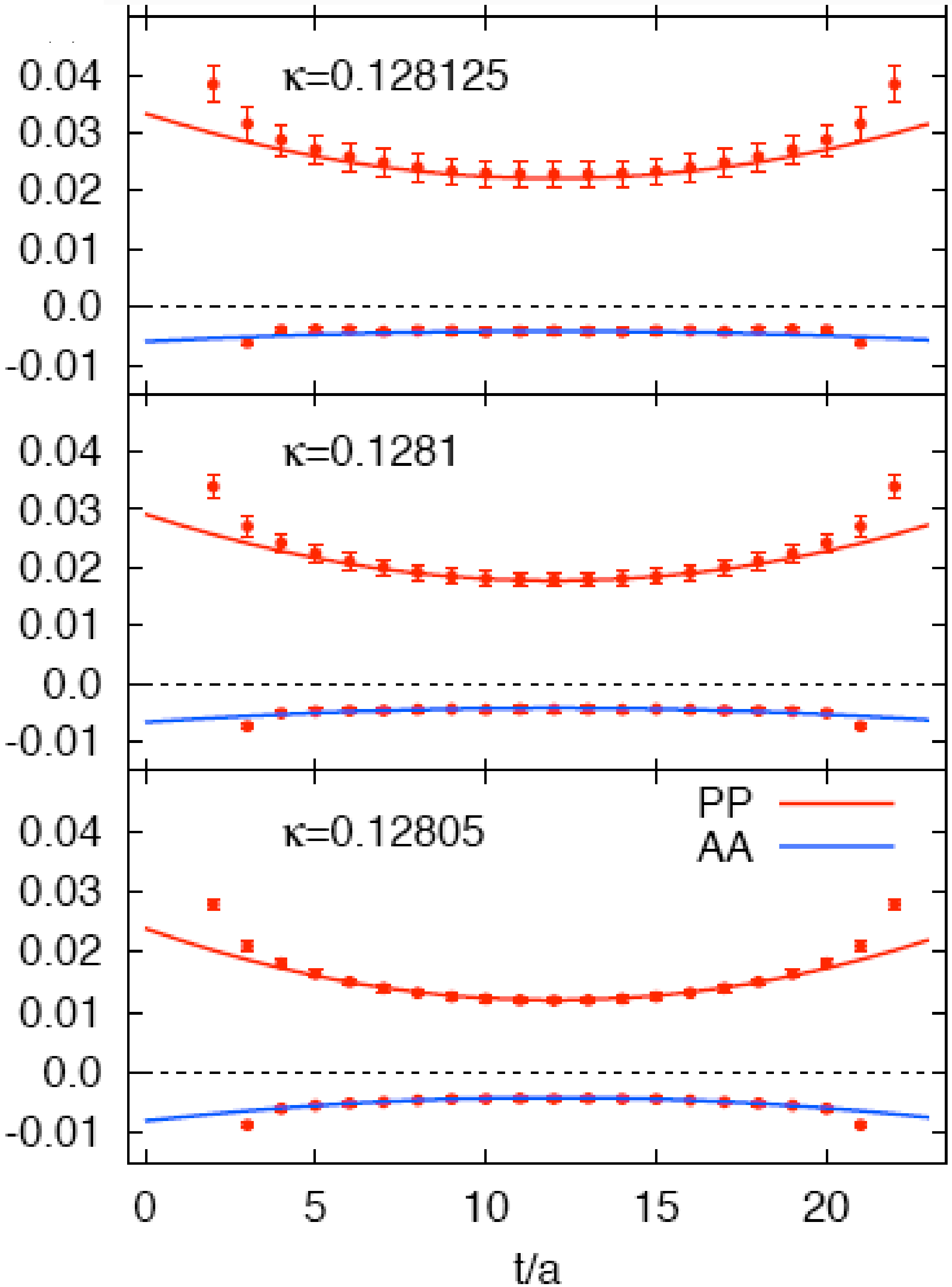
$N_f = 2$ tree-level improved nHYP Wilson fermions

$a \approx 0.115\,\mathrm{fm}$

$L \approx 2.8\,\mathrm{fm} \qquad N = 24$

$\mu \approx 2.1 \ldots 5.0$ (4 quark masses) $\mu = m_{\mathrm{PCAC}} \Sigma V$

Reanalysis of lattice data



Simultaneous fit to data for

- all 4 masses (3 shown)
- PP and AA correlator

$\chi^2/\mathrm{dof} = 0.3(1)$ (uncorr.)

Results for fit parameter

$$\Sigma^{1/3} = 250(4)\,\mathrm{MeV}^*$$

$$F = 87(3)\,\mathrm{MeV}$$

$$c_2 = 0.02(8)\,\mathrm{GeV}^4$$

* at scale 2 GeV

Reanalysis of lattice data

Main conclusions

- c_2 is compatible with zero
  ➡ Effects of explicit chiral symmetry breaking are negligible

- Results for F and Σ are unchanged for $c_2 \equiv 0$
  ➡ Data is in the GSM regime

- Question*: Why are the corrections so small ? Use of nHYP links ???

- Caution: Other lattice actions may have larger corrections

* recall the "but"

Comment on twisted mass Wilson ChPT

- Include twisted mass term: $m \longrightarrow m + i\mu\sigma^3$
- Essentially unchanged
  - epsilon expansion of correlation functions
  - high suppression of lattice artifacts to
    - NNLO in GSM regime
    - NLO in GSM* regime
    - LO in Aoki regime
- Slightly changed
  - details in the leading correction for the GSM* regime, among others breaking of isospin:

$$\Delta_{a^2} \longrightarrow \begin{cases} \Delta_{a^2}^{11,22} \\ \Delta_{a^2}^{33} \end{cases}$$

SU(2) flavor indices

Details in OB, S.Necco, A.Shindler, JHEP04 (2010) 053

Conclusions

- Epsilon expansion in Wilson ChPT shows
  - Lattice spacing corrections are highly suppressed, typically to NNLO
- May explain the good fit of recent lattice data by continuum ChPT
- Leading corrections for smaller quark masses are O(a^2) and
  - involve one unknown low-energy coupling c_2
  - are computable by standard methods
- Main conclusions
  - Epsilon regime simulations with Wilson fermions are in good shape
  - Spend time on large volumes, not on
    - exact chiral symmetry
    - small lattice spacings

Outlook

- Higher corrections (linear in a) for the GSM* regime
- Aoki regime $\rightarrow$ you do not want to be there in practice
  but the math is interesting
- Residual pion mass in a finite volume

$$M_{\pi,\mathrm{L}} = \frac{3}{2F^2L^3}\left[1 + \frac{4}{15}F^2L^3B^2m^2 + \ldots\right] \rightarrow \frac{3}{2F^2L^3} = M_{\pi,\mathrm{res}}$$

SU(2) Cont ChPT
Leutwyler 1987

Studied on the lattice with Wilson fermions by Bietenholz et.al.
arXiv:1002.1696[hep-lat]

Question: How modified in WChPT ??? Necessarily in the Aoki regime ...

Additional material

- Phase diagram

Digression: c_2

- c_2 determines the phase diagram — S.Sharpe, R.Singleton '99
  - $c_2 > 0$ Existence of Aoki phase separated by a 2nd order phase transition
    parity and flavor are spontaneously broken
    existence of massless pions $a \neq 0$
  - $c_2 < 0$ 1st order phase transition
    Pions have minimal pion mass $M^2_{\pi,\mathrm{min}} = 2|c_2|a^2$
- Hard to measure in practice
  - Isospin 2 scattering length
    Aoki, OB, Biedermann 2008
    $$a_0^2 = -\frac{1}{16\pi F^2}\left(M_\pi^2 + 2c_2a^2\right)$$
    tree-level !
  - Pion mass splitting
    in twisted mass QCD
    Scorzato 2004
    $$M^2_{\pi^0} - M^2_{\pi^\pm} = 2c_2a^2$$