

Closed $SU(N)$ Flux Tubes as Bosonic Strings

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Joint Lattice Seminar



* With Dr. Barak Bringoltz and Dr. Mike Teper

based on:

arXiv:0709.0693, arXiv:0812.0334, arXiv:0912.3238

and mostly on unpublished results



Overview

1. Introduction.
2. Theoretical Expectations.
 - Nambu Goto.
 - Effective String Theory.
3. Lattice Calculation.
4. $D = 2 + 1$.
 - Quantum Numbers/Operators.
 - Results.
5. $D = 3 + 1$.
 - Quantum Numbers/Operators.
 - Results.
6. Conclusions.
7. Future Work.

1. Introduction: General

General question:

→ What effective string theory describes flux tubes in $SU(N)$ gauge theories?

Two cases:

→ Open string

→ Closed string (torelon)

During the last decade:

→ $3D, 4D$ with $Z_2, Z_4, SU(N \leq 8)$ (Caselle and collaborators, Gliozzi and collaborators, Kuti and collaborators, Lüscher & Weisz, Majumdar and collaborators, Teper and collaborators)

Recently in $D = 2 + 1$: Nambu-Goto works well for:

→ Fundamental Closed String Spectrum. (A.A, B. Bringoltz, M. Teper)[arXiv:hep-lat/0709.0693]

→ $k = 2$ Closed String Spectrum. (A.A, B. Bringoltz, M. Teper)[arXiv:hep-lat/0812.0334]

First attempt to extract the $D = 3 + 1$ torelon spectrum in $SU(3)$:

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→ **Closed string** (torelon) ← in $D = 2 + 1$ & $D = 3 + 1$

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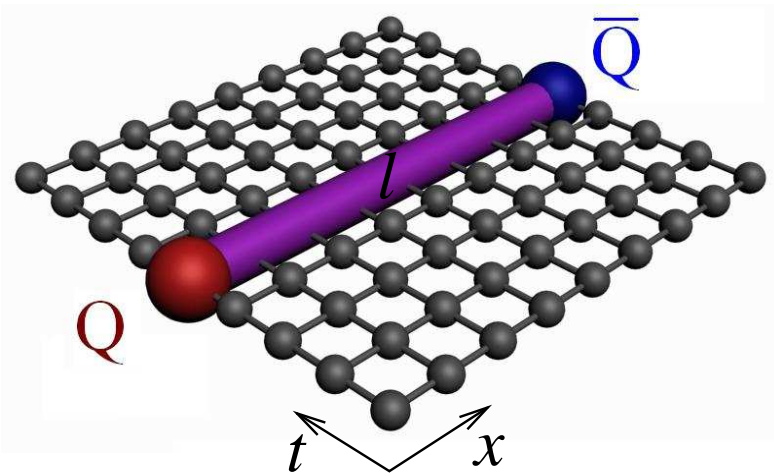
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1. Introduction: Open-Closed (torelons) flux tubes

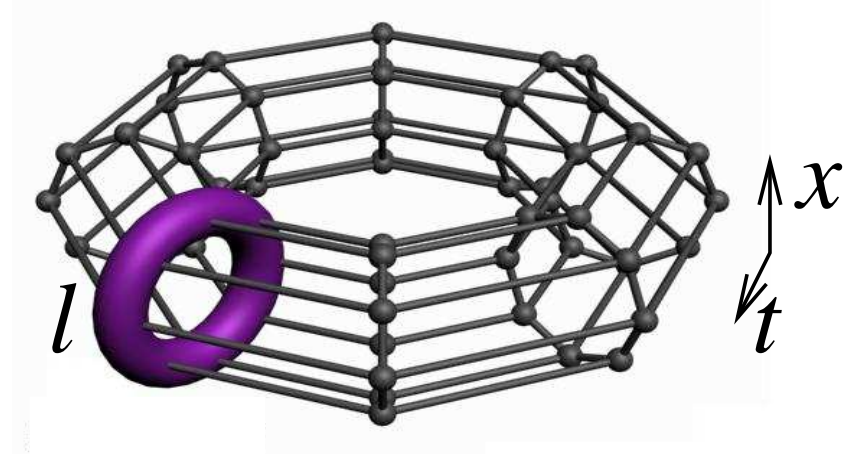
Open flux tube



$$\Phi(l, t) = \psi^\dagger(0, t)U(0, l; t)\psi(l, t)$$

Periodic B.C
→

Closed flux tube



$$\Phi(l, t) = \text{Tr}U(l; t)$$

2. Theoretical Expectations: Nambu-Goto String

→ Spectrum:

$$E_{N_L, N_R, q, w}^2 = (\sigma l w)^2 + 8\pi\sigma \left(\frac{N_L + N_R}{2} - \frac{D-2}{24} \right) + \left(\frac{2\pi q}{l} \right)^2.$$

→ Described by:

1. The winding number w ($w=1$),
2. The winding momentum $p_{\parallel} = 2\pi q/l$ with $q = 0, \pm 1, \pm 2, \dots$
3. The transverse momentum p_{\perp} ($p_{\perp} = 0$),
4. N_L and N_R connected through the relation: $N_L - N_R = qw$.

$$N_L = \sum_{k>0} \sum_{n_L(k)>0} n_L(k)k \quad \text{and} \quad N_R = \sum_{k'>0} \sum_{n_R(k')>0} n_R(k')k'$$

→ Construction of states:

$$(\alpha_{-k_1}^{i_1})^{n_L(k_1)} \dots (\alpha_{-k_{m_L}}^{i_{m_L}})^{n_L(k_{m_L})} (\bar{\alpha}_{-k'_1}^{i'_1})^{n_R(k'_1)} \dots (\bar{\alpha}_{-k'_{m_R}}^{i'_{m_R}})^{n_R(k'_{m_R})} | 0 \rangle$$

$$(i = 1, \dots, D-2)$$

$$\text{Example: } \alpha_{-2}\alpha_{-1}\bar{\alpha}_{-1} | 0 \rangle$$

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$$\text{Example: } \alpha_{-2}\alpha_{-1}\bar{\alpha}_{-1} | 0 \rangle \rightarrow N_L = 3, N_R = 1, q = 2 (w = 1)$$

2. Theoretical Expectations: Effective String Approaches

→ **Polchinski & Strominger at any D** [[Phys. Rev. Lett. 67 \(1991\) 1681](#)] (L.O):

$$E_n = \sigma l + \frac{4\pi}{l} \left(n - \frac{D-2}{24} \right) + \mathcal{O}(1/l^2).$$

→ **Lüscher & Weisz at $D = 2 + 1$** [[JHEP 0407 \(2004\) 014](#)](N.L.O):

$$E_n = \sigma l + \frac{4\pi}{l} \left(n - \frac{1}{24} \right) - \frac{8\pi^2}{\sigma l^3} \left(n - \frac{1}{24} \right)^2 + \mathcal{O}(1/l^4)$$

→ **Drummond and Dass&Matlock** [[arXiv:hep-th/0411017/0606265](#)](N.L.O):

$$E_n = \sigma l + \frac{4\pi}{l} \left(n - \frac{D-2}{24} \right) - \frac{8\pi^2}{\sigma l^3} \left(n - \frac{D-2}{24} \right)^2 + \mathcal{O}(1/l^4)$$

→ **Aharony&Karzbrun at any D** [[arXiv:0903.1927](#)](N.N.L.O):

$$E_n = \sigma l + \frac{4\pi}{l} \left(n - \frac{D-2}{24} \right) - \frac{8\pi^2}{\sigma l^3} \left(n - \frac{D-2}{24} \right)^2 + \frac{32\pi^3}{\sigma^2 l^5} \left(n - \frac{D-2}{24} \right)^3 + \mathcal{O}(1/l^7)$$

→ **Recently Dass&collaborators at any D** [[arXiv:0911.3236](#)](All.O):

$$E_n = \sigma l + \frac{4\pi}{l} \left(n - \frac{D-2}{24} \right) - \frac{8\pi^2}{\sigma l^3} \left(n - \frac{D-2}{24} \right)^2 + \dots + \mathcal{O}(1/l^\infty) = E_{nN.G}$$

3. Lattice Calculation: General Strategy

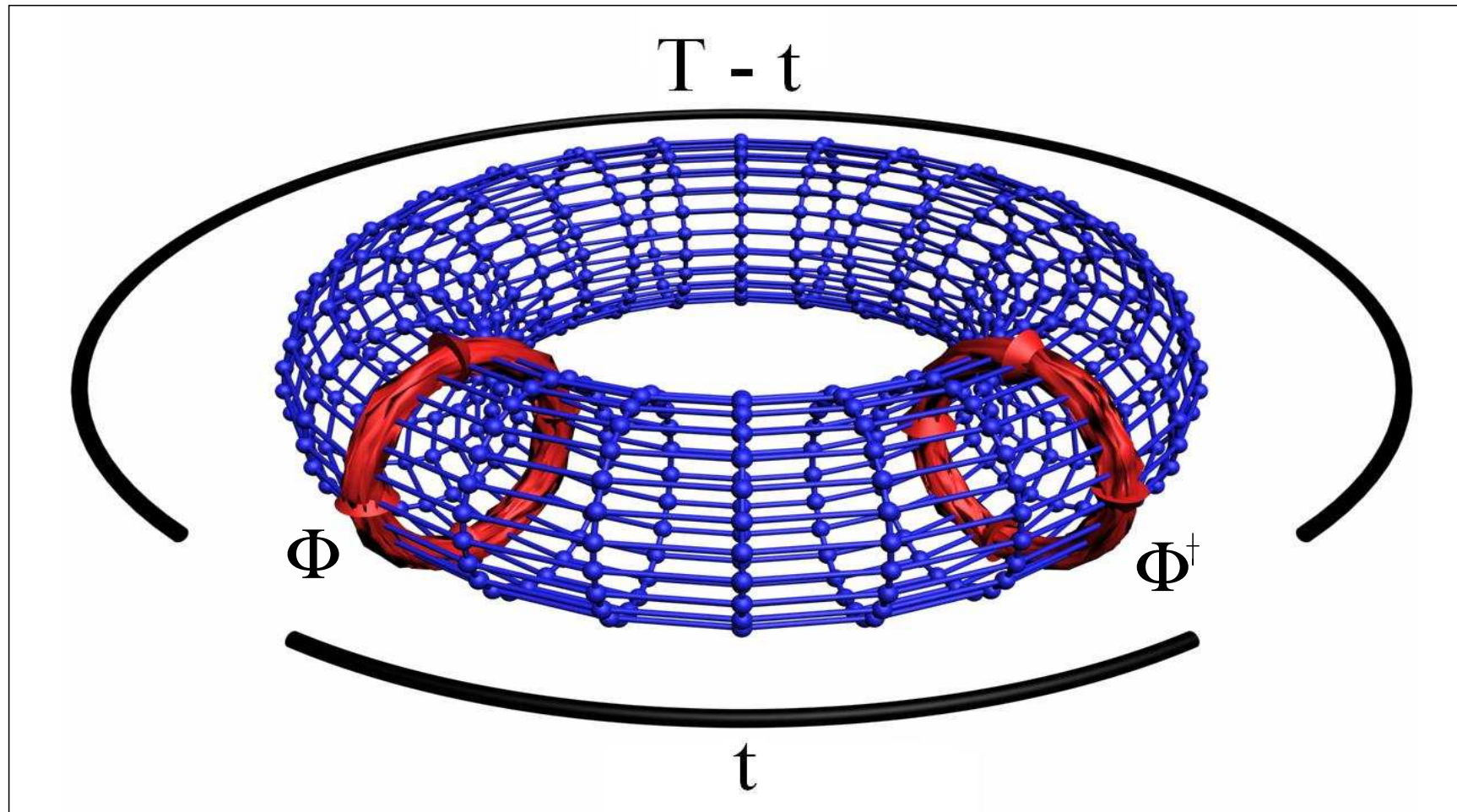
- Use the standard Wilson action
- Cabbibo-Marinary & Kennedy-Pendelton heat bath
- Construct a basis of operators $\Phi_i : i = 1, 2, \dots$ described by the right quantum numbers
- Calculate the correlation function (Matrix) $C_{ij}(t) = \langle \Phi_i^\dagger(t) \Phi_j(0) \rangle$
- Apply the variational technique
- Extract the correlators for a number of excited states $C_n(t) = \langle \Psi_n^\dagger(t) \Psi_n(0) \rangle$
- Energies can be calculated using the correlation functions of specific operators:

$$\begin{aligned} C(t) &= \langle \Psi^\dagger(t) \Psi(0) \rangle = \langle \Psi^\dagger(0) e^{-Ht} \Psi(0) \rangle \\ &= |\langle 0 | \Psi(0) | vac \rangle|^2 e^{-E_0 t} + \sum_{n=1} |\langle n | \Psi(0) | vac \rangle|^2 e^{-E_n t} \xrightarrow{t \rightarrow \infty} |\langle 0 | \Psi(0) | vac \rangle|^2 e^{-E_0 t} \end{aligned}$$

- By fitting the results, we extract the mass (energy) for each state.

3. Lattice Calculation: Correlation Function

Pictorialisation of the Correlation Function



4. $D=2+1$

4. D=2+1: Lattice Calculation

→ We define our theory on a 3D Euclidean lattice with volume:

$$L_{\parallel} \times L_{\perp_1} \times L_T.$$

→ We use the standard Wilson Action:

$$S_L = \beta \sum_p \left\{ 1 - \frac{1}{N} \text{ReTr} U_p \right\} \quad \text{with} \quad \beta = \frac{2N}{ag^2}$$

→ Monte Carlo simulations:

- $SU(3)$ at $\beta = 21$, $a\sqrt{\sigma} \simeq 0.174$
- $SU(3)$ at $\beta = 40$, $a\sqrt{\sigma} \simeq 0.087$
- $SU(4)$ at $\beta = 50$, $a\sqrt{\sigma} \simeq 0.131$
- $SU(5)$ at $\beta = 80$, $a\sqrt{\sigma} \simeq 0.130$
- $SU(6)$ at $\beta = 80$, $a\sqrt{\sigma} \simeq 0.172$
- $SU(6)$ at $\beta = 171$, $a\sqrt{\sigma} \simeq 0.086$

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- **SU(6) at $\beta = 171$, $a\sqrt{\sigma} \simeq 0.086$ Today:**
 - * $q = 0$
 - * $q = 1$
 - * $q = 2$

4. $D=2+1$: Symmetries

→ Flux tube:

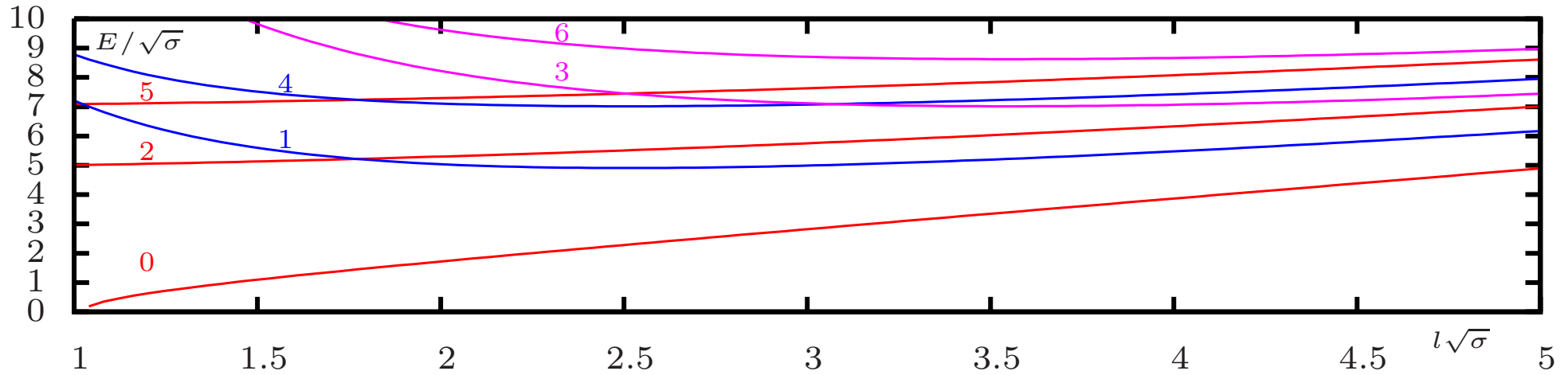
- Lattice Symmetry of Rotations in $D = 2 + 1 \rightarrow$ Reflections on transverse line
- States are described by the quantum number of Parity $P = \pm$
- Need to construct operators transforming irreducibly under P

→ Bosonic String:

- String states can be characterized as irreducible representations of $SO(D - 2)$.
- In $D = 2 + 1$ this becomes the transverse parity ($P = \pm$).
- Under Parity: $\alpha_{-k} \xrightarrow{P} -\alpha_{-k}$ and $\bar{\alpha}_{-k} \xrightarrow{P} -\bar{\alpha}_{-k}$
- Even number of phonons $\rightarrow P = + (\alpha_{-2}\bar{\alpha}_{-1} | 0\rangle)$
- Odd number of phonons $\rightarrow P = - (\alpha_{-1} | 0\rangle)$

4. D=2+1: Symmetries: Bosonic String

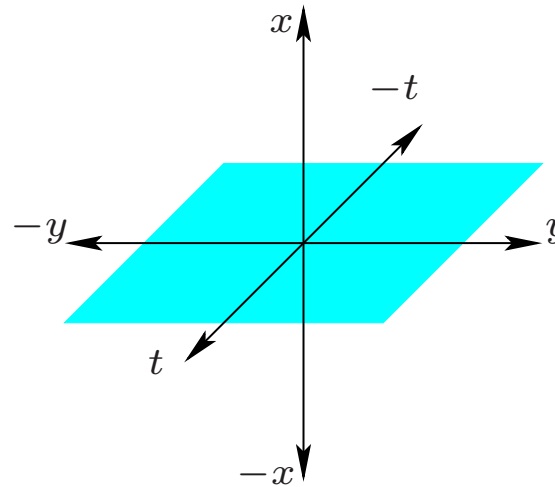
The seven lowest ($q = 0, 1, 2$) NG energy levels for the $w = 1$ closed string



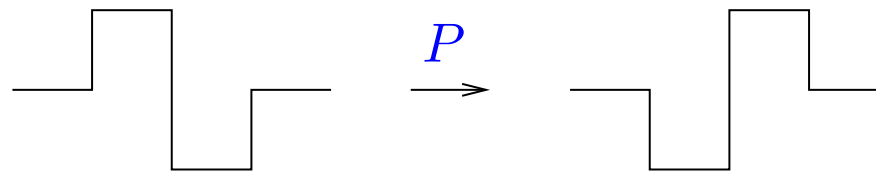
level	N_L	N_R	q	$P = +$	$P = -$
0	0	0	0	$ 0\rangle$	
1	1	0	1		$\alpha_{-1} 0\rangle$
2	1	1	0	$\alpha_{-1}\bar{\alpha}_{-1} 0\rangle$	
3	2	0	2	$\alpha_{-1}\alpha_{-1} 0\rangle$	$\alpha_{-2} 0\rangle$
4	2	1	1	$\alpha_{-2}\bar{\alpha}_{-1} 0\rangle$	$\alpha_{-1}\alpha_{-1}\bar{\alpha}_{-1} 0\rangle$
5	2	2	0	$\alpha_{-2}\bar{\alpha}_{-2} 0\rangle, \alpha_{-1}\alpha_{-1}\bar{\alpha}_{-1}\bar{\alpha}_{-1} 0\rangle$	$\alpha_{-2}\bar{\alpha}_{-1}\bar{\alpha}_{-1} 0\rangle, \alpha_{-1}\alpha_{-1}\bar{\alpha}_{-2} 0\rangle$
6	3	1	2	$\alpha_{-3}\bar{\alpha}_{-1} 0\rangle, \alpha_{-1}\alpha_{-1}\alpha_{-1}\bar{\alpha}_{-1} 0\rangle$	$\alpha_{-2}\alpha_{-1}\bar{\alpha}_{-1} 0\rangle$

4. D=2+1: Quantum Numbers: Parity

Parity reflection plane



Example:



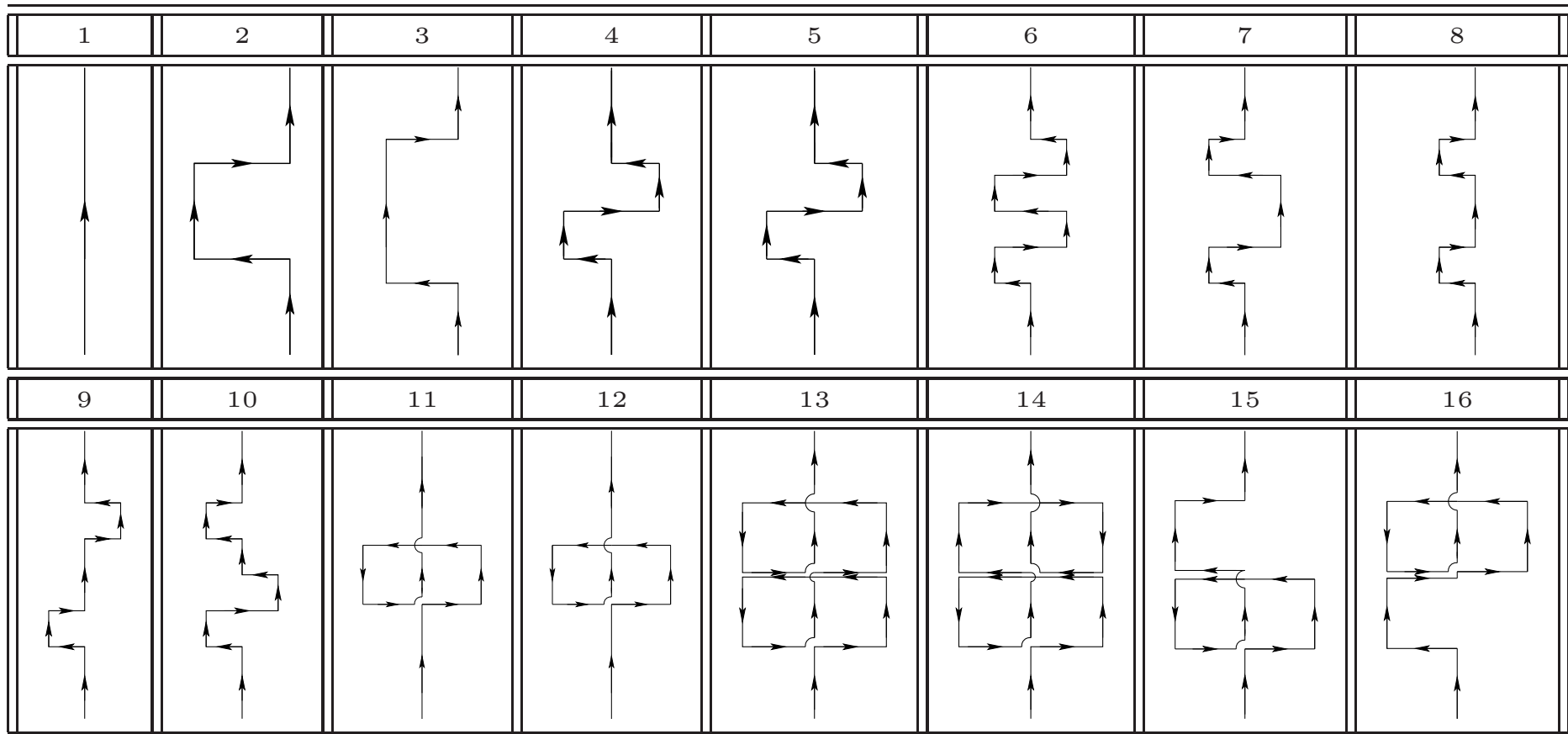
We can project onto $P = \pm$ using operators such as:

$$\Phi(q_{||}, P) = \frac{1}{L_{||}L_{\perp}} \sum_{x_{||}, x_{\perp}} \{ \text{Tr} \{ \text{---} \text{---} \} \pm \text{Tr} \{ \text{---} \text{---} \} \} e^{i2\pi q_{||} x_{||} / L_{||}}$$

4. $D=2+1$: Transverse deformations

→ ~ 200 operators

Tranverse deformations in 2 spatial dimensions:



4. D=2+1 Results: $N_L = N_R = 0, q = 0, P = +$ (Trivial)

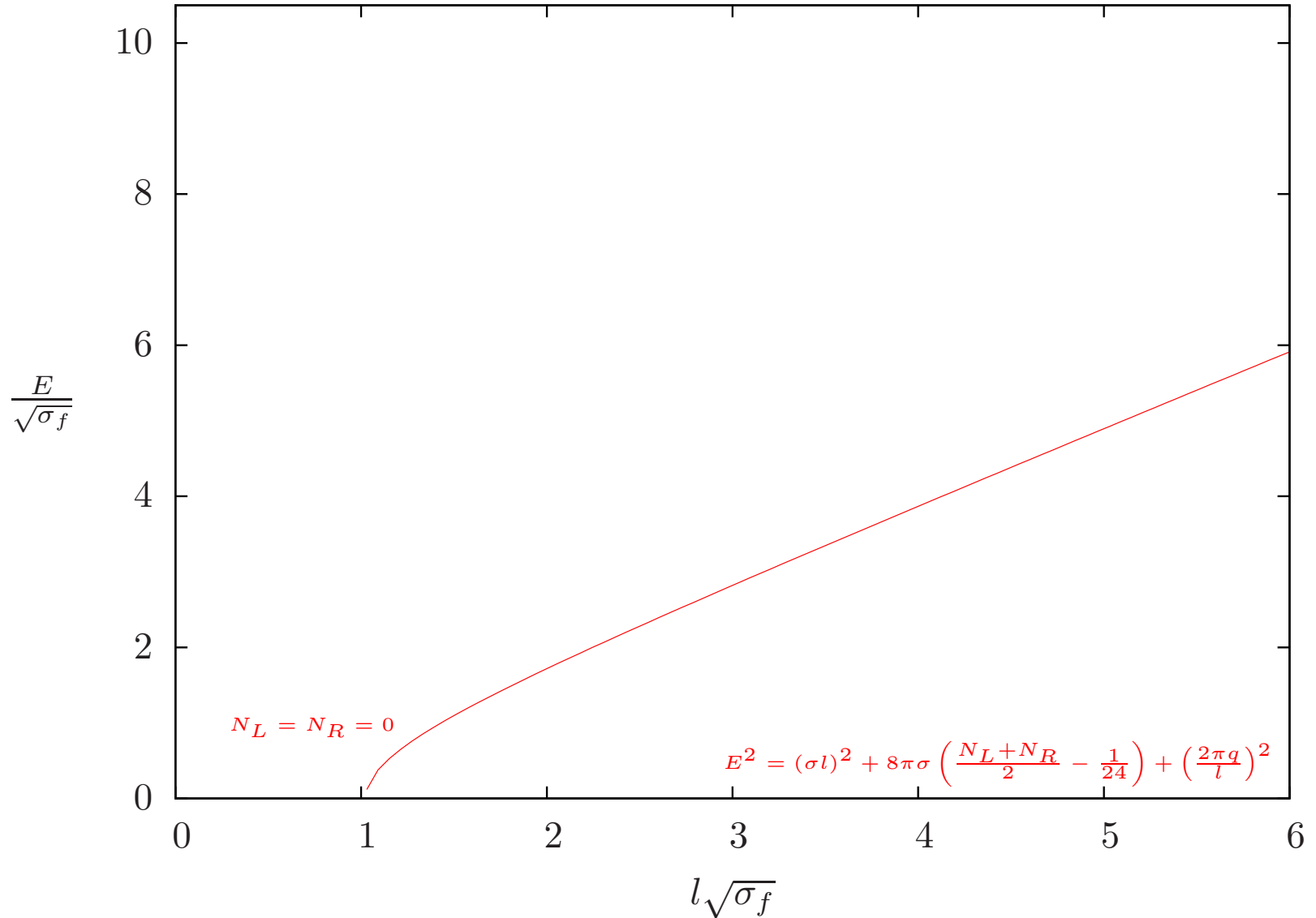
- $N_L = N_R = 0$ corresponds to the Ground State (No movers)
- String theoretically: $|0\rangle$
- Even number of phonons
- Expected to appear as $\{P = +, q = 0\}$ ground state
- Apply variational technique to the correlation matrix $P = +, q = 0$.
- Extraction of the ground state.
 - Fit data with Nambu-Goto or L.O/N.L.O/N.N.L.O $\rightarrow a^2\sigma_f$ (string tension)
 - This provides parameter free predictions for the energies in N.G:

$$E_{N_L, N_R, q, w}^2 = (\sigma_f l w)^2 + 8\pi\sigma_f \left(\frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left(\frac{2\pi q}{l} \right)^2.$$

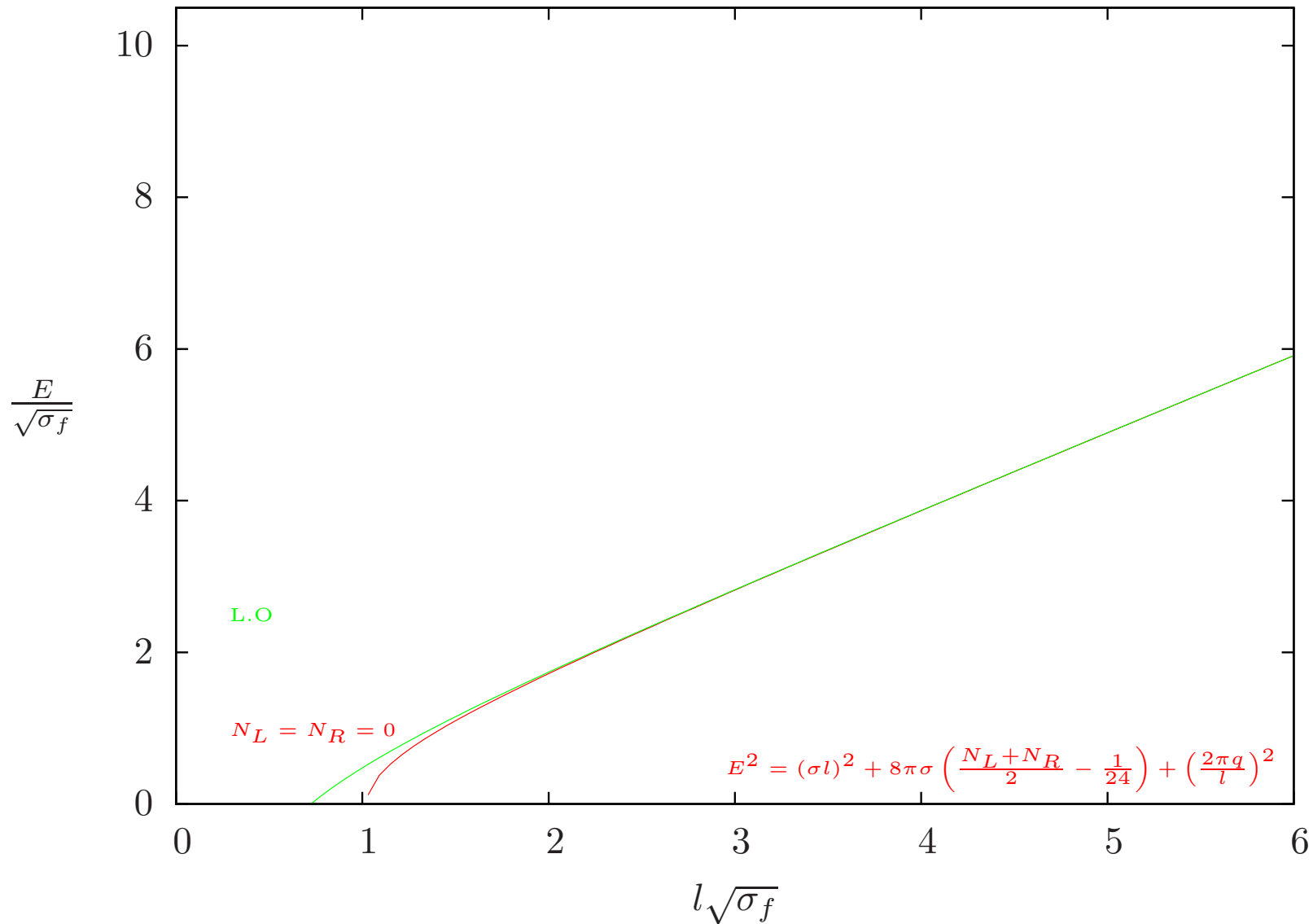
In lattice units:

$$a^2 E_{N_L, N_R, q, w}^2 = (a^2 \sigma_f L w)^2 + 8\pi a^2 \sigma_f \left(\frac{N_L + N_R}{2} - \frac{1}{24} \right) + \left(\frac{2\pi q}{L} \right)^2.$$

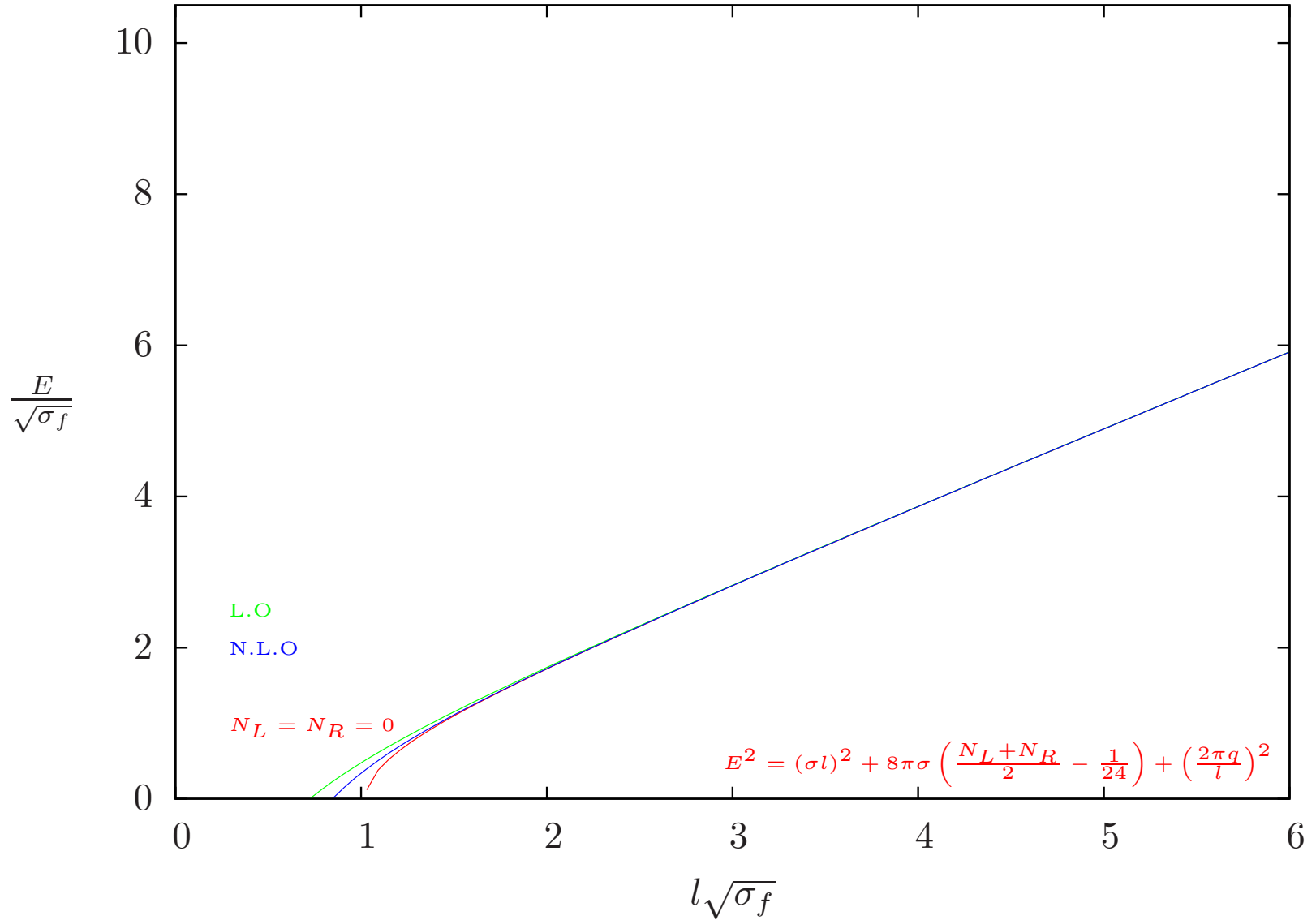
4. D=2+1 Results: $N_L = N_R = 0$, $q = 0$, $P = +$



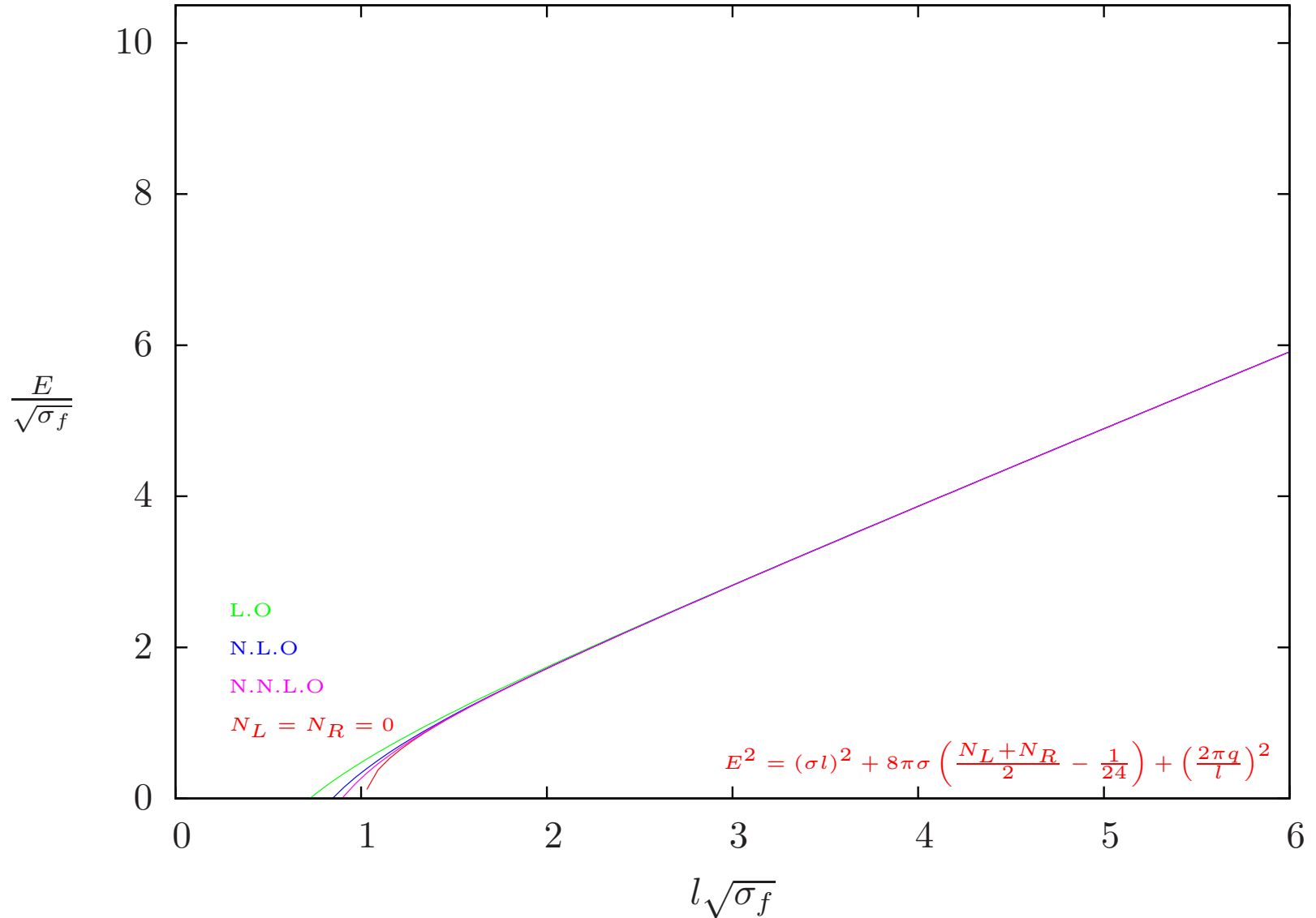
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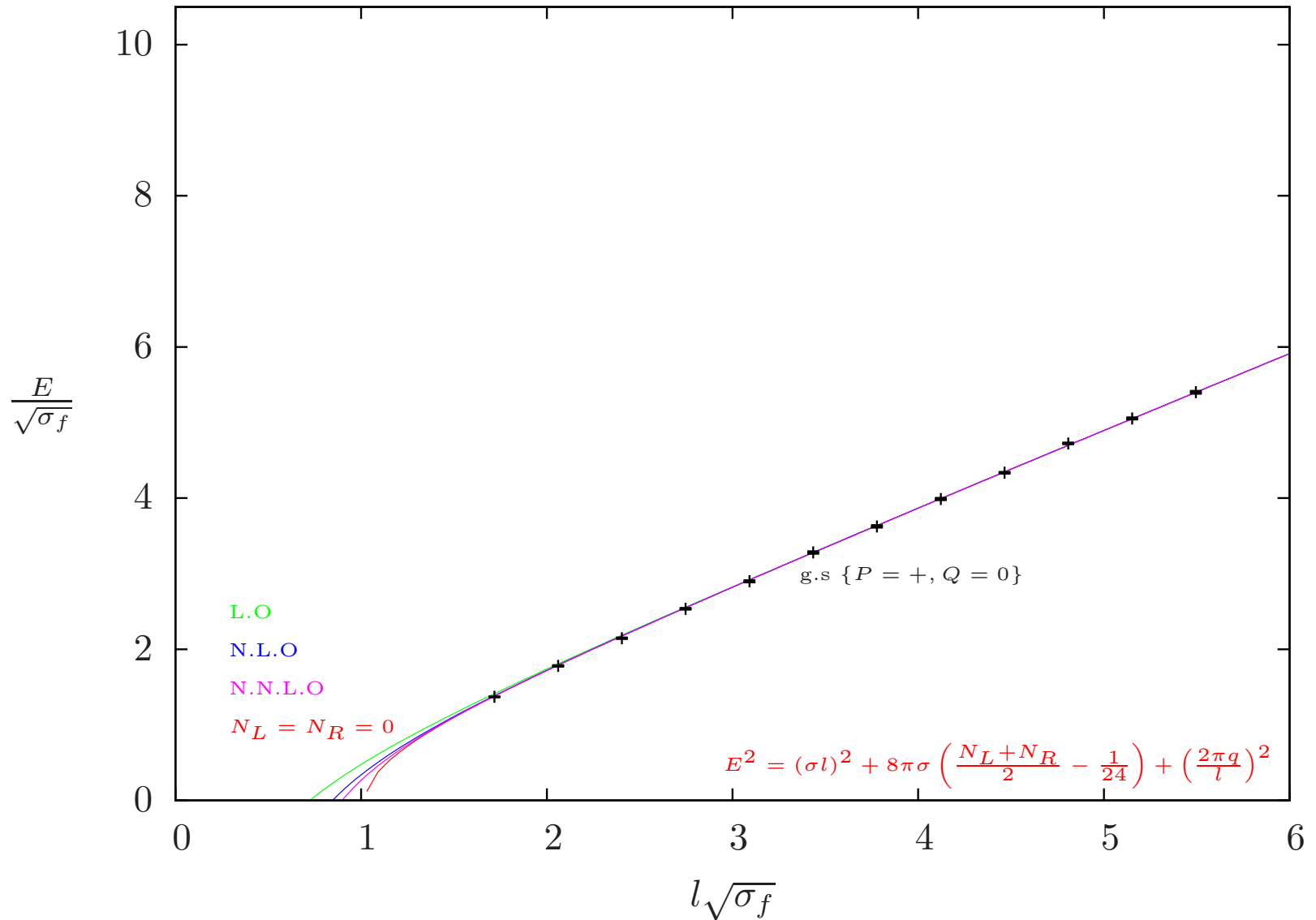
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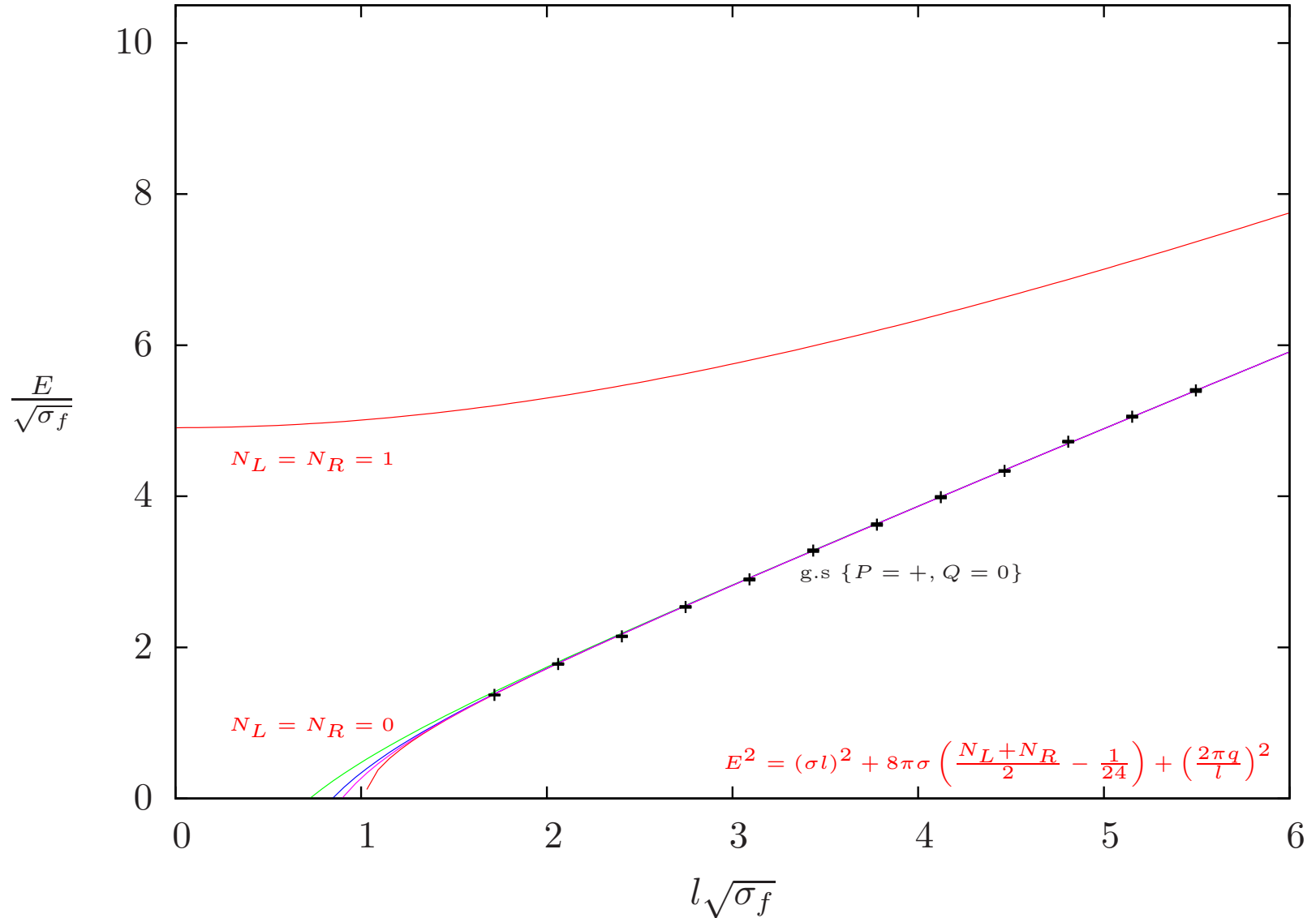
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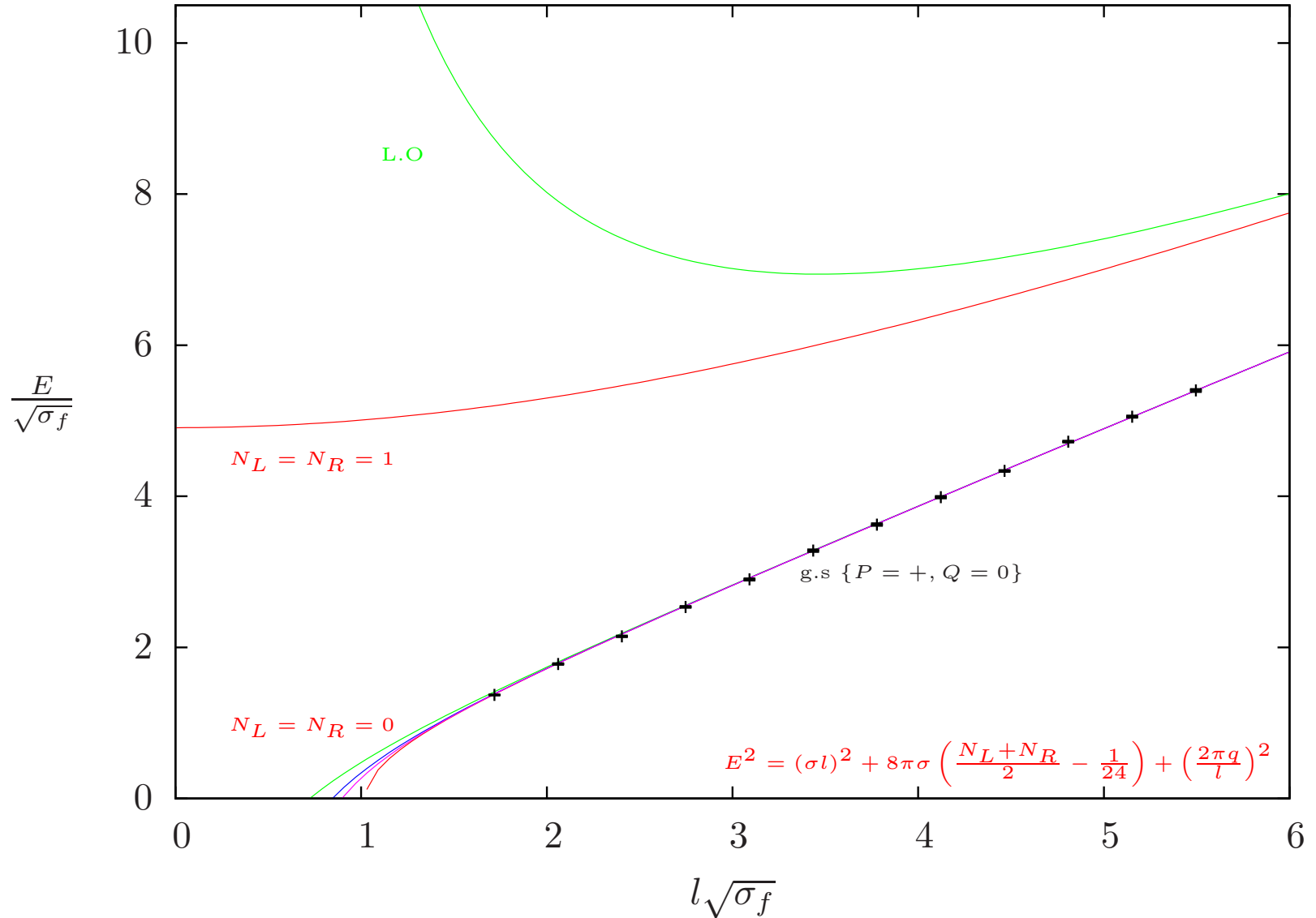
4. D=2+1 Results: $N_L = N_R = 1, q = 0, P = +$

- $N_L = N_R = 1$ corresponds to the $q = 0$ first excited state.
- String theoretically: $\alpha_{-1}\bar{\alpha}_{-1} | 0\rangle$
- Even number of phonons
- Expected to appear as $\{P = +, q = 0\}$ first excited state.
- Apply variational technique to the correlation matrix $\{P = +, q = 0\}$.
- Extraction of the first excited state.

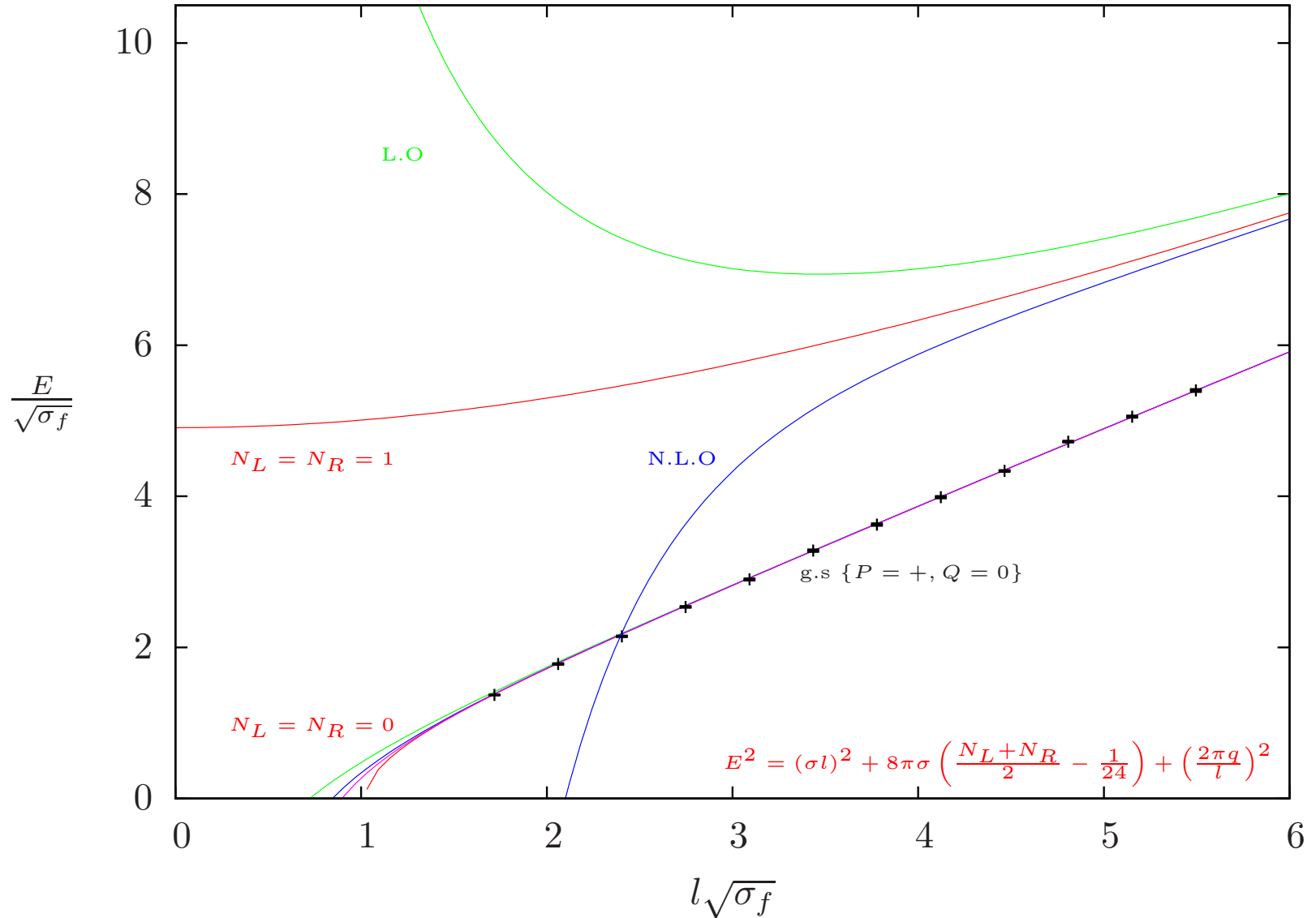
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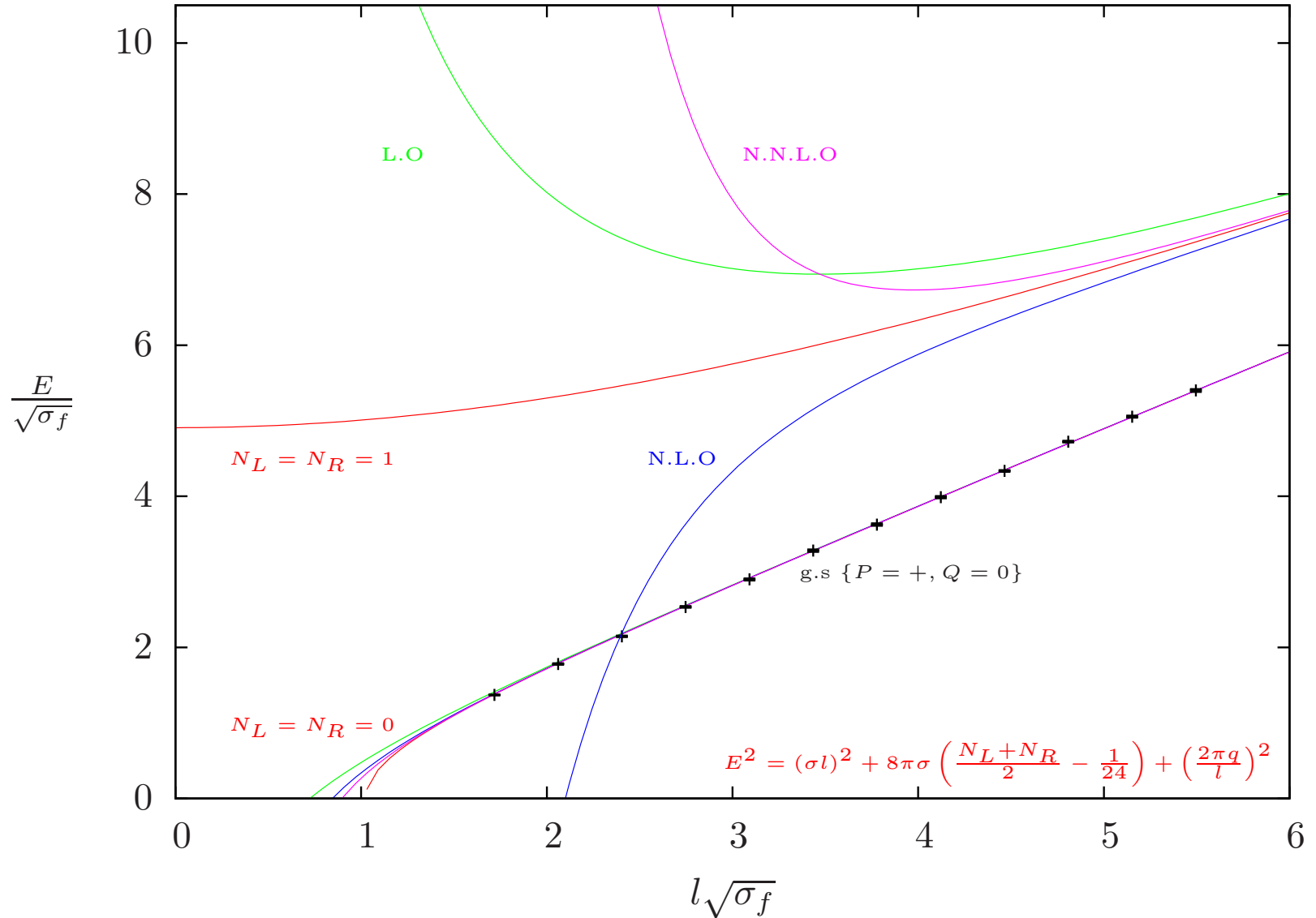
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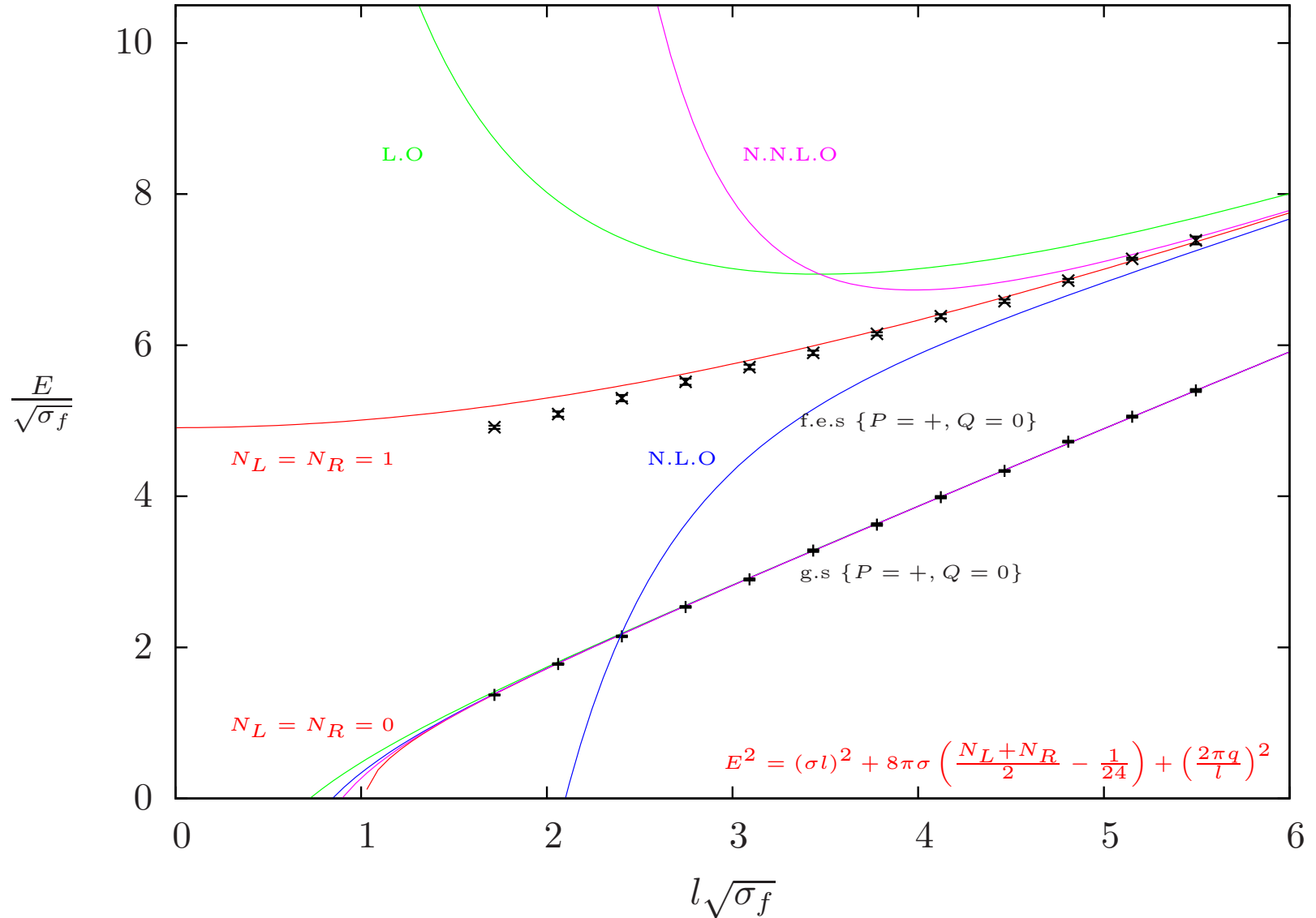
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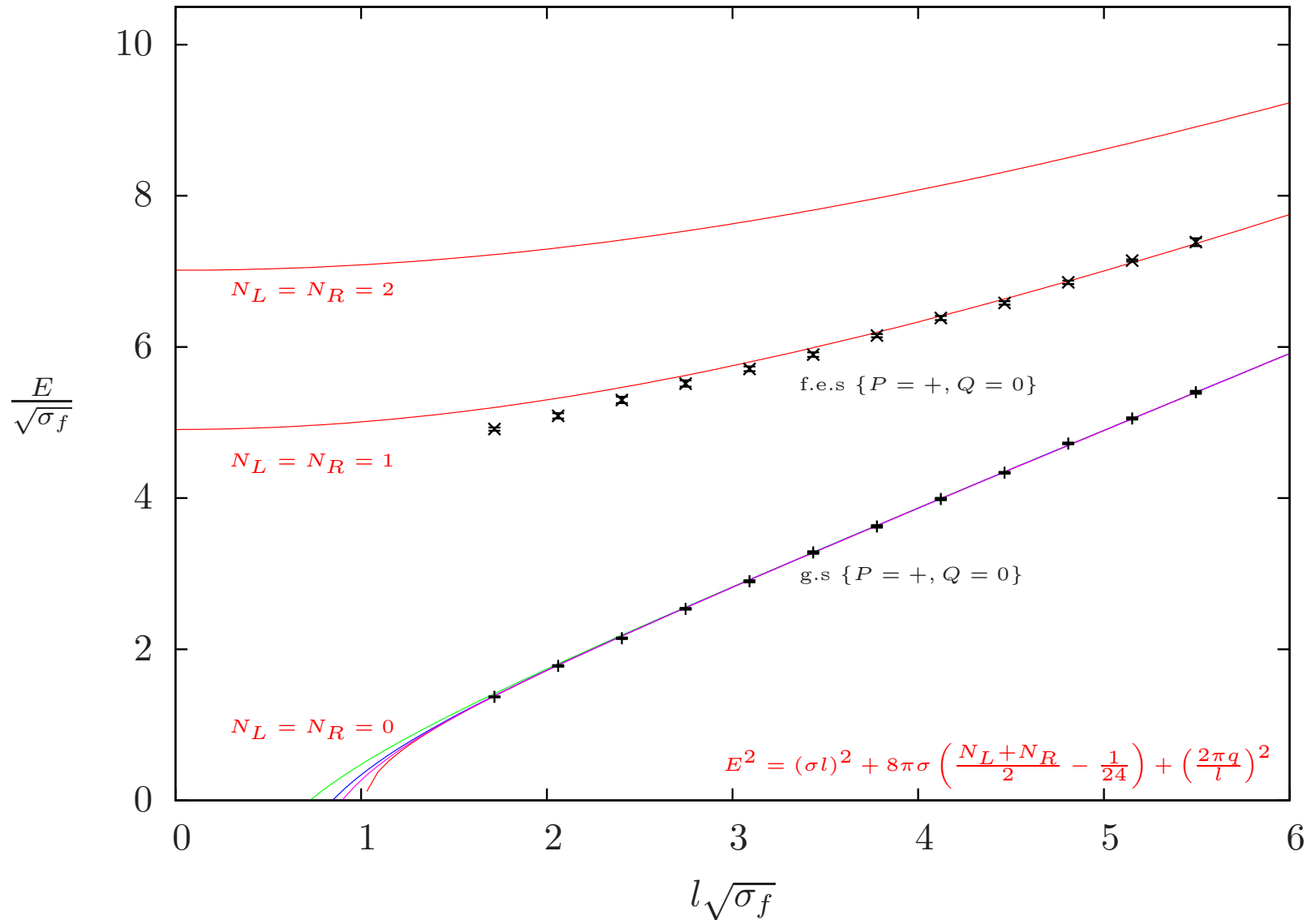
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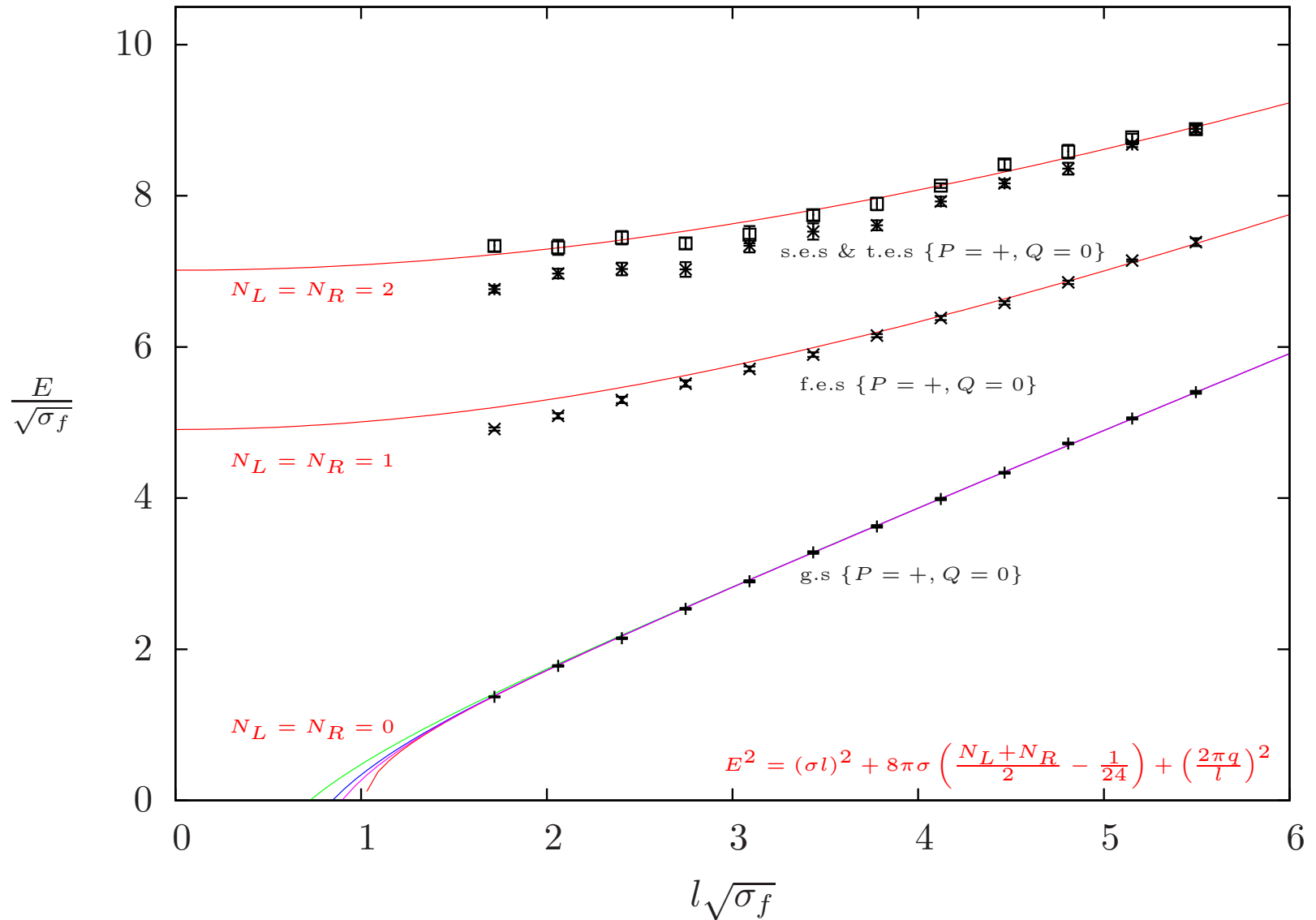
4. D=2+1 Results: $N_L = N_R = 2, q = 0, P = \pm$

- $N_L = N_R = 2$ corresponds to the $q = 0$ second excited state.
- String theoretically, four states can be constructed:
 - $\alpha_{-2}\bar{\alpha}_{-2} | 0 \rangle$
 - $\alpha_{-1}\alpha_{-1}\bar{\alpha}_{-1}\bar{\alpha}_{-1} | 0 \rangle$
 - $\alpha_{-2}\bar{\alpha}_{-1}\bar{\alpha}_{-1} | 0 \rangle$
 - $\alpha_{-1}\alpha_{-1}\bar{\alpha}_{-2} | 0 \rangle$
- Two states with even number of phonons, expected to appear as $\{P = +, q = 0\}$ second and third excited states.
- Two states with odd number of phonons, expected to appear as $\{P = -, q = 0\}$ ground and first excited states.
- Apply variational technique to the correlation matrices $\{P = \pm, q = 0\}$.

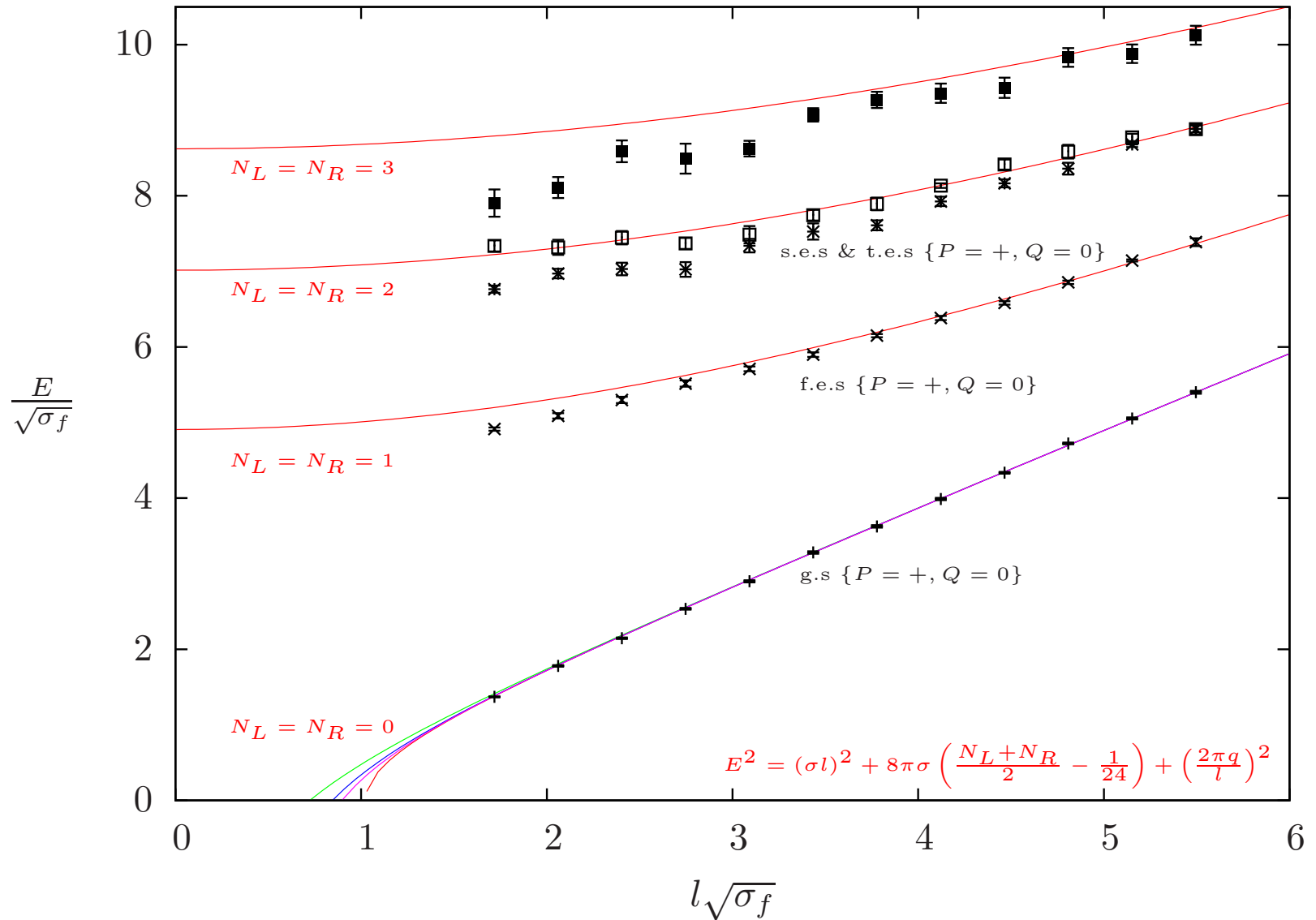
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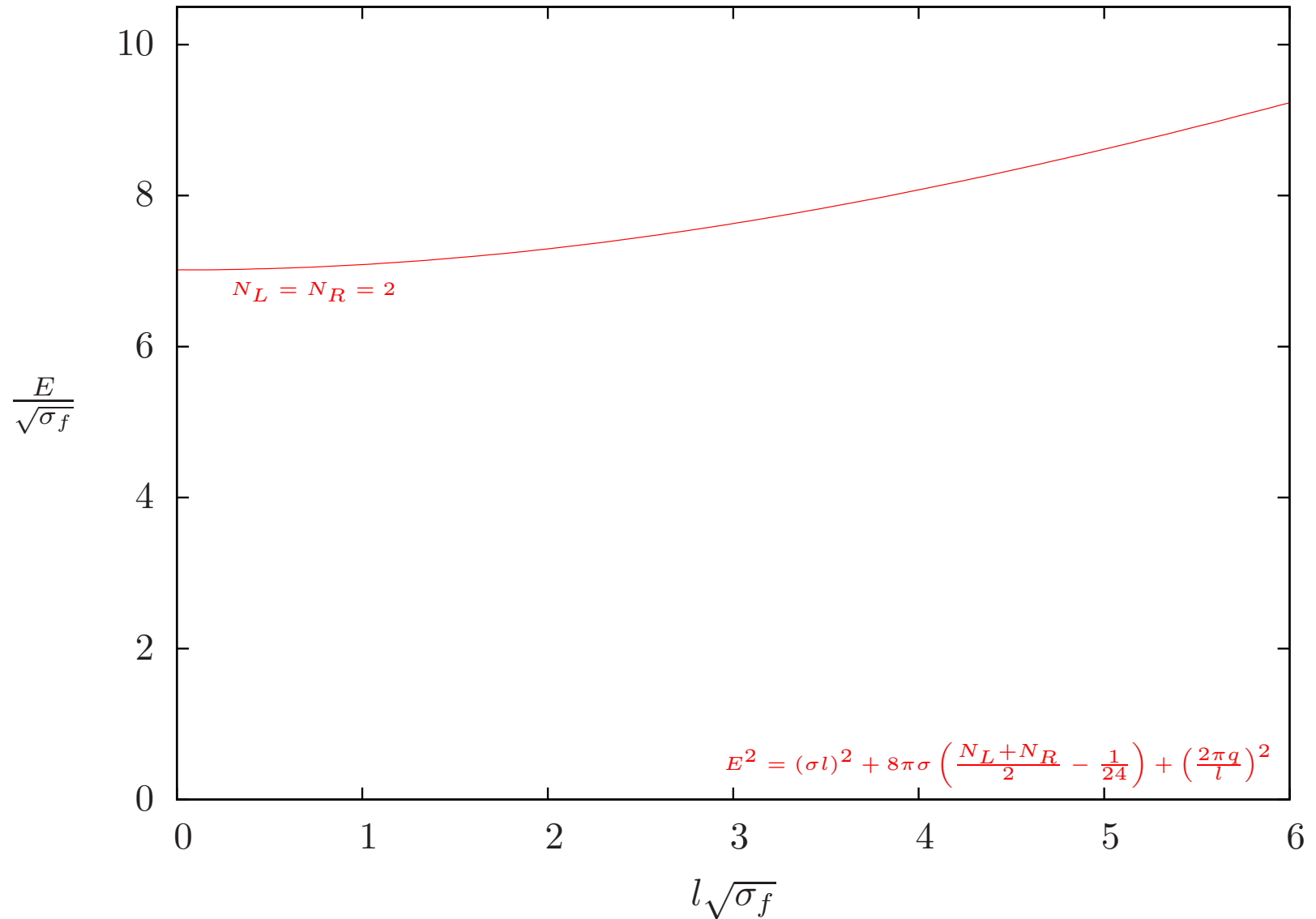
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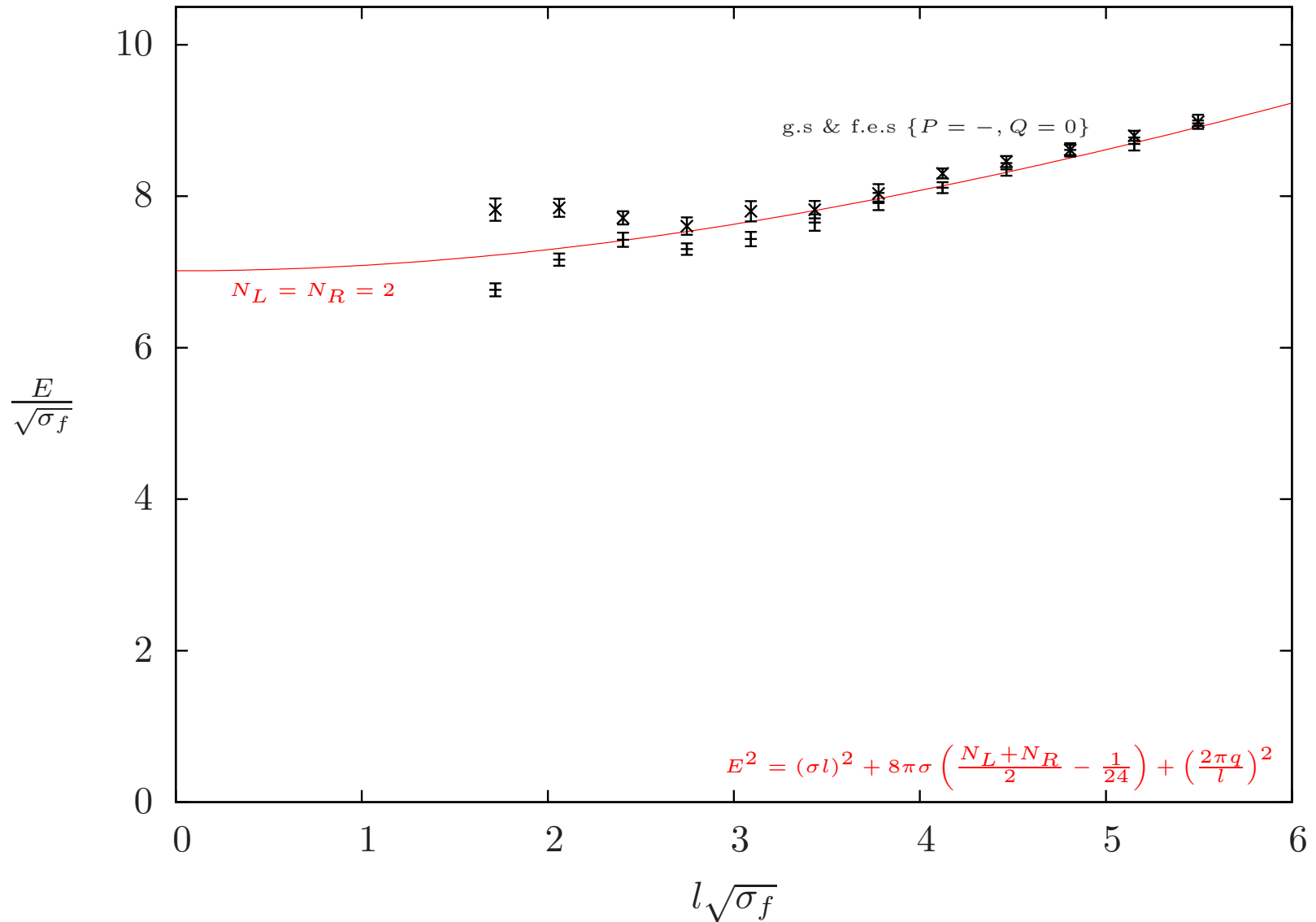
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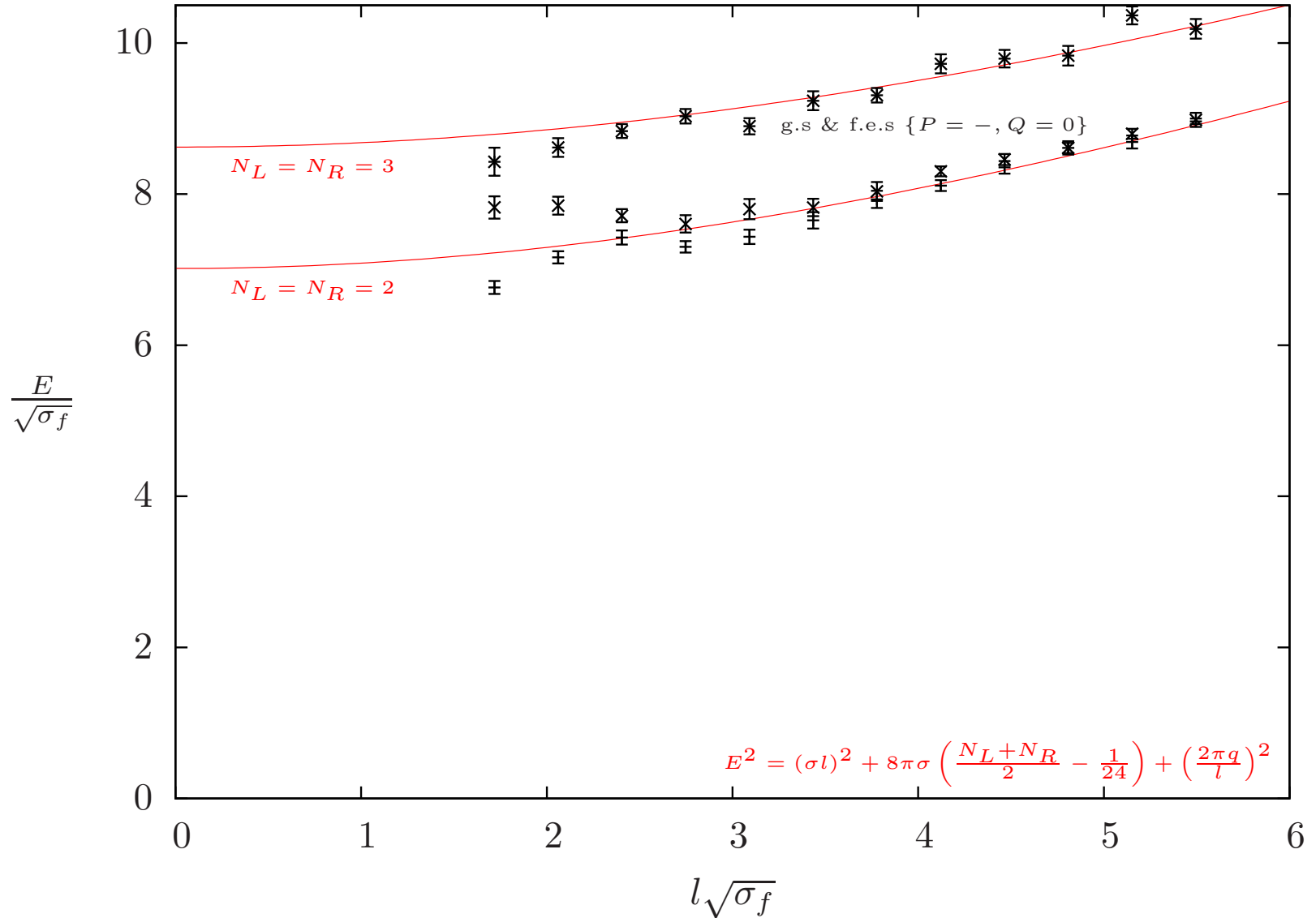
4. D=2+1 Results: $N_L = N_R = 2, q = 0, P = -$



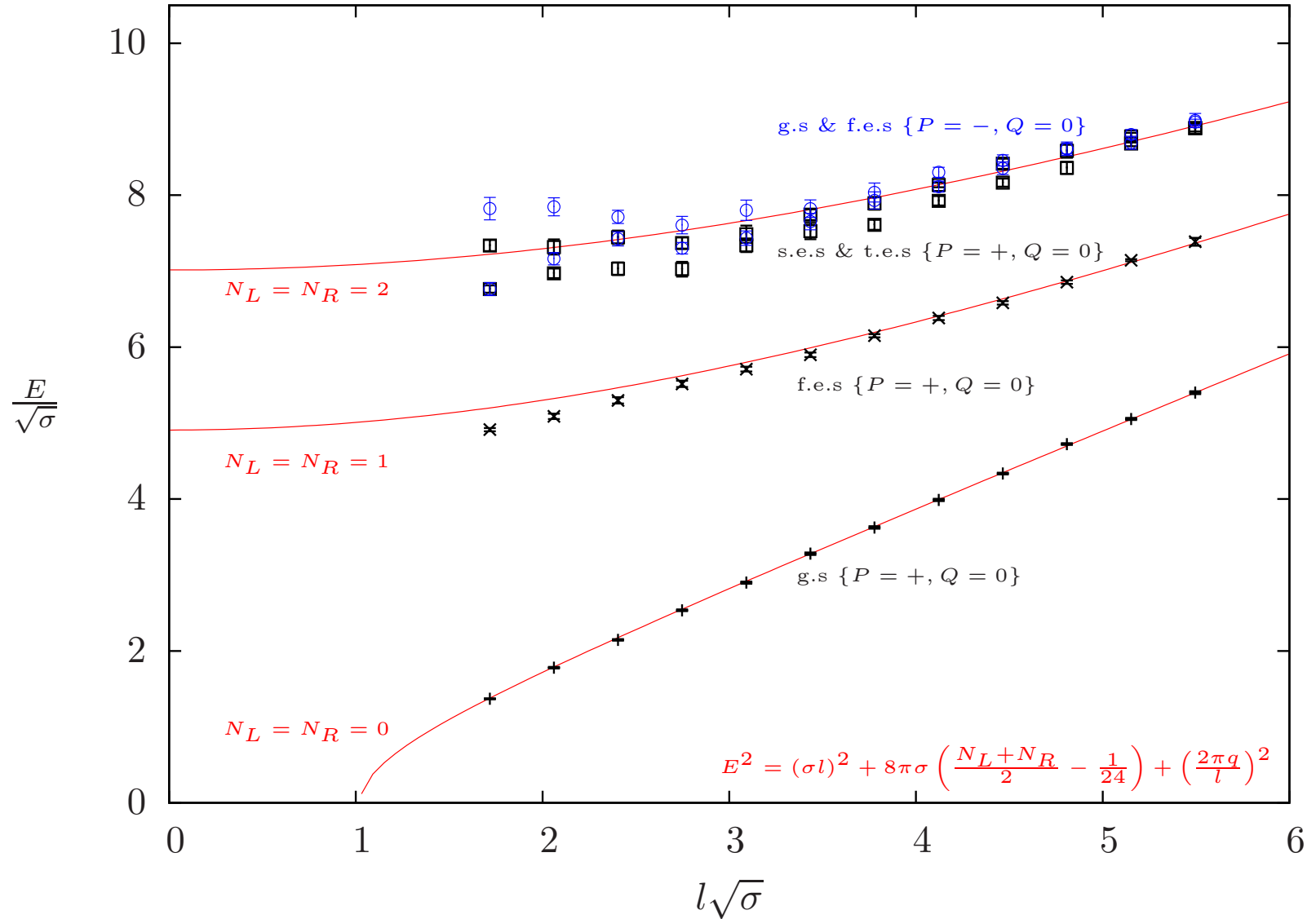
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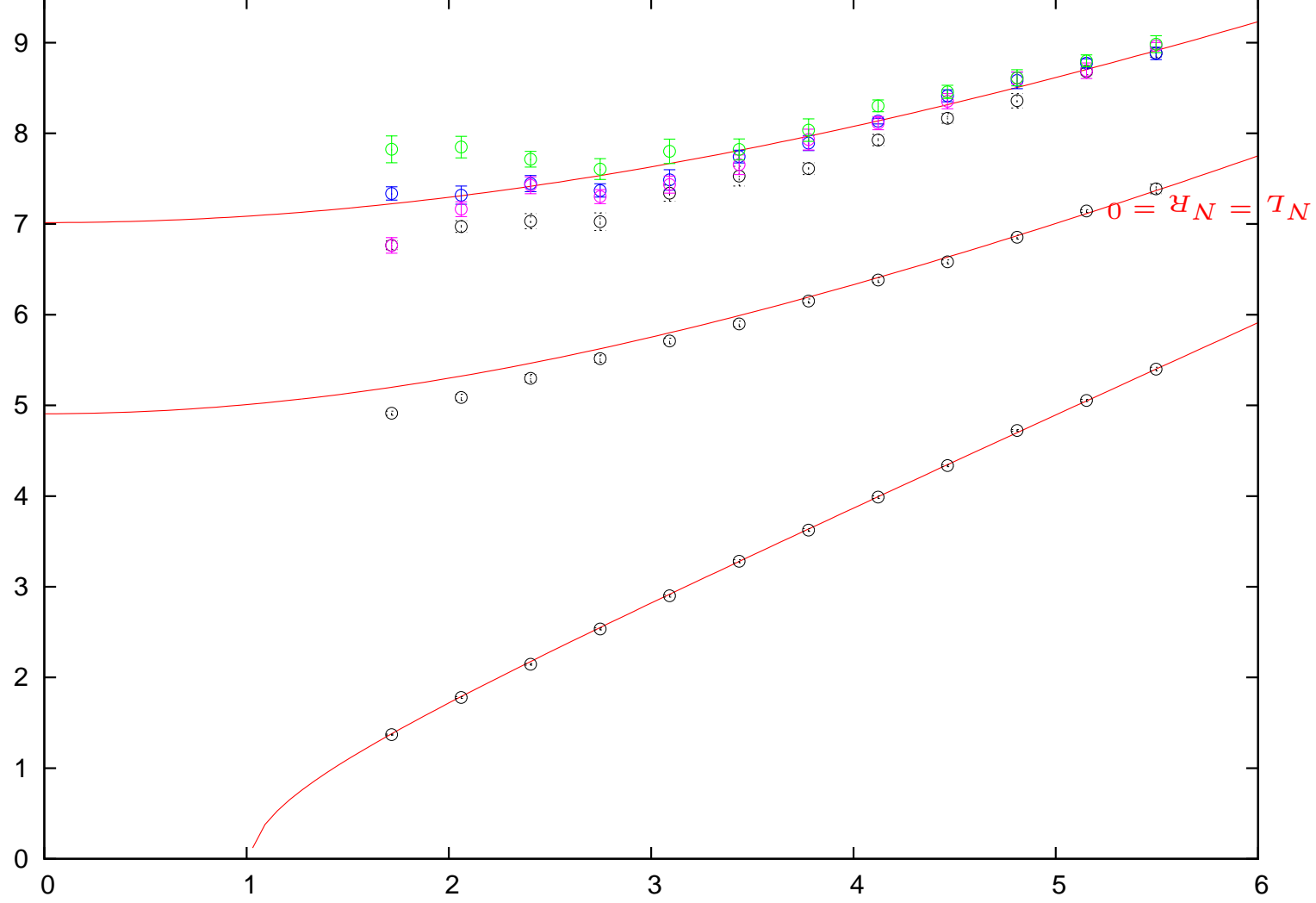
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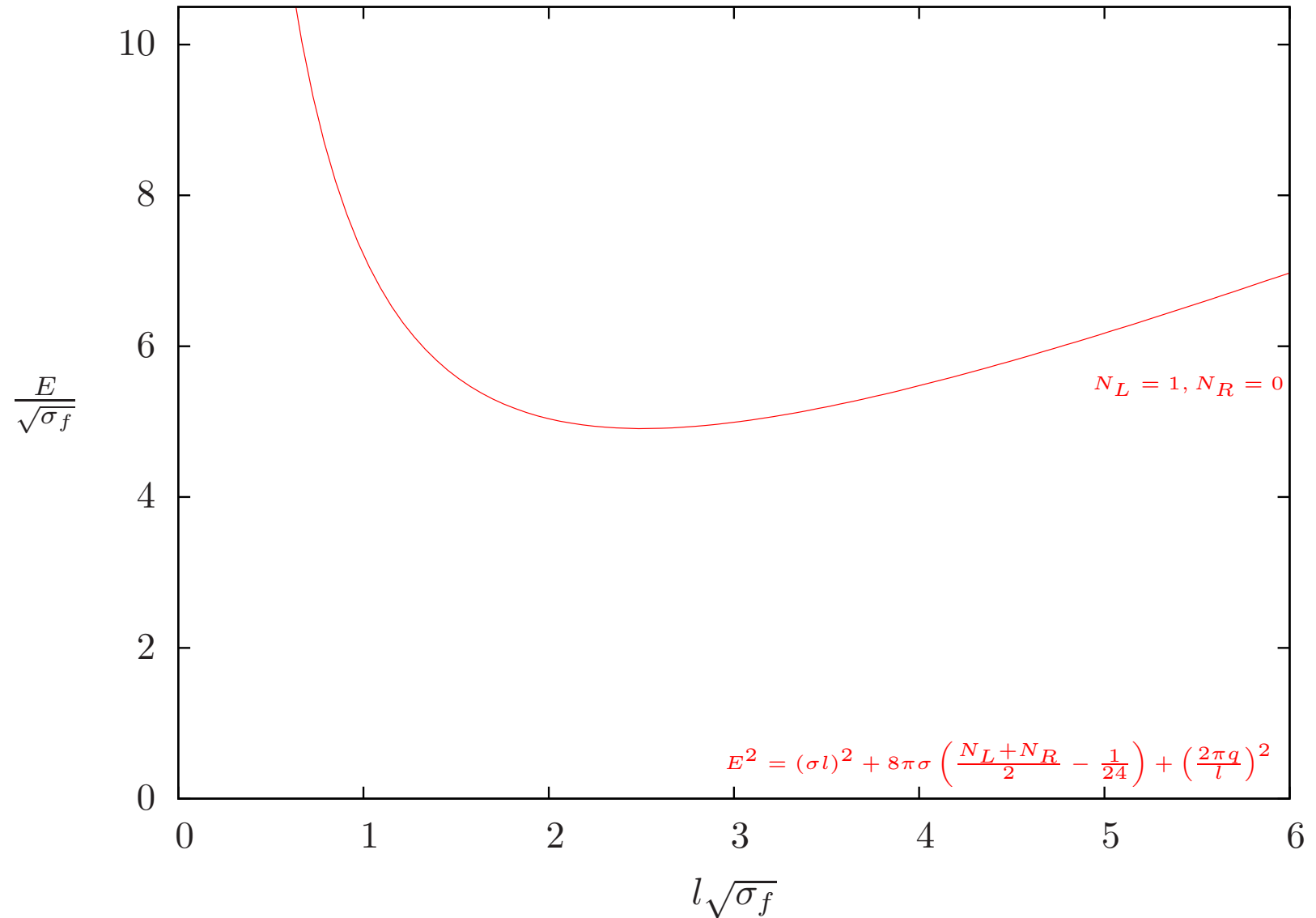
4. $D=2+1$ Results: $N_L = N_R = 2, q = 0$



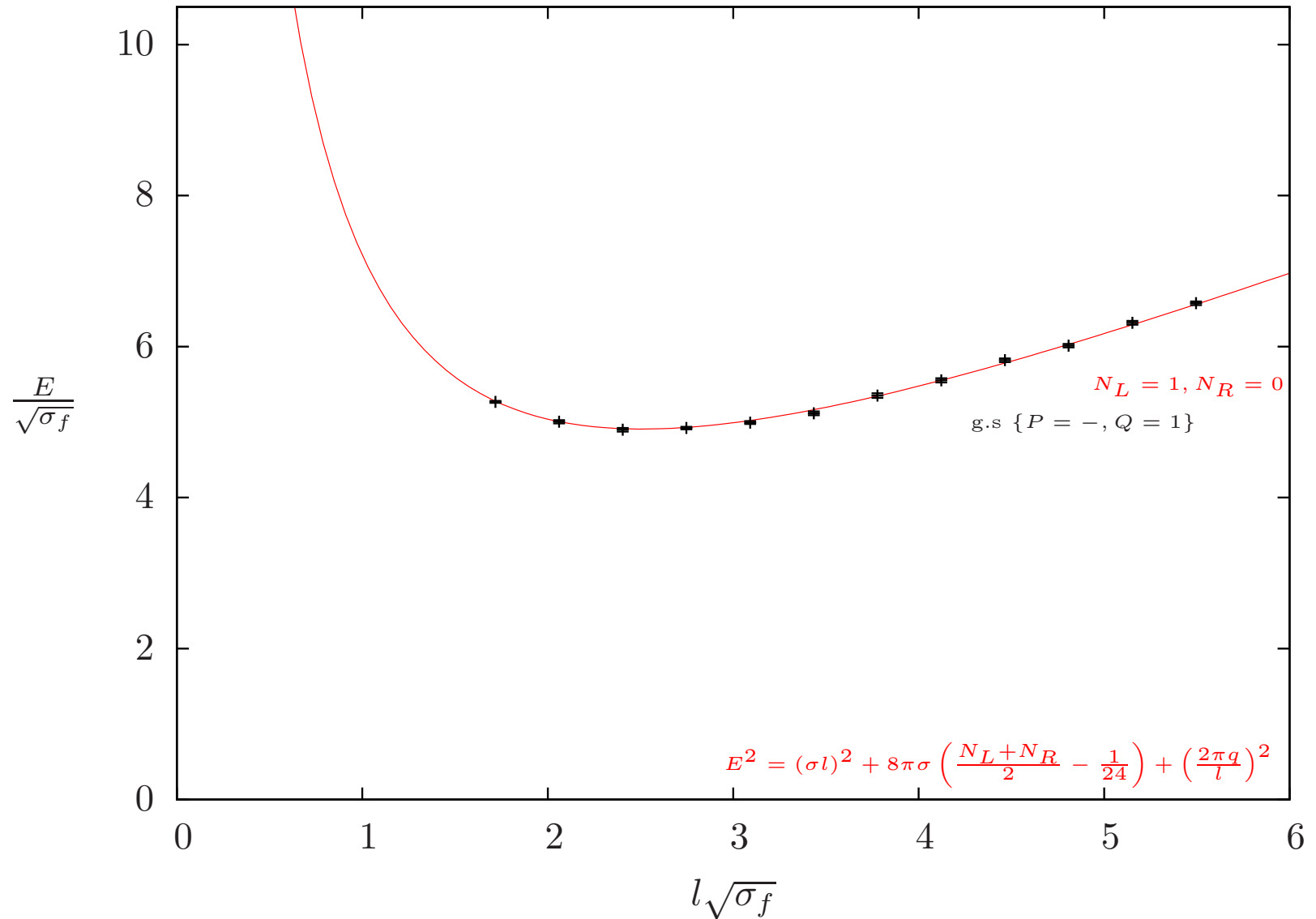
4. D=2+1 Results: $N_L = 1, N_R = 0, q = 1, P = -$

- $N_L = 1, N_R = 0$ corresponds to the $q = 1$ ground state.
- String theoretically: $\alpha_{-1} | 0 \rangle$
- Odd number of phonons
- Expected to appear as $\{P = -, q = 1\}$ ground state
- Apply variational technique to the correlation matrix $\{P = -, q = 1\}$.
- Extraction of the ground state.

4. D=2+1 Results: $N_L = 1, N_R = 0, q = 1, P = -$



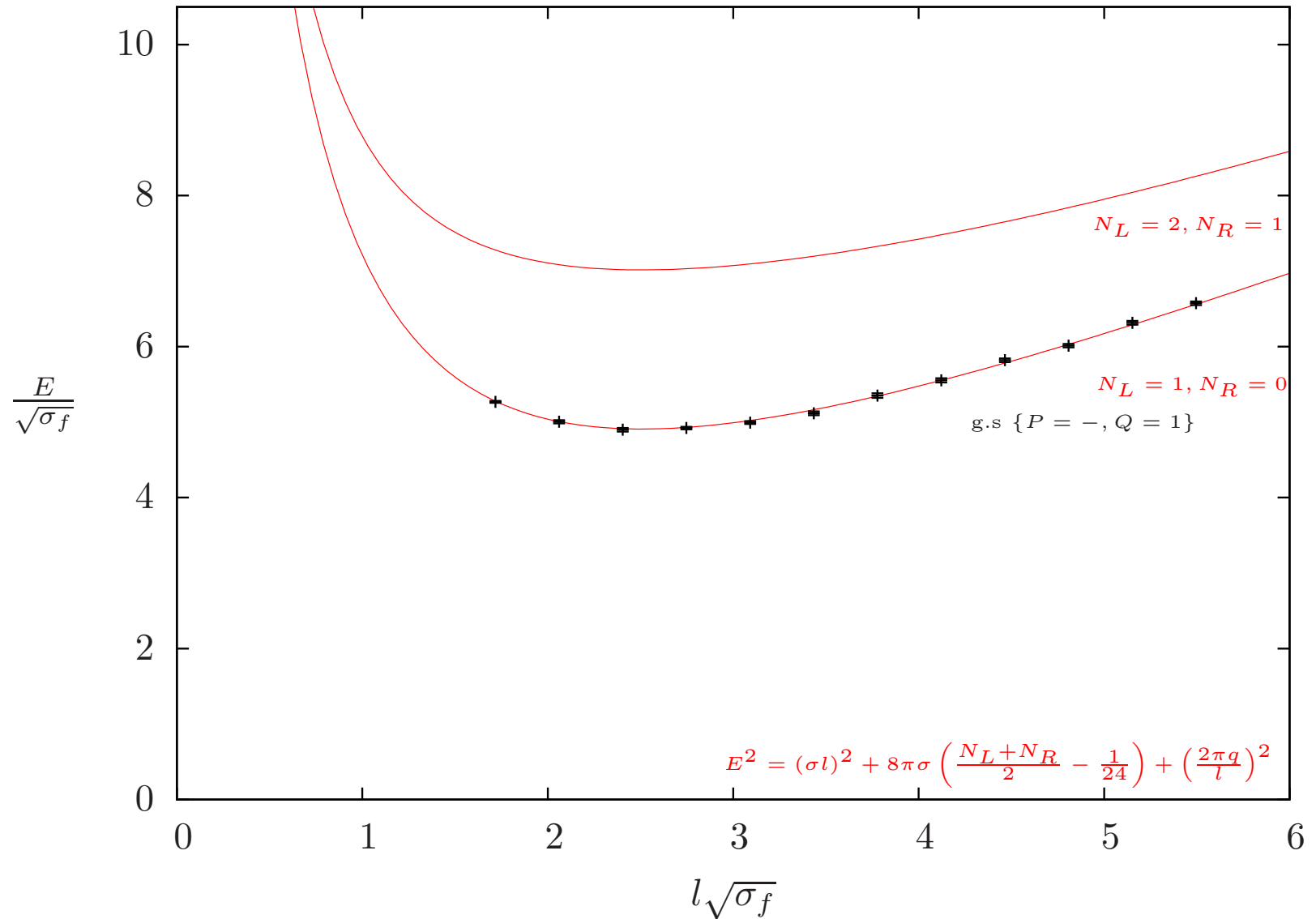
4. D=2+1 Results: $N_L = 1, N_R = 0, q = 1, P = -$



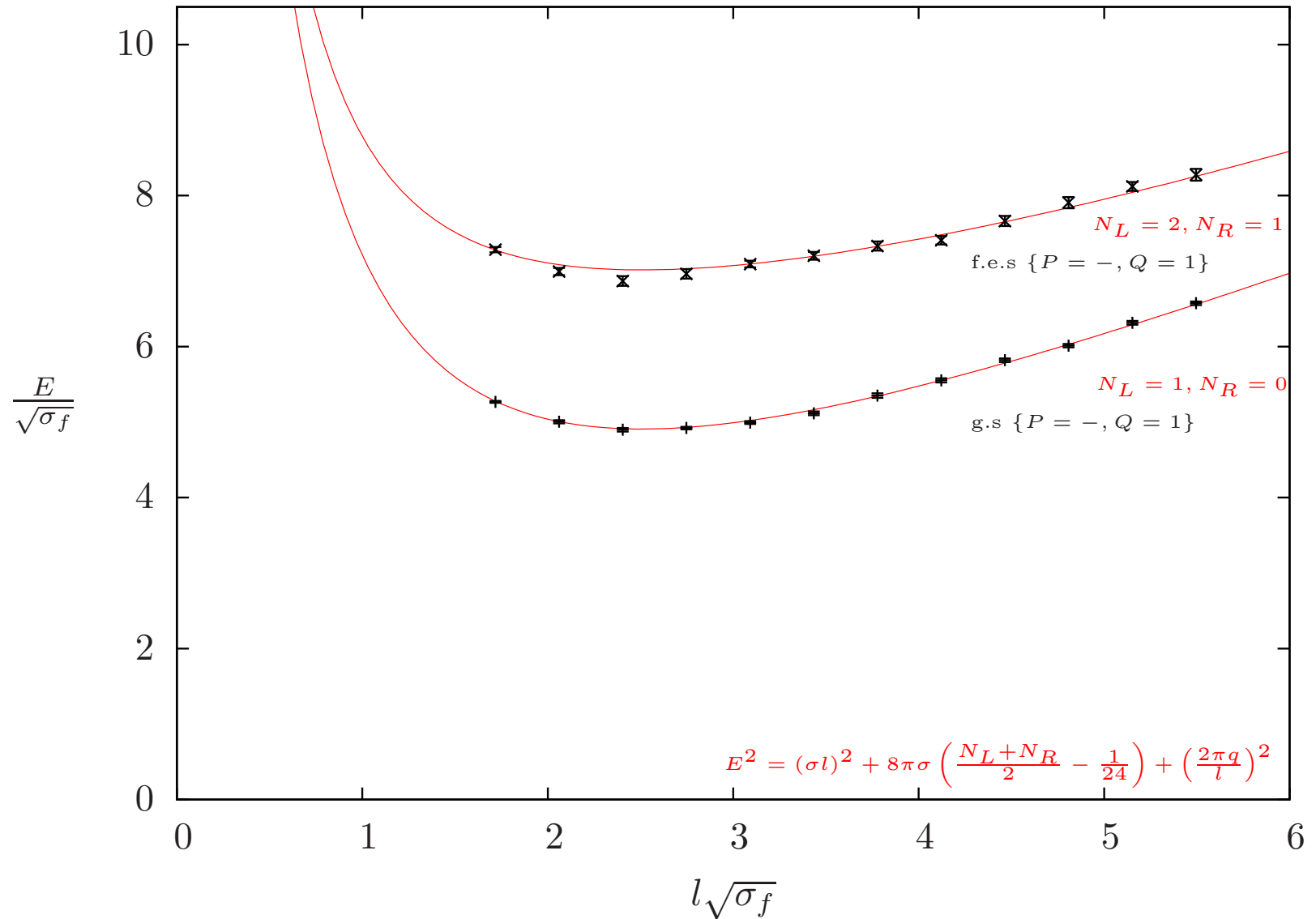
4. D=2+1 Results: $N_L = 2, N_R = 1, q = 1, P = \pm$

- $N_L = 2, N_R = 1$ corresponds to the $q = 1$ first excited state.
- String theoretically, two states can be constructed:
 - $\alpha_{-2}\bar{\alpha}_{-1} | 0 \rangle$
 - $\alpha_{-1}\alpha_{-1}\bar{\alpha}_{-1} | 0 \rangle$
- One state with odd number of phonons, expected to appear as $\{P = -, q = 1\}$ first excited state.
- One state with even number of phonons, expected to appear as $\{P = +, q = 1\}$ ground state.
- Apply variational technique to the correlation matrices $\{P = \pm, q = 1\}$.

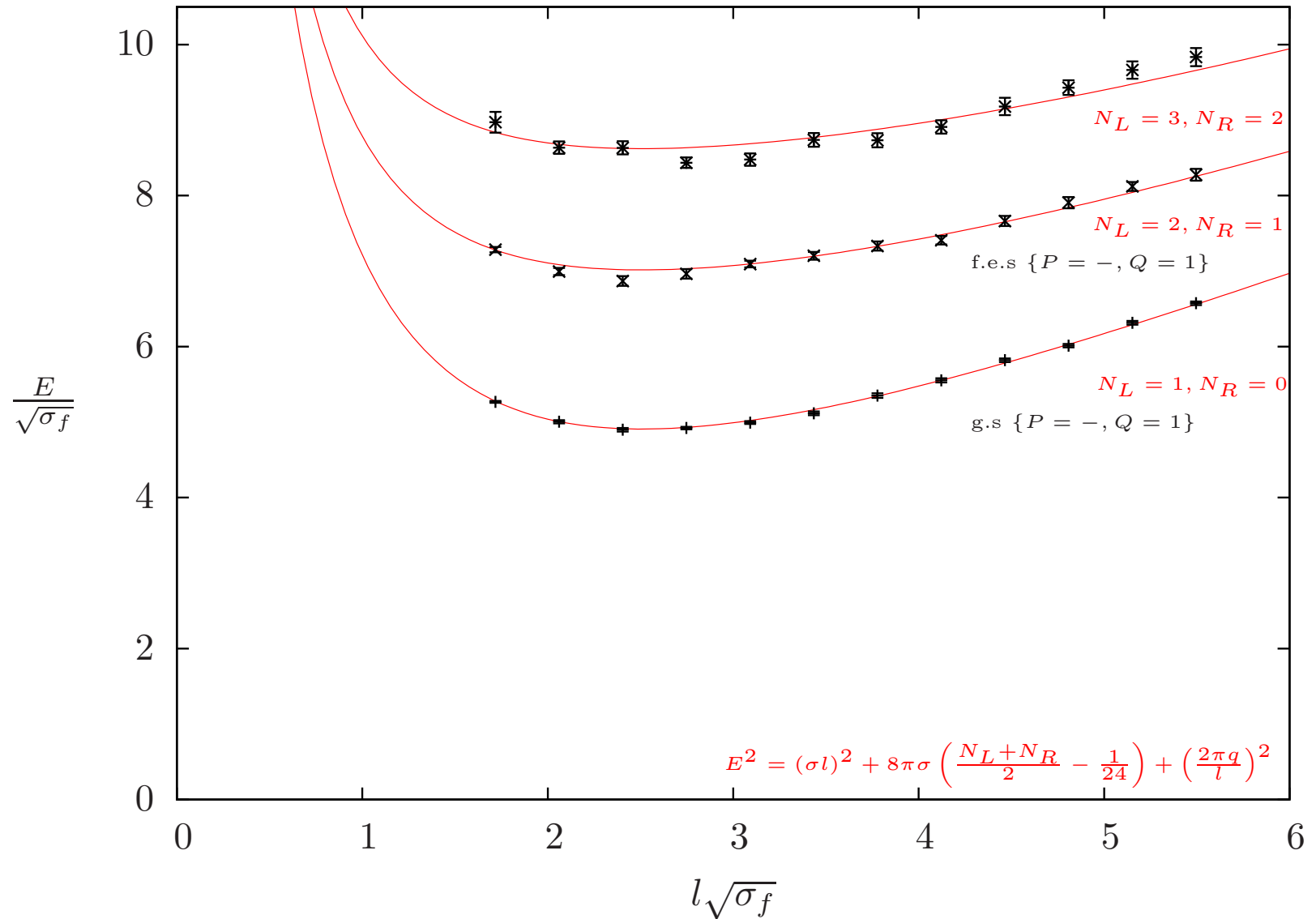
4. D=2+1 Results: $N_L = 2, N_R = 1, q = 1, P = -$



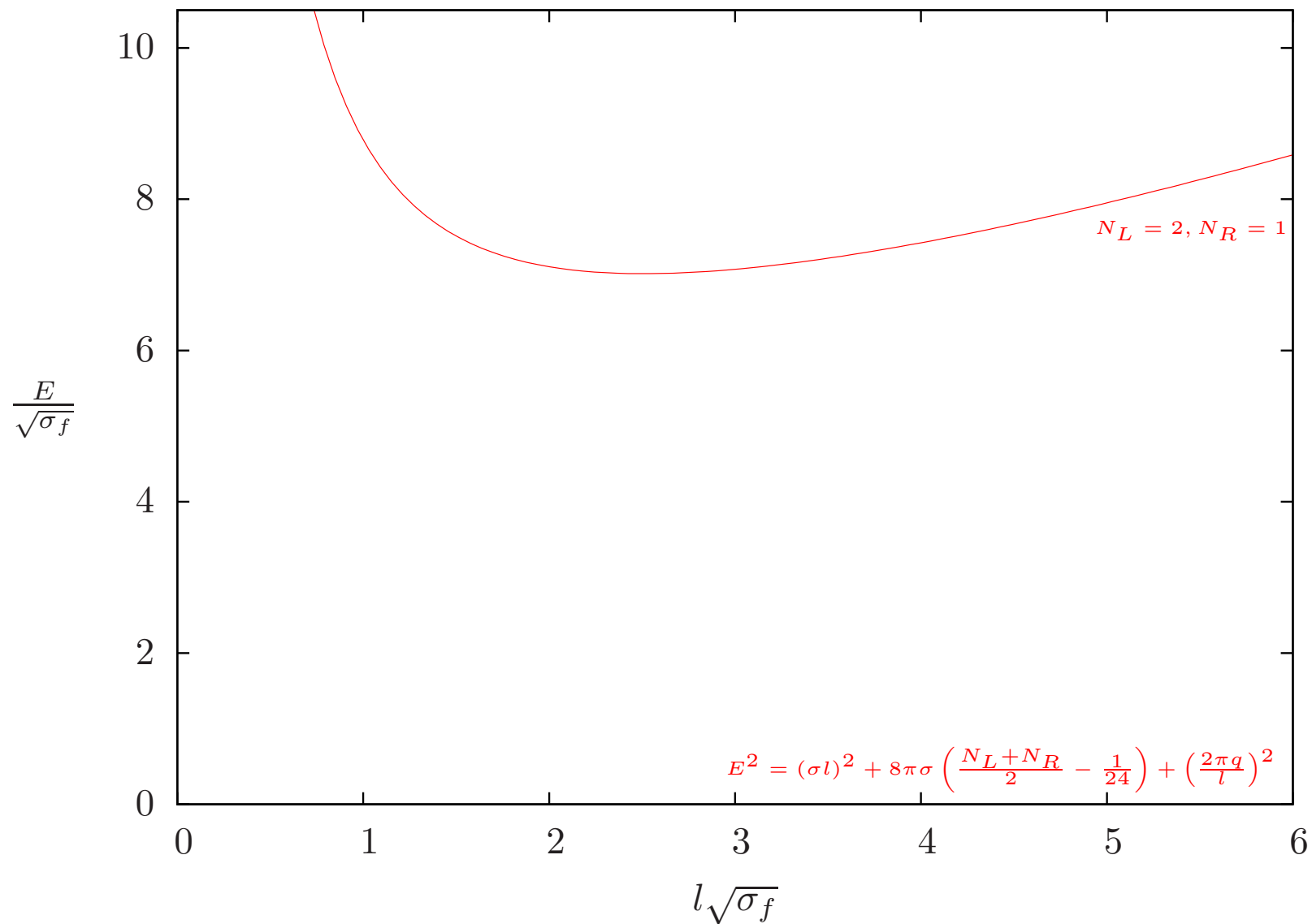
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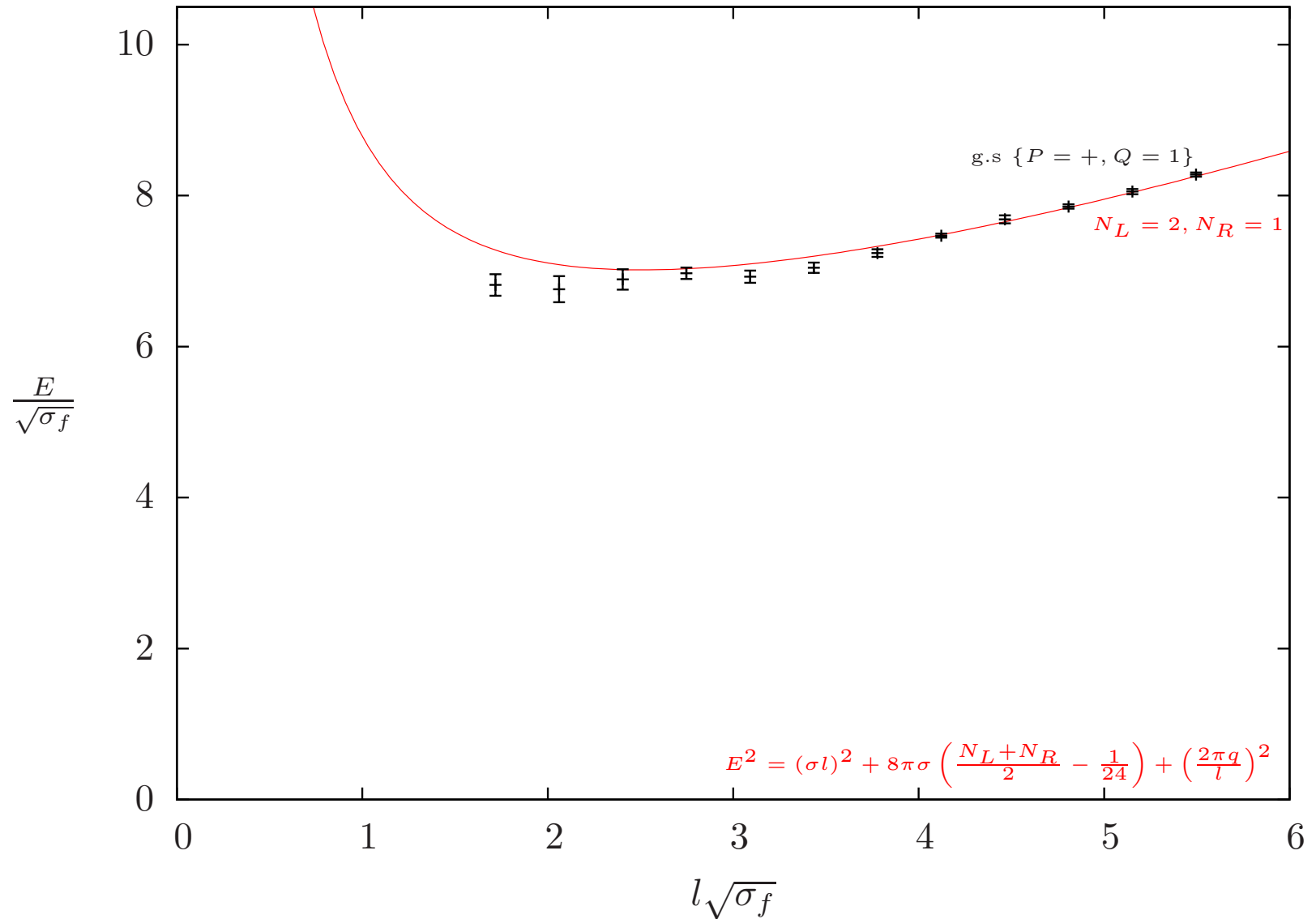
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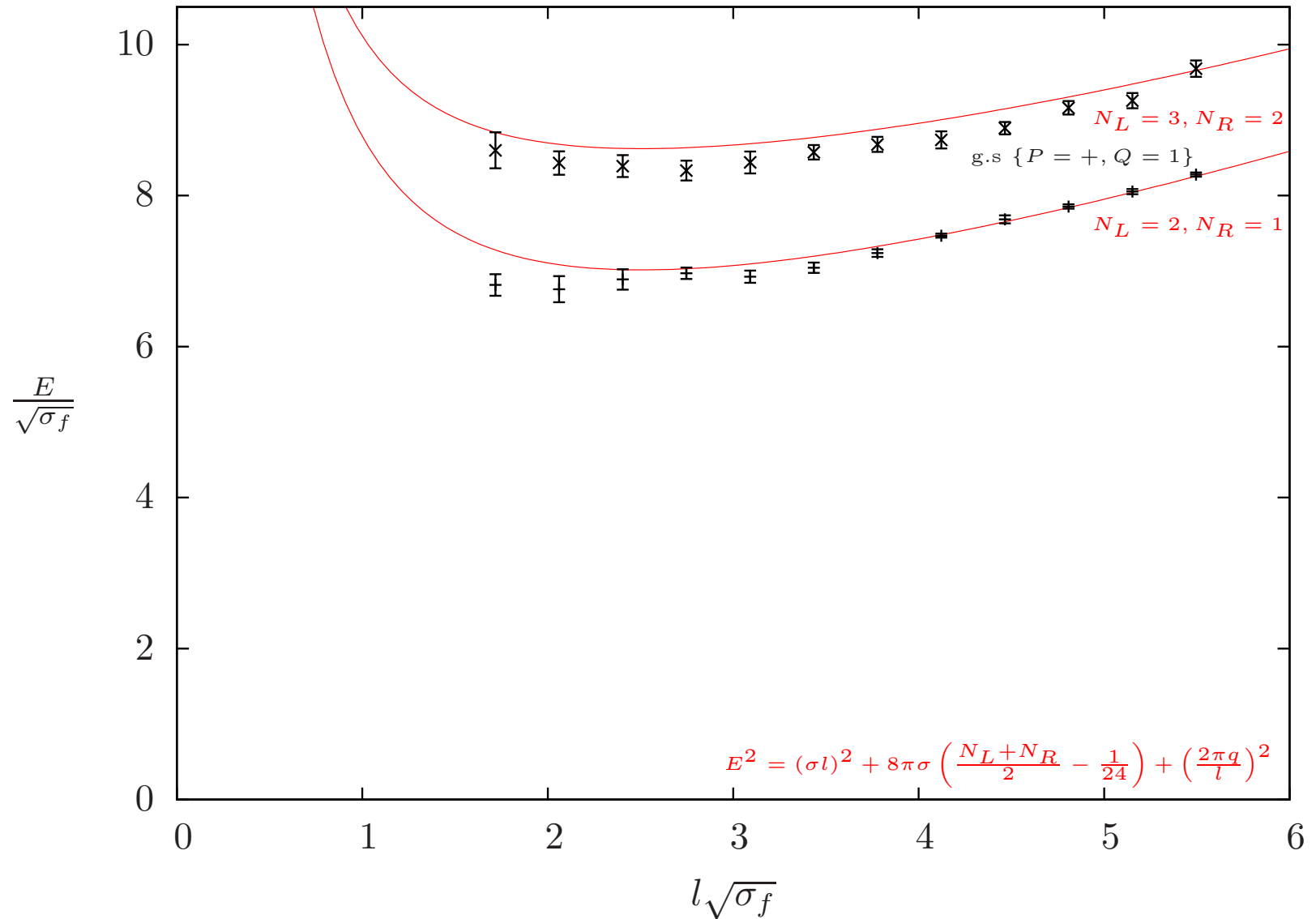
4. D=2+1 Results: $N_L = 2, N_R = 1, q = 1, P = +$



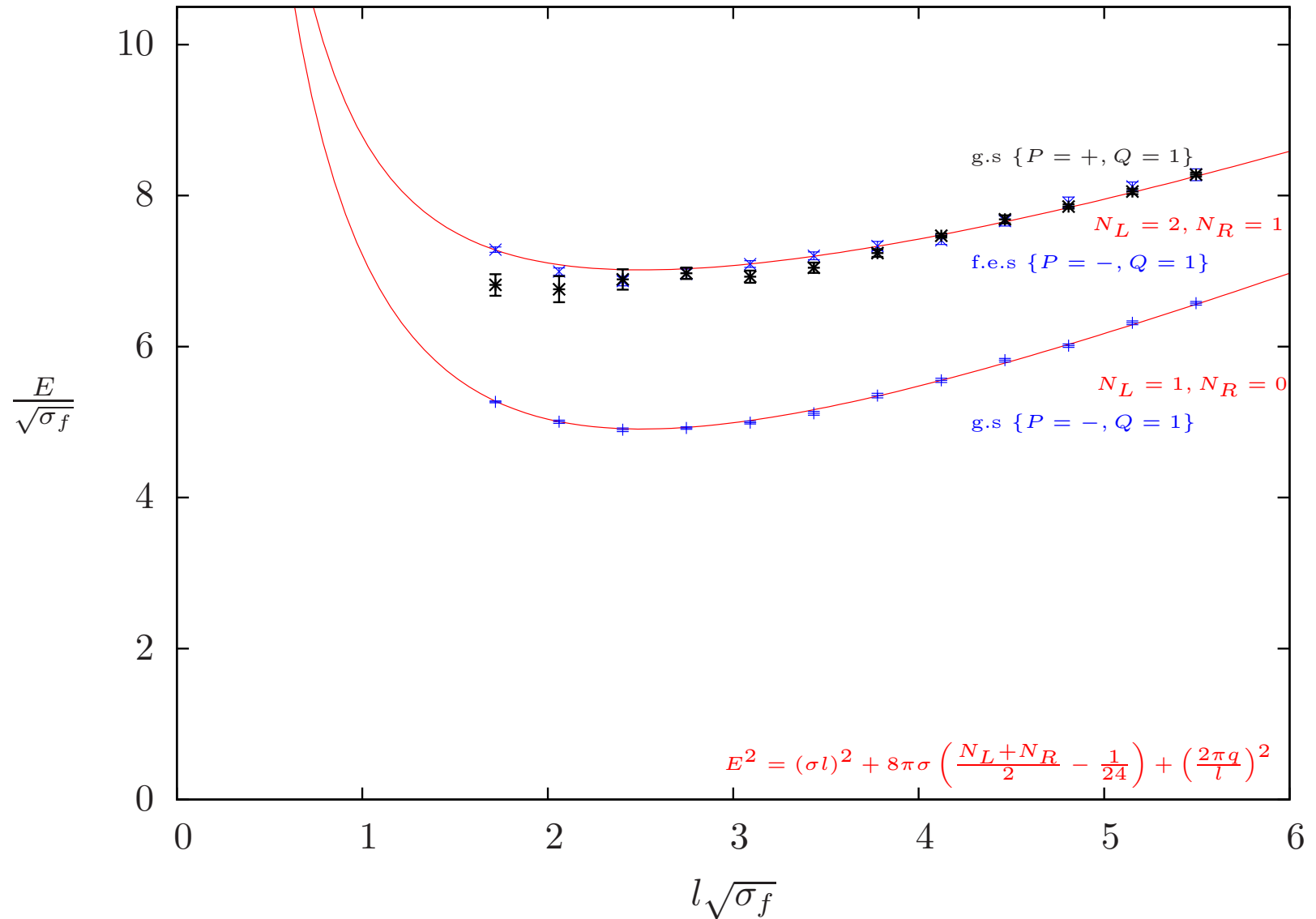
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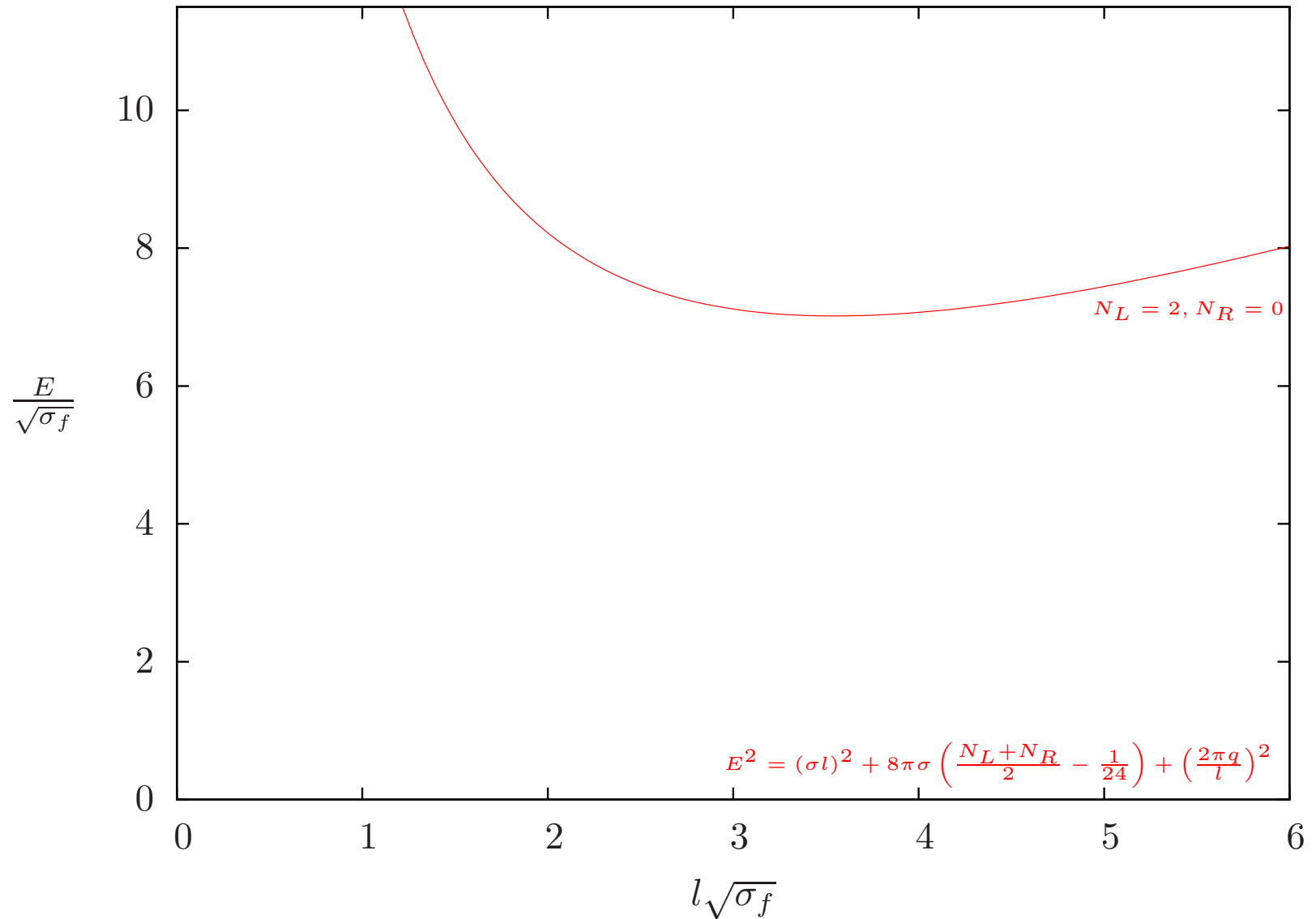
4. D=2+1 Results: $N_L = 2, N_R = 1, q = 1, P = \pm$



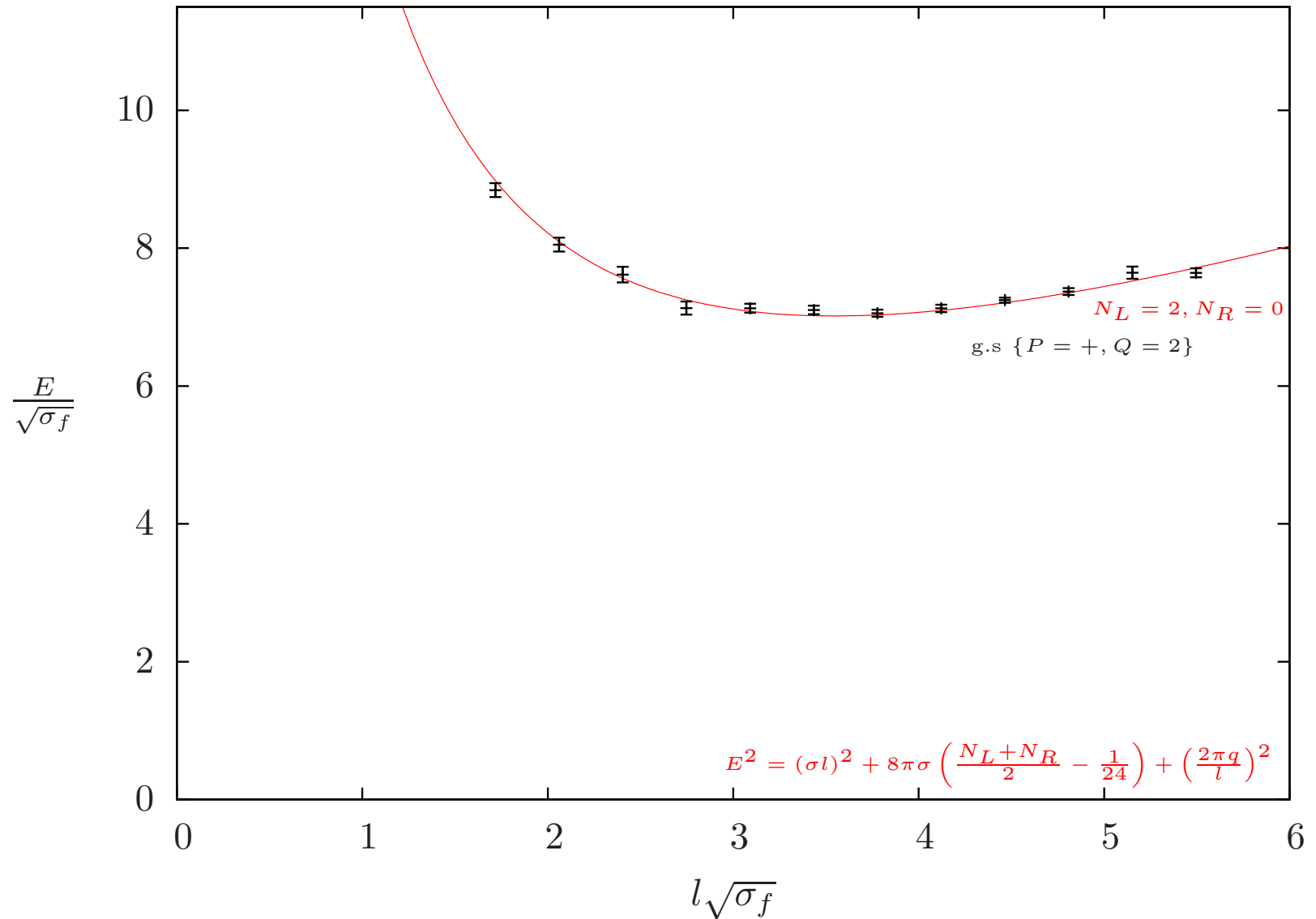
4. D=2+1 Results: $N_L = 2, N_R = 0, q = 2, P = \pm$

- $N_L = 2, N_R = 0$ corresponds to the $q = 2$ ground state.
- String theoretically, two states can be constructed:
 - $\alpha_{-2} | 0 \rangle$
 - $\alpha_{-1} \alpha_{-1} | 0 \rangle$
- One state with odd number of phonons, expected to appear as $\{P = -, q = 2\}$ ground state.
- One state with even number of phonons, expected to appear as $\{P = +, q = 2\}$ ground state.
- Apply variational technique to the correlation matrices $\{P = \pm, q = 2\}$.

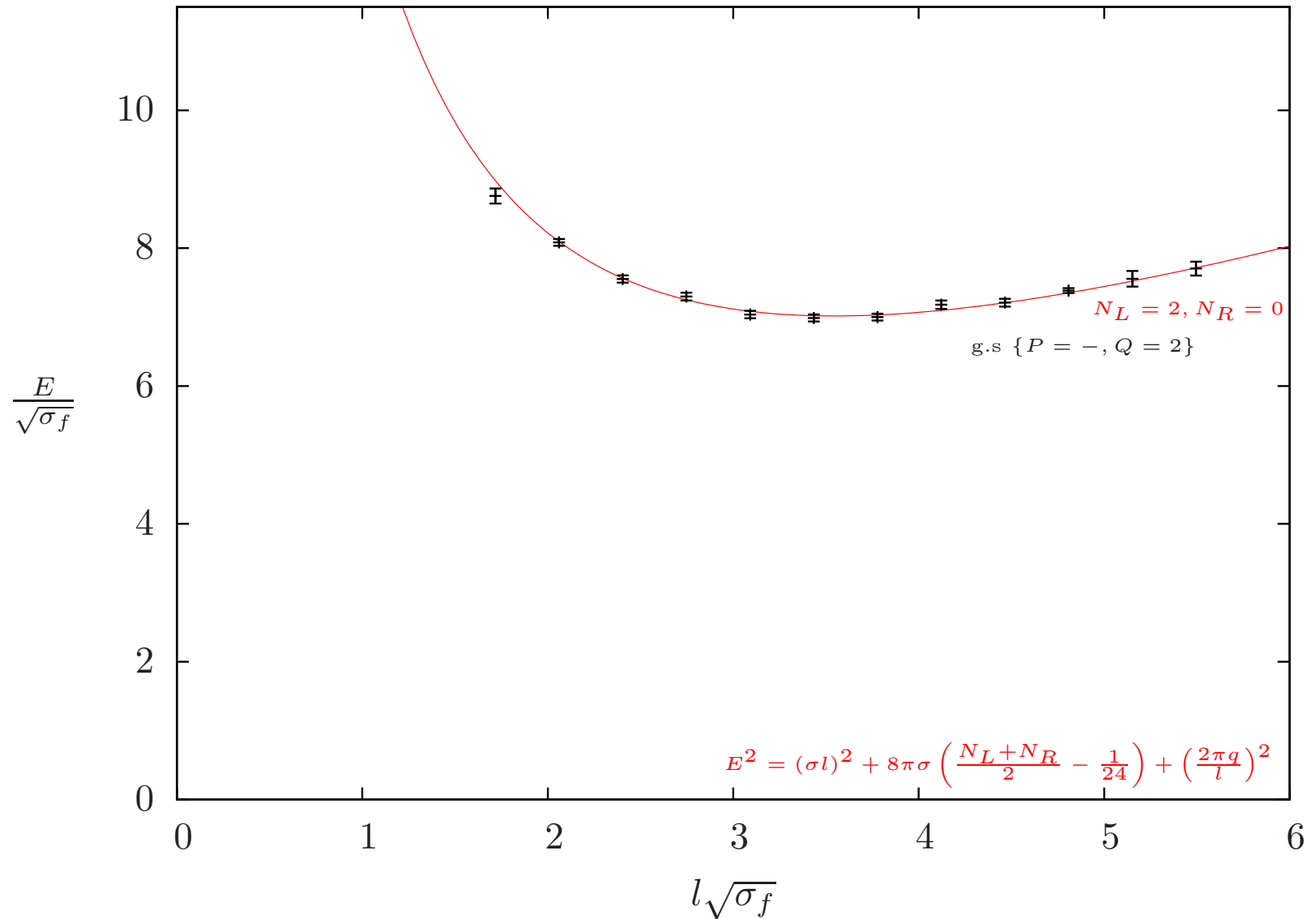
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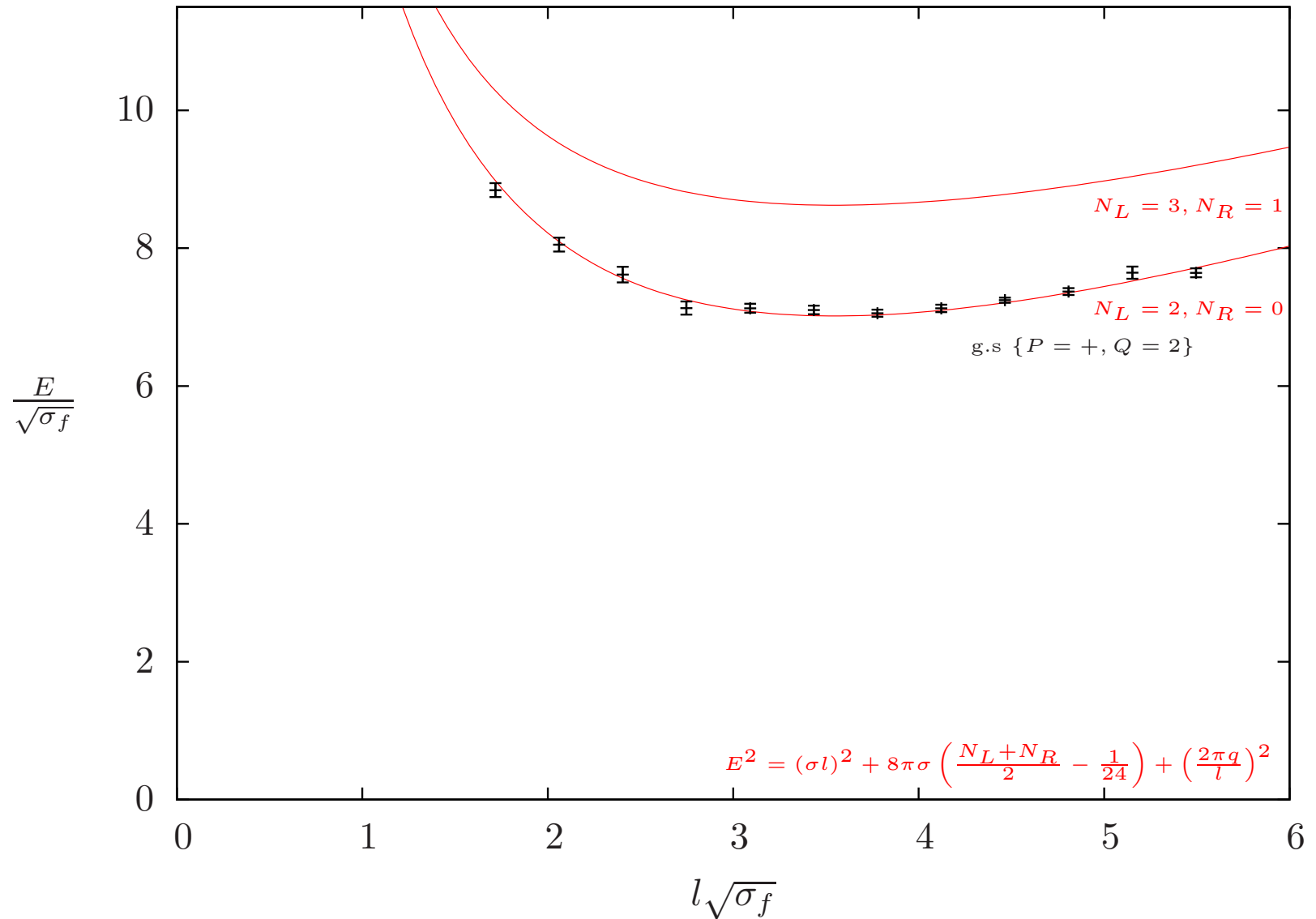
4. D=2+1 Results: $N_L = 2, N_R = 0, q = 2, P = -$



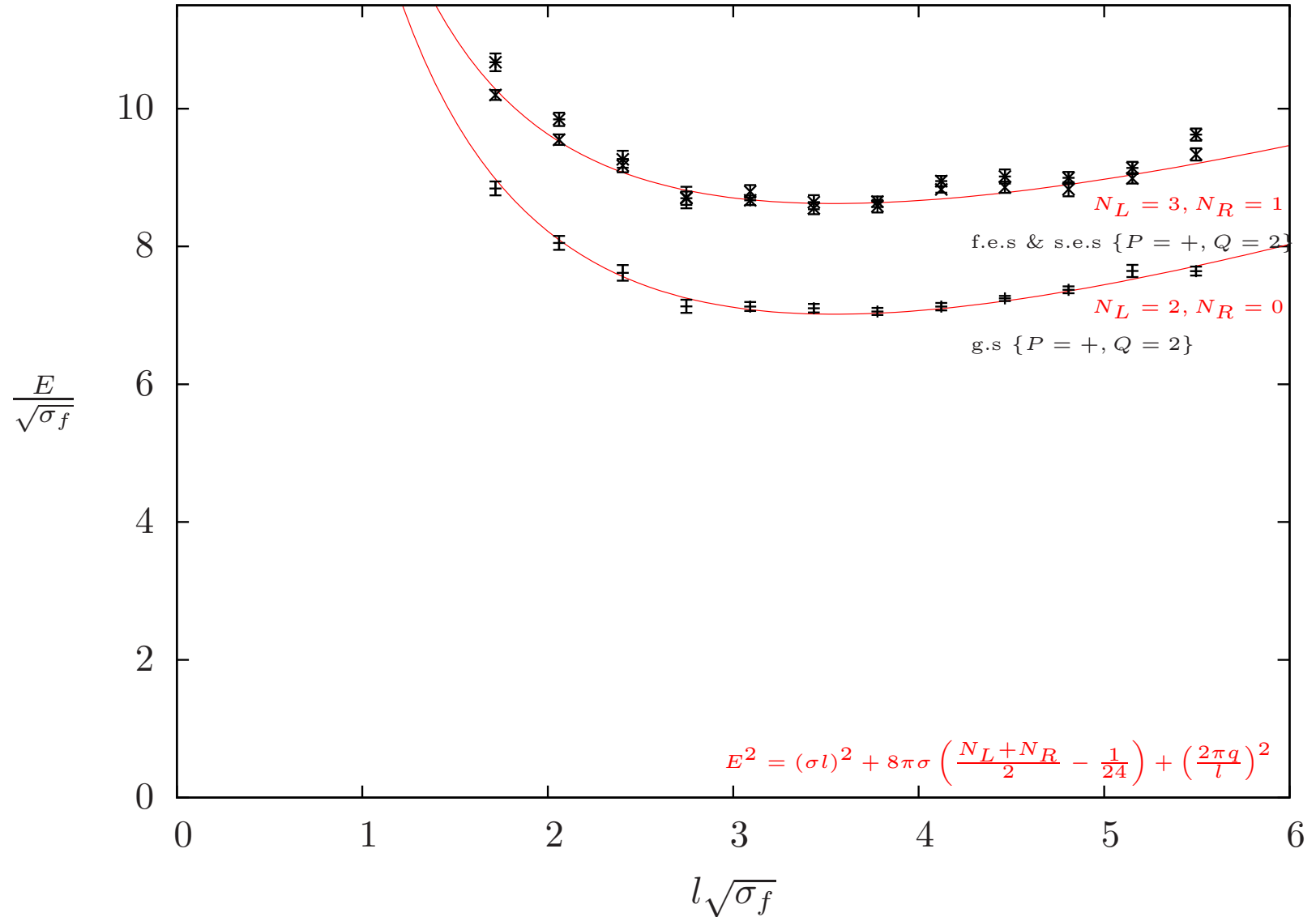
4. D=2+1 Results: $N_L = 3, N_R = 1, q = 2, P = \pm$

- $N_L = 3, N_R = 1$ corresponds to the $q = 2$ first excited state.
- String theoretically, three states can be constructed:
 - $\alpha_{-3}\bar{\alpha}_{-1} | 0 \rangle$
 - $\alpha_{-2}\alpha_{-1}\bar{\alpha}_{-1} | 0 \rangle$
 - $\alpha_{-1}\alpha_{-1}\alpha_{-1}\bar{\alpha}_{-1} | 0 \rangle$
- One state with odd number of phonons, expected to appear as $\{P = -, q = 2\}$ first excited state.
- Two states with even number of phonons, expected to appear as $\{P = +, q = 2\}$ first and second excited states.
- Apply variational technique to the correlation matrices $\{P = \pm, q = 2\}$.

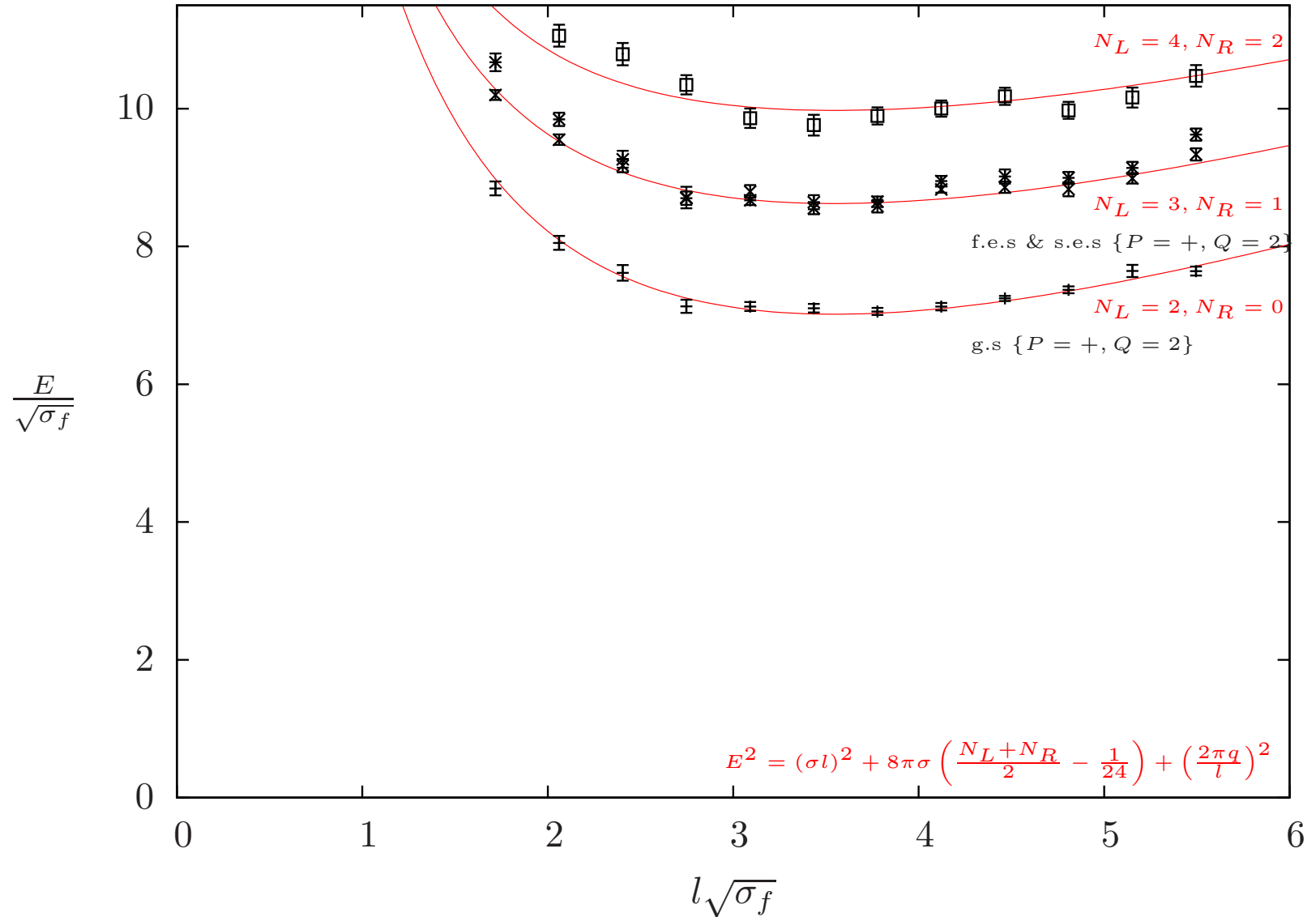
4. D=2+1 Results: $N_L = 3, N_R = 1, q = 2, P = +$



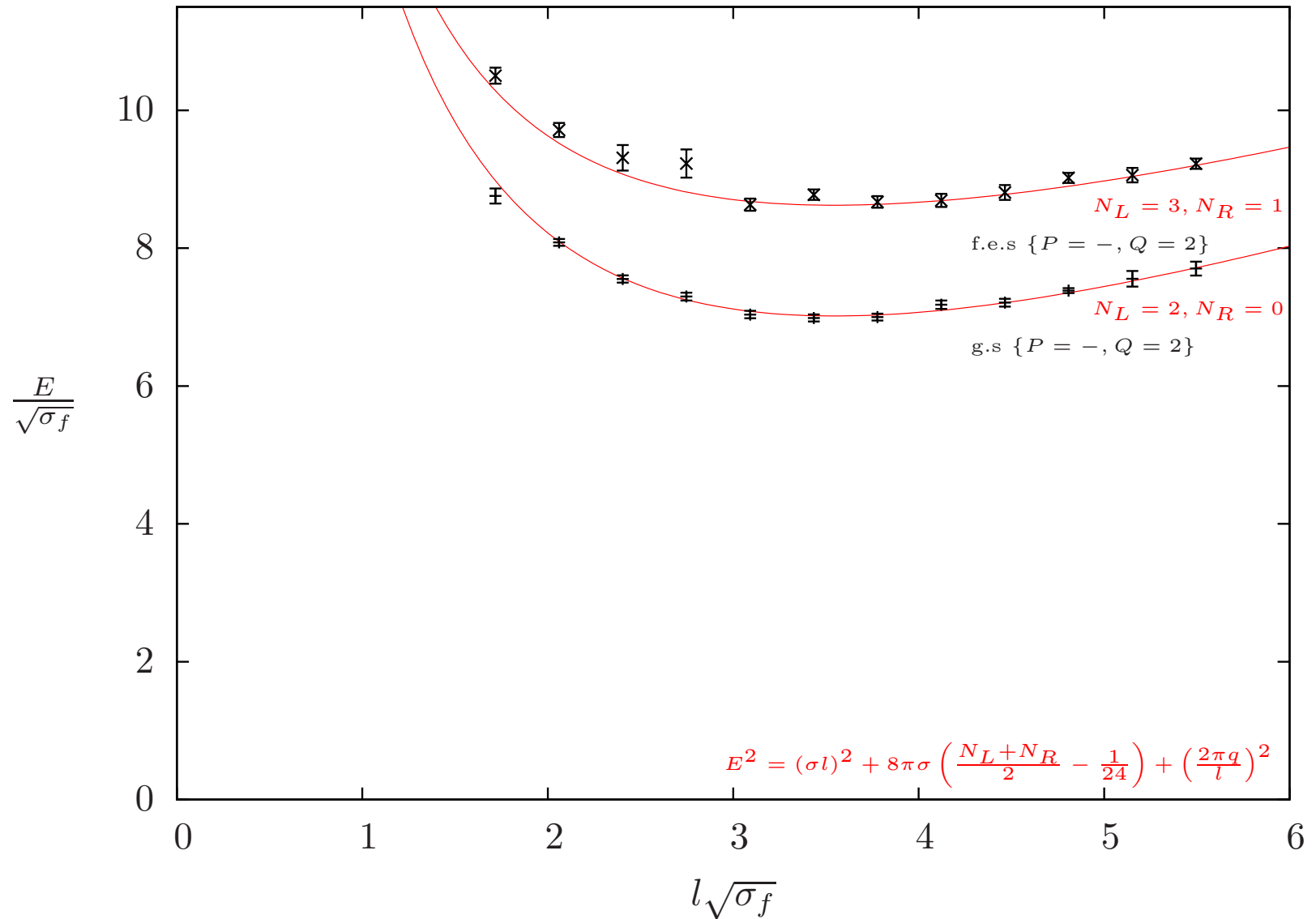
4. D=2+1 Results: $N_L = 3, N_R = 1, q = 2, P = +$



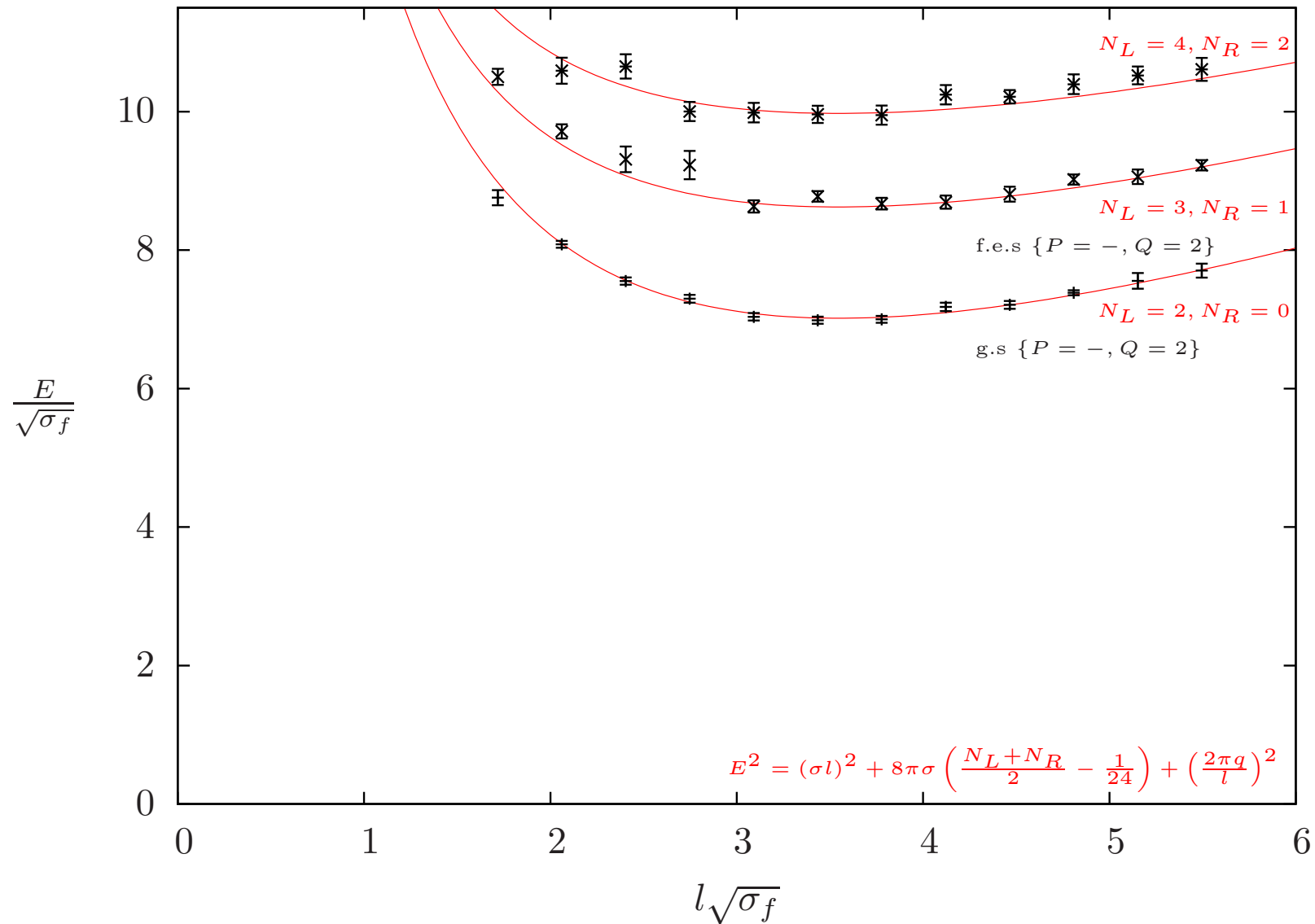
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5. $D=3+1$

5. D=3+1: Lattice Calculation

→ We define our theory on a 4D Euclidean lattice with volume:

$$L_{\parallel} \times L_{\perp_1} \times L_{\perp_2} \times L_T.$$

→ We use the standard Wilson Action:

$$S_L = \beta \sum_p \left\{ 1 - \frac{1}{N} \text{ReTr} U_p \right\} \quad \text{with} \quad \beta = \frac{2N}{g^2}$$

→ Monte Carlo simulations:

- First: $SU(3)$, $\beta = 6.0625$ with $a\sqrt{\sigma_f} \simeq 0.195$ ($a \simeq 0.09\text{fm}$).
- $a \rightarrow 0$: $SU(3)$, $\beta = 6.3380$ with $a\sqrt{\sigma_f} \simeq 0.129$ ($a \simeq 0.06\text{fm}$).
- $N \rightarrow \infty$: $SU(5)$, $\beta = 17.630$ with $a\sqrt{\sigma_f} \simeq 0.197$ ($a \simeq 0.09\text{fm}$).

→ Our Approach:

- Create a large basis of operators $\Phi_{i,q,J,P_{\mathcal{R}},P_{\mathcal{P}}} : i = 1, 2, \dots, N_O, (\sim 700)$.
- Calculate the correlation matrix
$$C_{ij,q,J,P_{\mathcal{R}},P_{\mathcal{P}}}(t) = \langle \Phi_{i,q,J,P_{\mathcal{R}},P_{\mathcal{P}}}^\dagger(t) \Phi_{j,q,J,P_{\mathcal{R}},P_{\mathcal{P}}}(0) \rangle$$
- Use the variational technique to extract correlators of different states.

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5. D=3+1: Symmetries: Flux-Tubes

More complicated structure than $D = 2 + 1$:

- Lattice Symmetry of Rotations about the string axis.
- $C_{4\nu} \otimes Z(\mathcal{R})$ for zero longitudinal momentum.
 - Rotations of $\pi/2 \rightarrow$ angular momentum J
 - Reflections in orthogonal plane (\mathcal{P} -Parity)
 - Reflections about the mid-point on the principal axis (\mathcal{R} -Parity)
- 10 irreducible representations \equiv 10 correlation matrices
- $C_{4\nu}$ for non-zero longitudinal momentum.
 - Rotations of $\pi/2 \rightarrow$ angular momentum J
 - Reflections in orthogonal plane (\mathcal{P} -Parity)
- 5 irreducible representations \equiv 5 correlation matrices
- $A_1 \equiv (|J| = 0, 4, \dots, 4N, P_{\mathcal{P}} = +)$, $A_2 \equiv (|J| = 0, 4, \dots, 4N, P_{\mathcal{P}} = -)$
- $E \equiv (|J| = 1, 3, \dots, \pm 2N + 1, P_{\mathcal{P}} = \pm)$
- $B_1 \equiv (|J| = 2, 6, \dots, 4N + 2, P_{\mathcal{P}} = +)$, $B_2 \equiv (|J| = 2, 6, \dots, 4N + 2, P_{\mathcal{P}} = -)$

5. D=3+1: Symmetries: Bosonic String

Two transverse directions:

→ Define α_{-k}^+ and α_{-k}^- as (x, y are the transverse directions):

$$- \alpha_{-k}^+ = \alpha_{-k}^x + i\alpha_{-k}^y$$

$$- \alpha_{-k}^- = \alpha_{-k}^x - i\alpha_{-k}^y$$

→ Spin J .

$$- J = | \#(+) - \#(-) |$$

→ \mathcal{P} -Parity

$$- \text{Under } \mathcal{P}\text{-Parity: } \alpha_{-k}^+ \xleftrightarrow{P_{\mathcal{P}}} \alpha_{-k}^- \text{ \& } \bar{\alpha}_{-k}^+ \xleftrightarrow{P_{\mathcal{P}}} \bar{\alpha}_{-k}^-$$

→ \mathcal{R} -Parity

$$- \text{Under } \mathcal{R}\text{-Parity: } \alpha_{-k}^{\pm} \xleftrightarrow{P_{\mathcal{R}}} \bar{\alpha}_{-k}^{\pm}$$

• Example: $(\alpha_{-1}^+ \bar{\alpha}_{-1}^+ \pm \alpha_{-1}^- \bar{\alpha}_{-1}^-) | 0 \rangle$

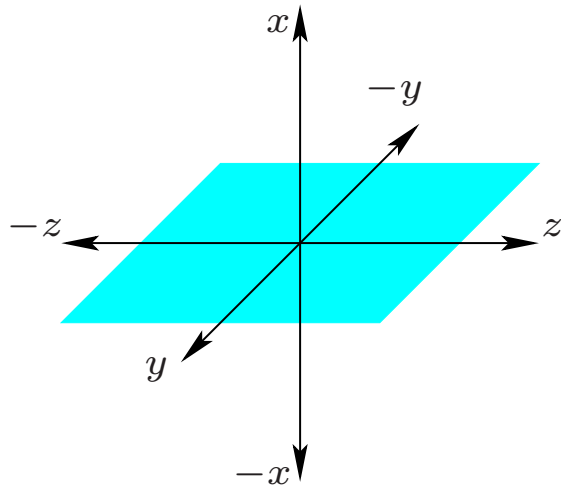
$$- J = 2$$

$$- P_{\mathcal{P}} = \pm$$

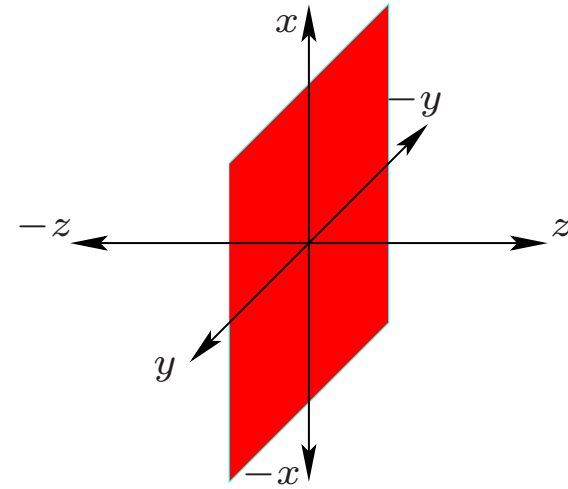
$$- P_{\mathcal{R}} = +$$

5. D=3+1: Quantum Numbers: Parity

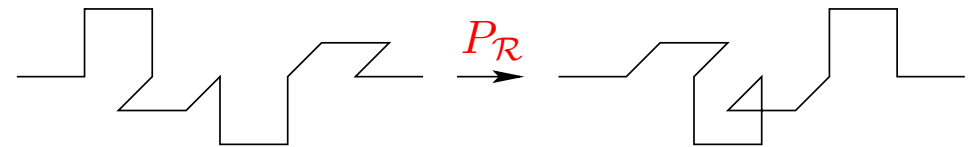
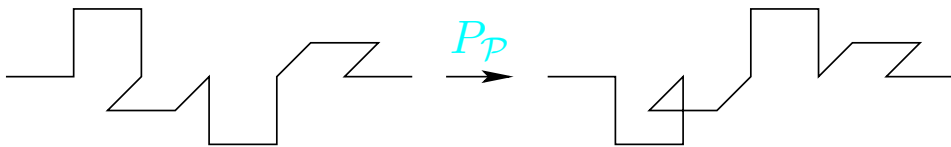
\mathcal{P} -Parity reflection plane



\mathcal{R} -Parity reflection plane

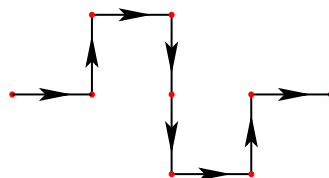


Example:



5. D=3+1: Quantum Numbers: Spin

→ Operator ϕ is given by the trace of path ordered product of blocked links



→ We can then form an operator of spin J :

$$\text{Continuum : } \phi(J) = \int d\theta e^{iJ\theta} \phi_\theta$$

$$\text{Lattice : } \phi_L(J) = \sum_n e^{iJn\frac{\pi}{2}} \phi_{n\frac{\pi}{2}}$$

→ Example $J = 1$:

$$\phi_L(J = 1) = i\phi_{\frac{\pi}{2}} - \phi_\pi - i\phi_{\frac{3\pi}{2}} + \phi_{2\pi}$$

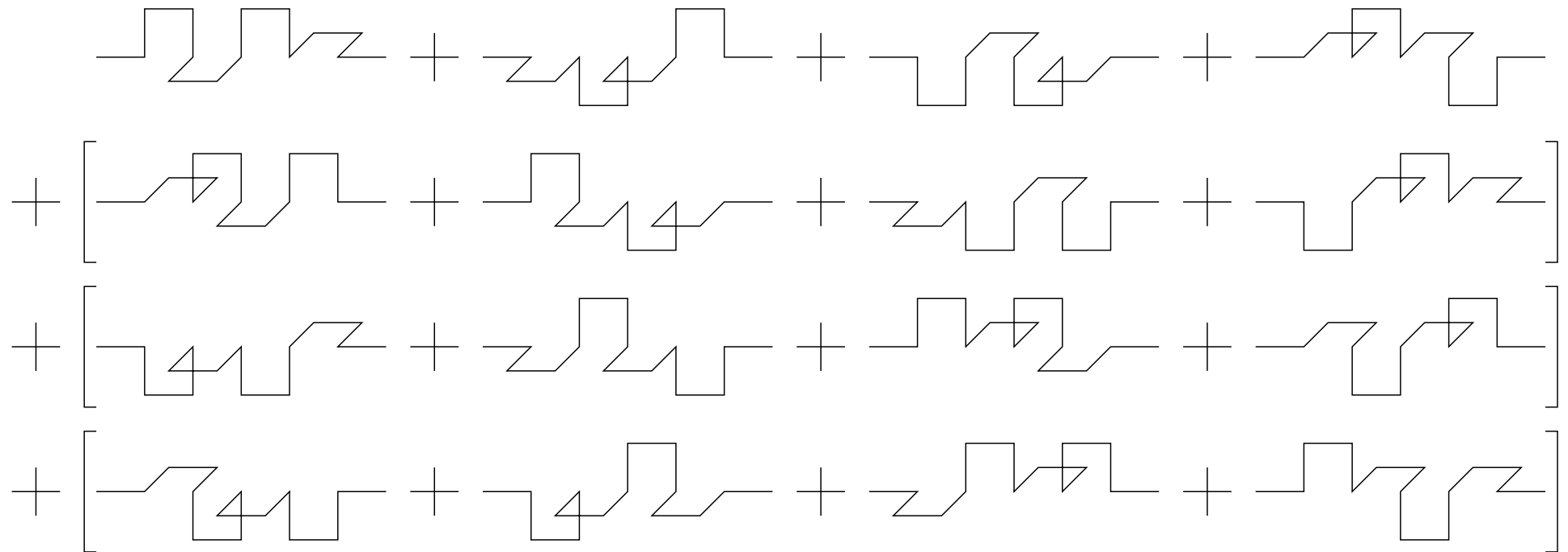
→ If $\phi_{\theta=0} \equiv \text{Tr} \left\{ \begin{array}{c} \text{---} \uparrow \text{---} \\ \text{---} \downarrow \text{---} \end{array} \right\}$

$$\phi_L(J = 1) = \text{Tr} \left\{ \begin{array}{c} \text{---} \uparrow \text{---} \\ \text{---} \downarrow \text{---} \end{array} + i \begin{array}{c} \text{---} \nearrow \text{---} \\ \text{---} \searrow \text{---} \end{array} - \begin{array}{c} \text{---} \downarrow \text{---} \\ \text{---} \uparrow \text{---} \end{array} - i \begin{array}{c} \text{---} \searrow \text{---} \\ \text{---} \nearrow \text{---} \end{array} \right\}$$

5. D=3+1: Operators: Example $J = 0, P_{\mathcal{P}} = +, P_{\mathcal{R}} = +$

Irreducible representation: $A_1, P_{\mathcal{R}} = +$.

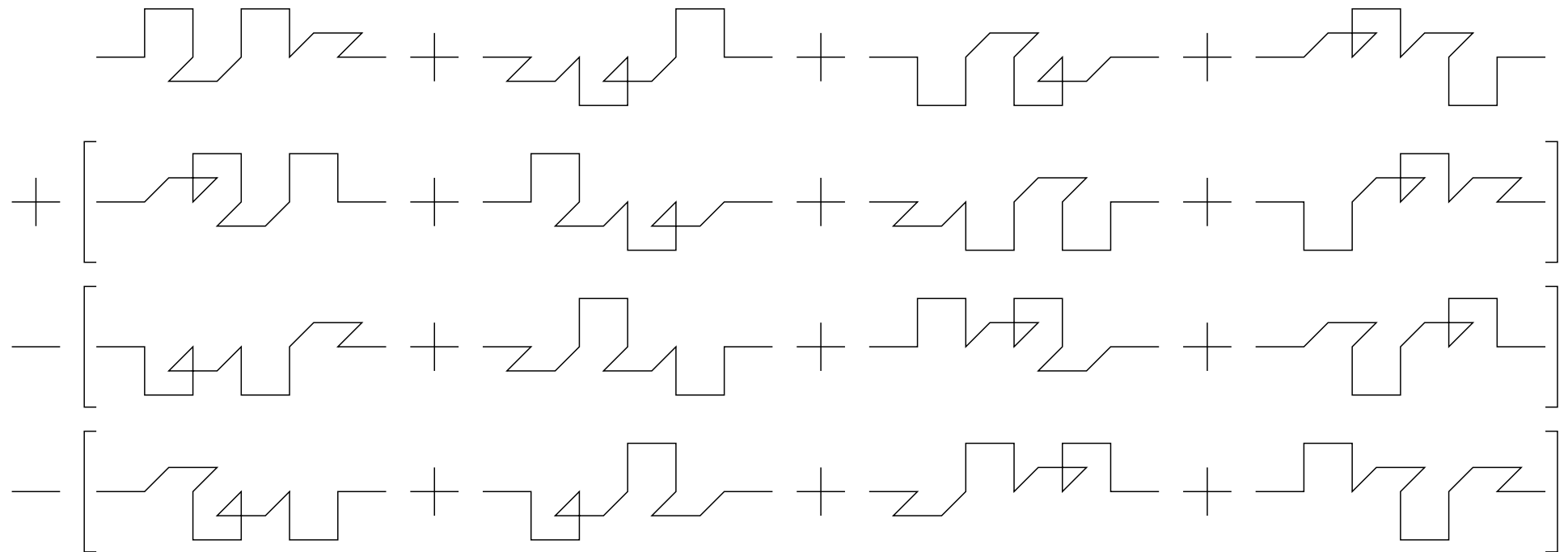
Operator before being traced:



5. D=3+1: Operators: Example $J = 0, P_{\mathcal{P}} = -, P_{\mathcal{R}} = +$

Irreducible representation: $A_2, P_{\mathcal{R}} = +$.

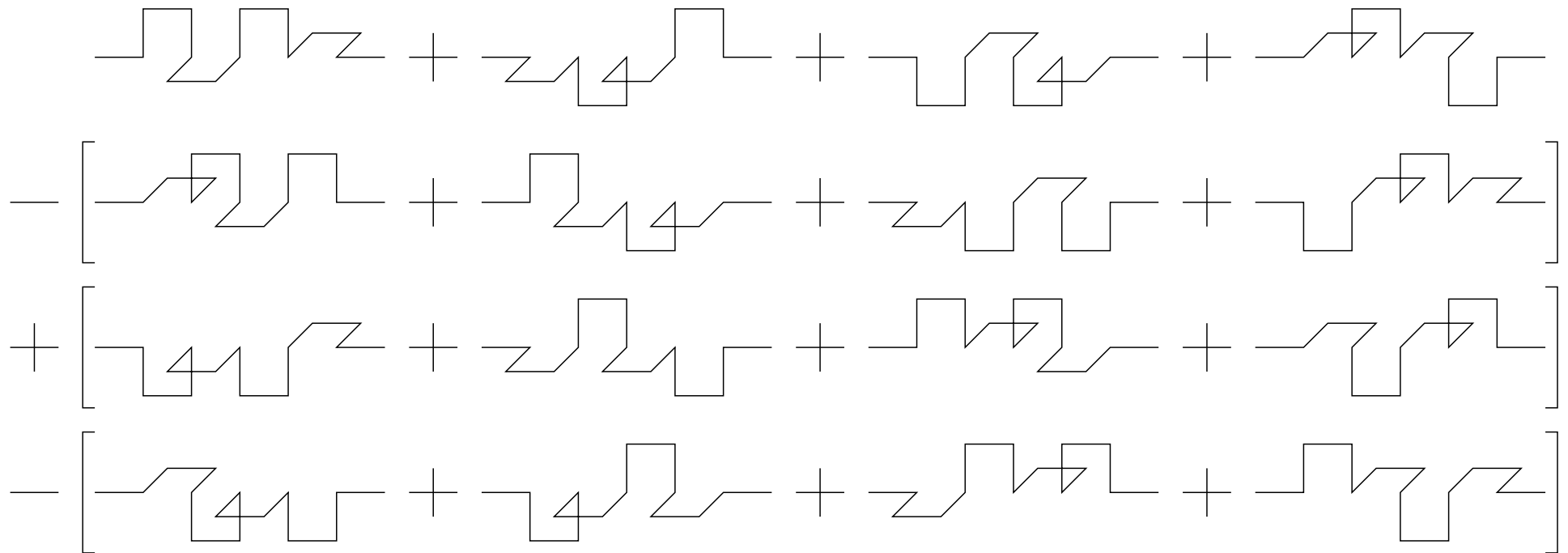
Operator before being traced:



5. D=3+1: Operators: Example $J = 0, P_{\mathcal{P}} = +, P_{\mathcal{R}} = -$

Irreducible representation: $A_1, P_{\mathcal{R}} = -$.

Operator before being traced:



5. D=3+1: Operators: Example $J = 0, P_{\mathcal{P}} = -, P_{\mathcal{R}} = -$

Irreducible representation: $A_2, P_{\mathcal{R}} = -$.


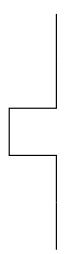
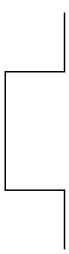
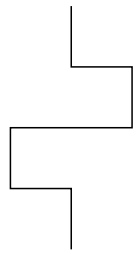

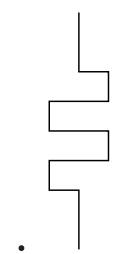
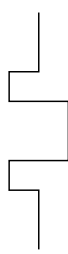
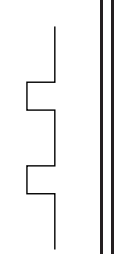
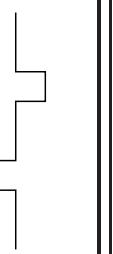
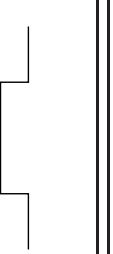
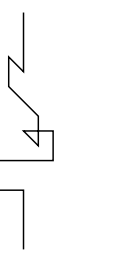
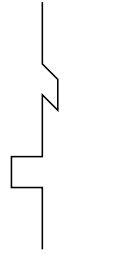
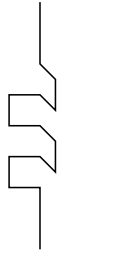
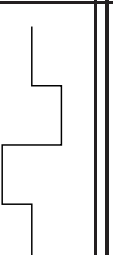
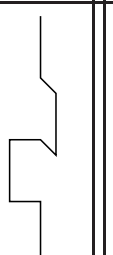
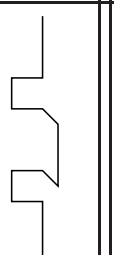
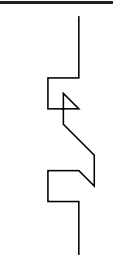

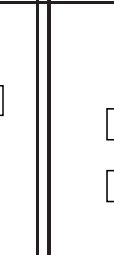

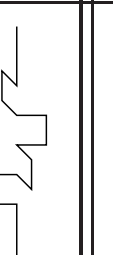
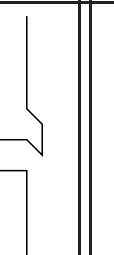
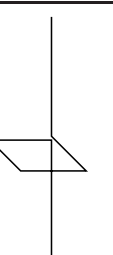
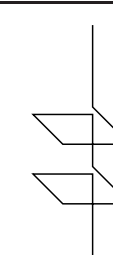

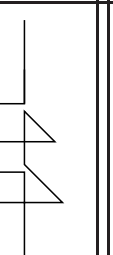
Operator before being traced:

The image displays a set of 20 Feynman diagrams, organized into four rows. Each diagram is a black line with various step-like and loop-like structures. The first row contains four diagrams with plus signs between them. The second row contains four diagrams enclosed in square brackets, preceded by a minus sign. The third row contains four diagrams enclosed in square brackets, preceded by a minus sign. The fourth row contains four diagrams enclosed in square brackets, preceded by a plus sign.

5. D=3+1: Operators: Transverse Deformations

→ ~ 700 operators

Tranverse deformations in 3 spatial dimensions:

1	2	3	4	5	6	7	8	9	10	11	12	13
												
14	15	16	17	18	19	20	21	22	23	24	25	26
												

5. D=3+1 Results: $N_L = N_R = 0$

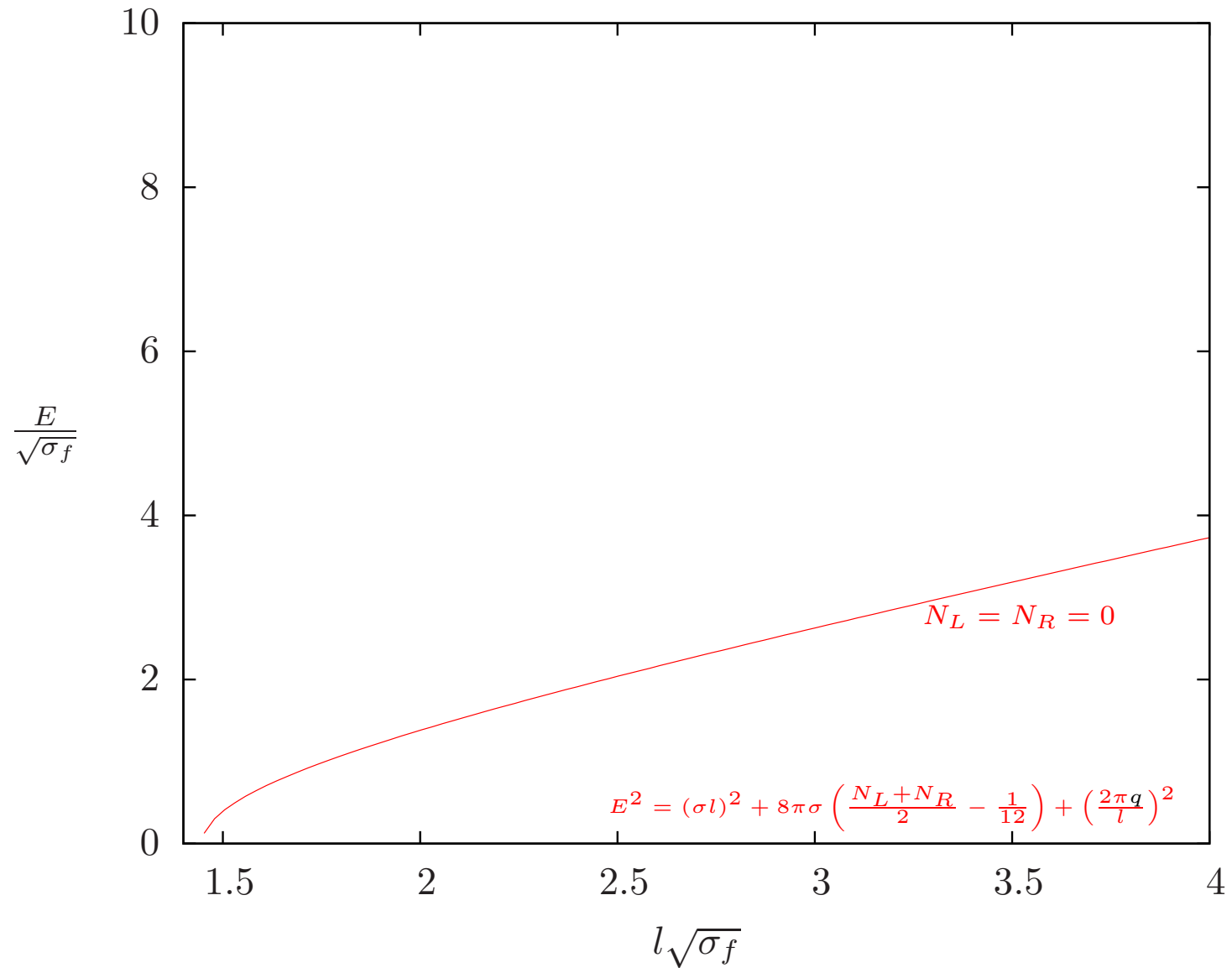
- $N_L = N_R = 0$ corresponds to the ground state (No movers)
- String theoretically: $|0\rangle$
- No phonons
- Expected to appear as $\{J = 0, P_{\mathcal{P}} = +, P_{\mathcal{R}} = +, q = 0\}$ ground state
- Apply variational technique to the correlation matrix $\{A_1, P_{\mathcal{R}} = +, q = 0\}$.
- Extraction of the ground state.
 - Fit data with Nambu-Goto or L.O/N.L.O/N.N.L.O $\rightarrow a^2\sigma_f$ (string tension)
 - This provides parameter free predictions for the energies in N.G:

$$E_{N_L, N_R, q, w}^2 = (\sigma_f l w)^2 + 8\pi\sigma_f \left(\frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left(\frac{2\pi q}{l} \right)^2.$$

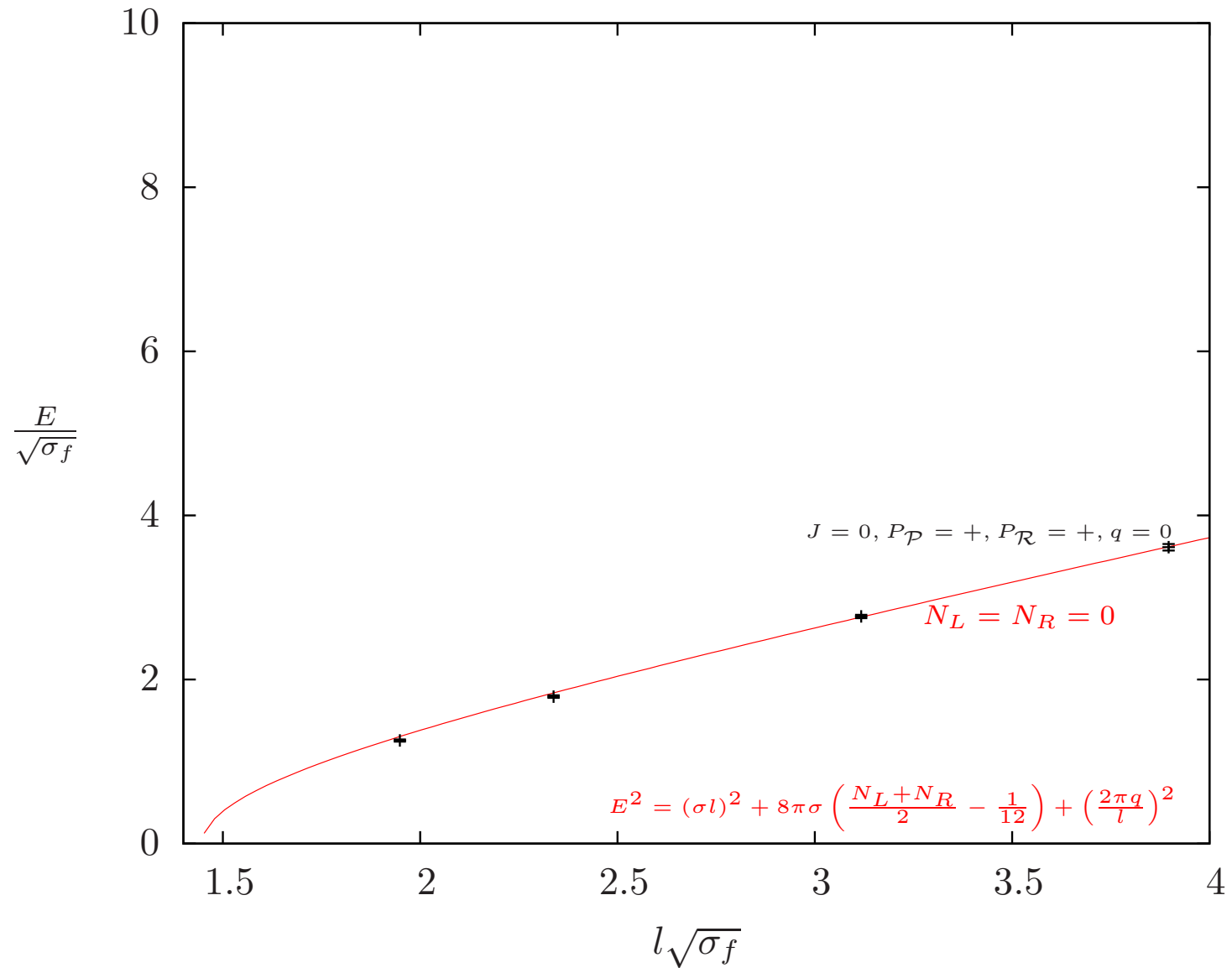
In lattice units:

$$a^2 E_{N_L, N_R, q, w}^2 = (a^2\sigma_f L w)^2 + 8\pi a^2\sigma_f \left(\frac{N_L + N_R}{2} - \frac{1}{12} \right) + \left(\frac{2\pi q}{L} \right)^2.$$

5. D=3+1 Results: $N_L = N_R = 0$



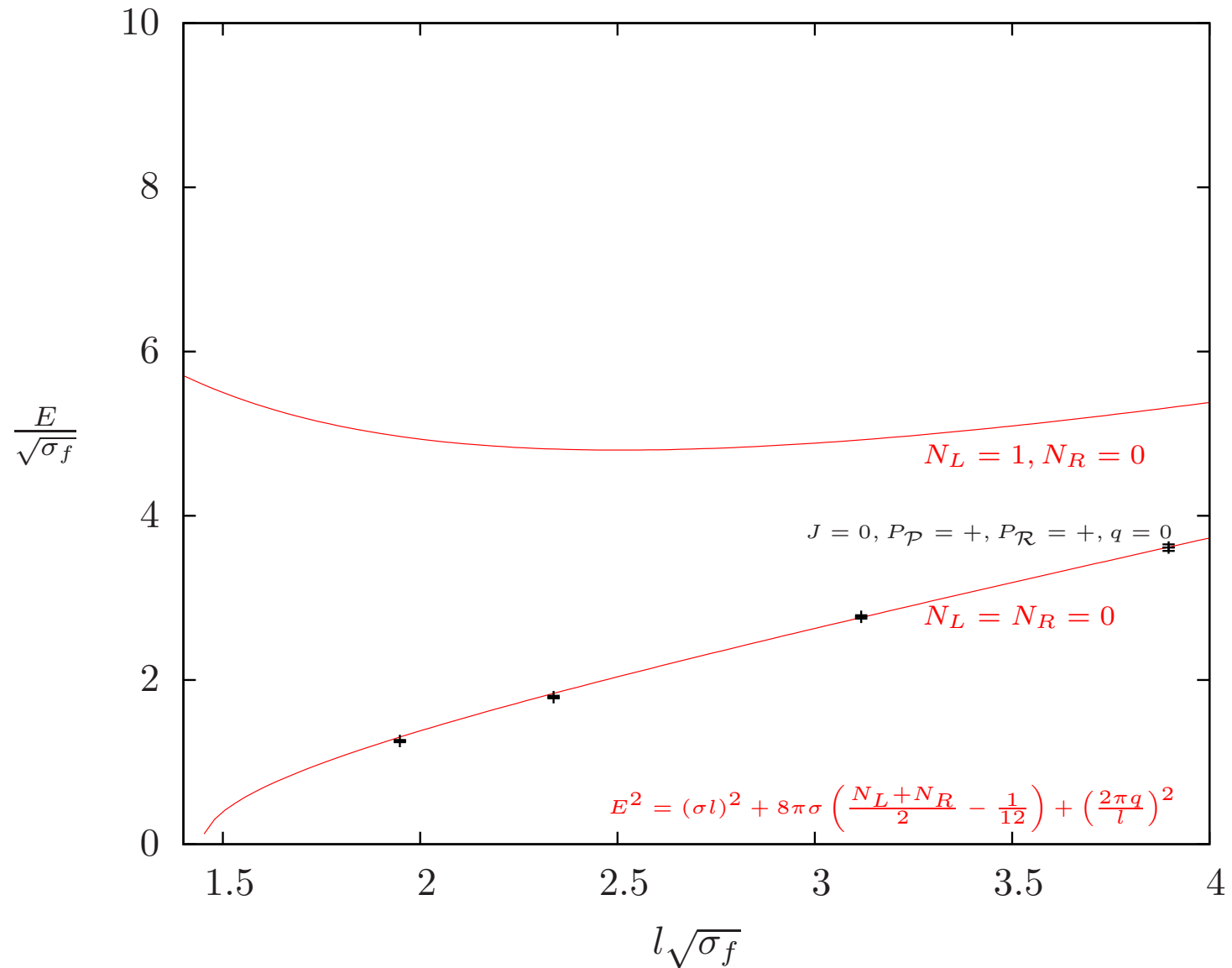
5. D=3+1 Results: $N_L = N_R = 0$



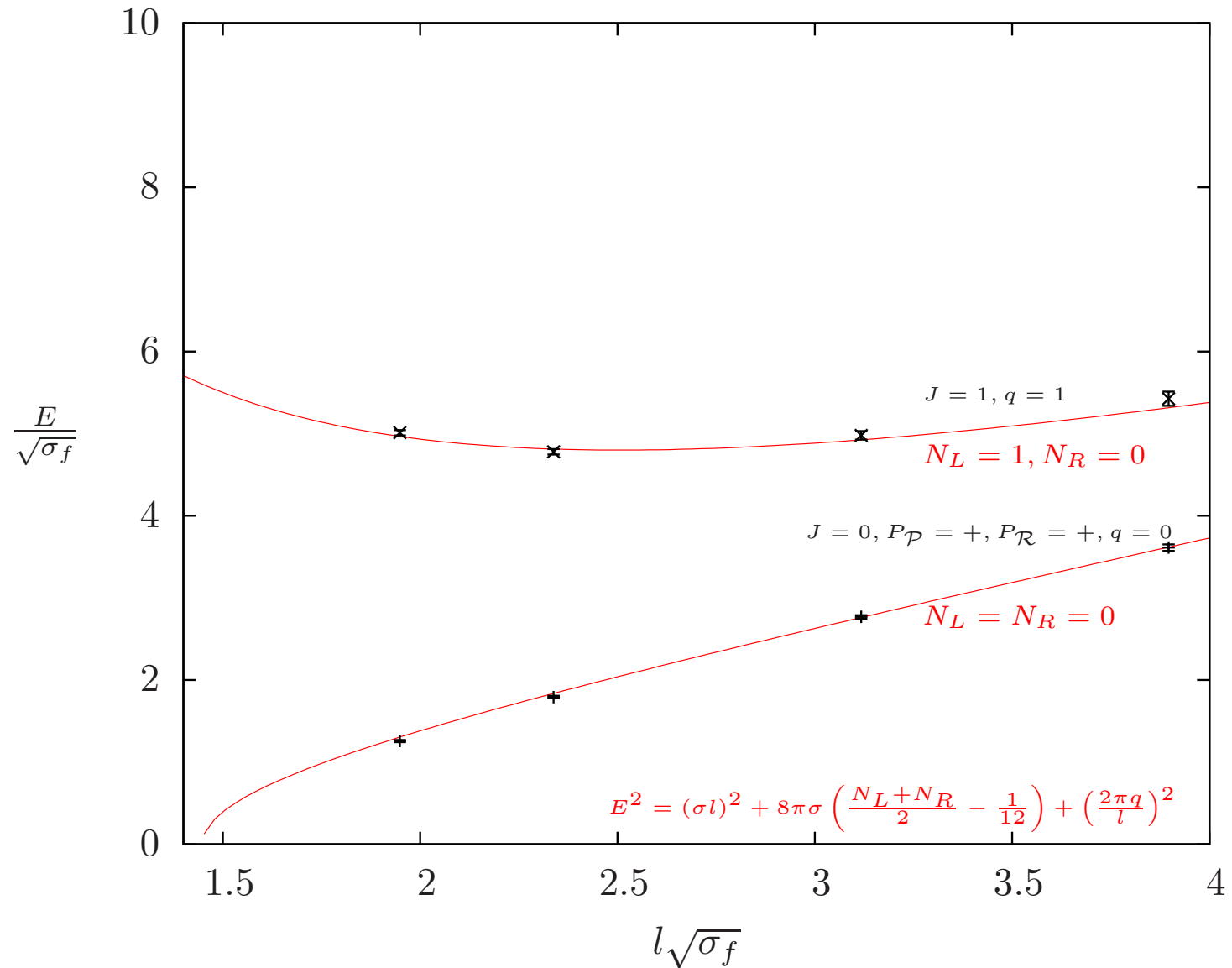
5. D=3+1 Results: $N_L = 1, N_R = 0$

- $N_L = 1, N_R = 0$ corresponds to the $q = 1$ ground state
- String theoretically: $(\alpha_{-1}^+ + \alpha_{-1}^-, \alpha_{-1}^+ - \alpha_{-1}^-) | 0 \rangle$
- One phonon $\rightarrow J = 1$
- Expected to appear as $\{J = 1, q = 1\}$ ground state
- Apply variational technique to the correlation matrix $\{E, q = 1\}$.
- Extraction of the $\{E, q = 1\}$ ground state.

5. D=3+1 Results: $N_L = 1, N_R = 0$



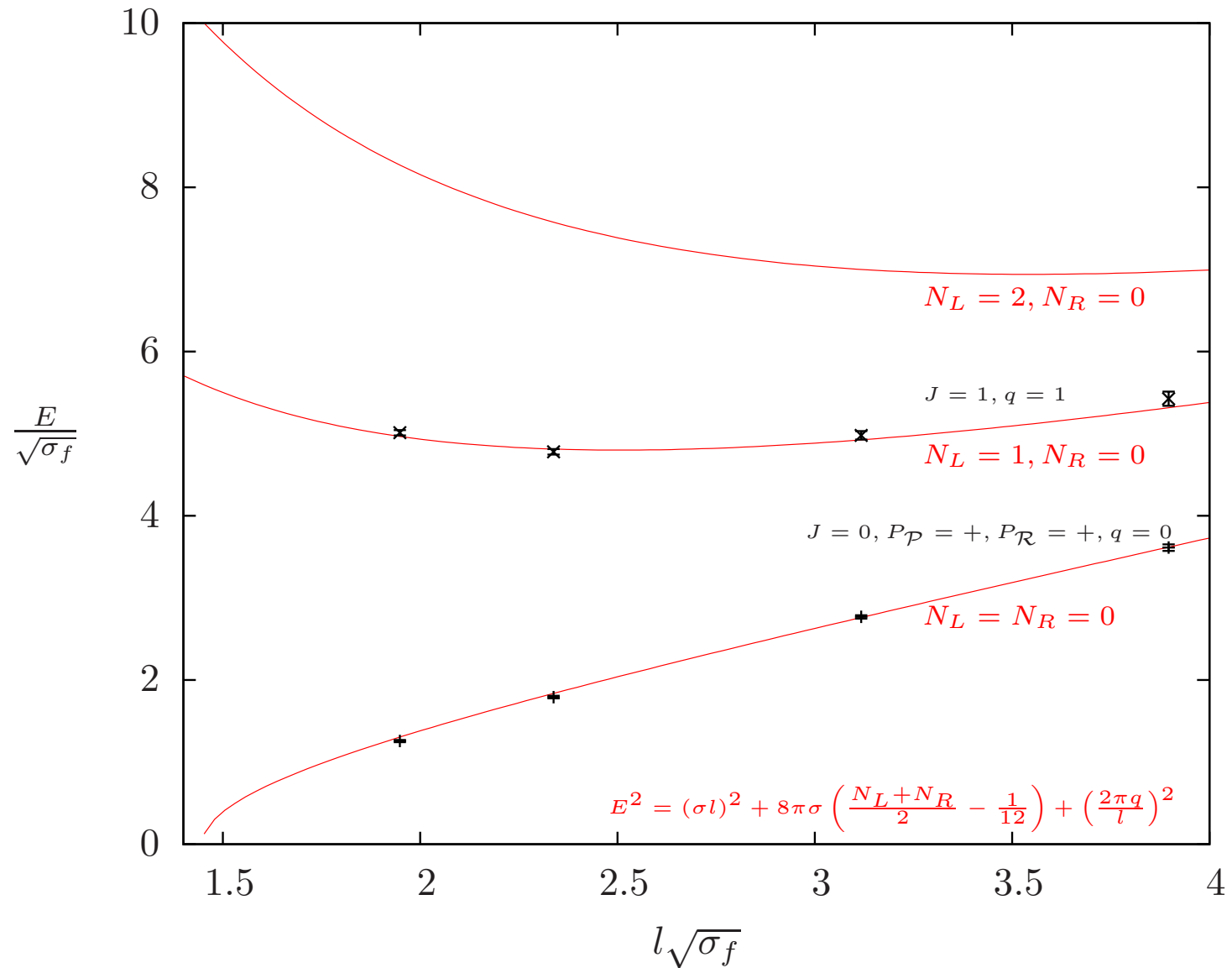
5. D=3+1 Results: $N_L = 1, N_R = 0$



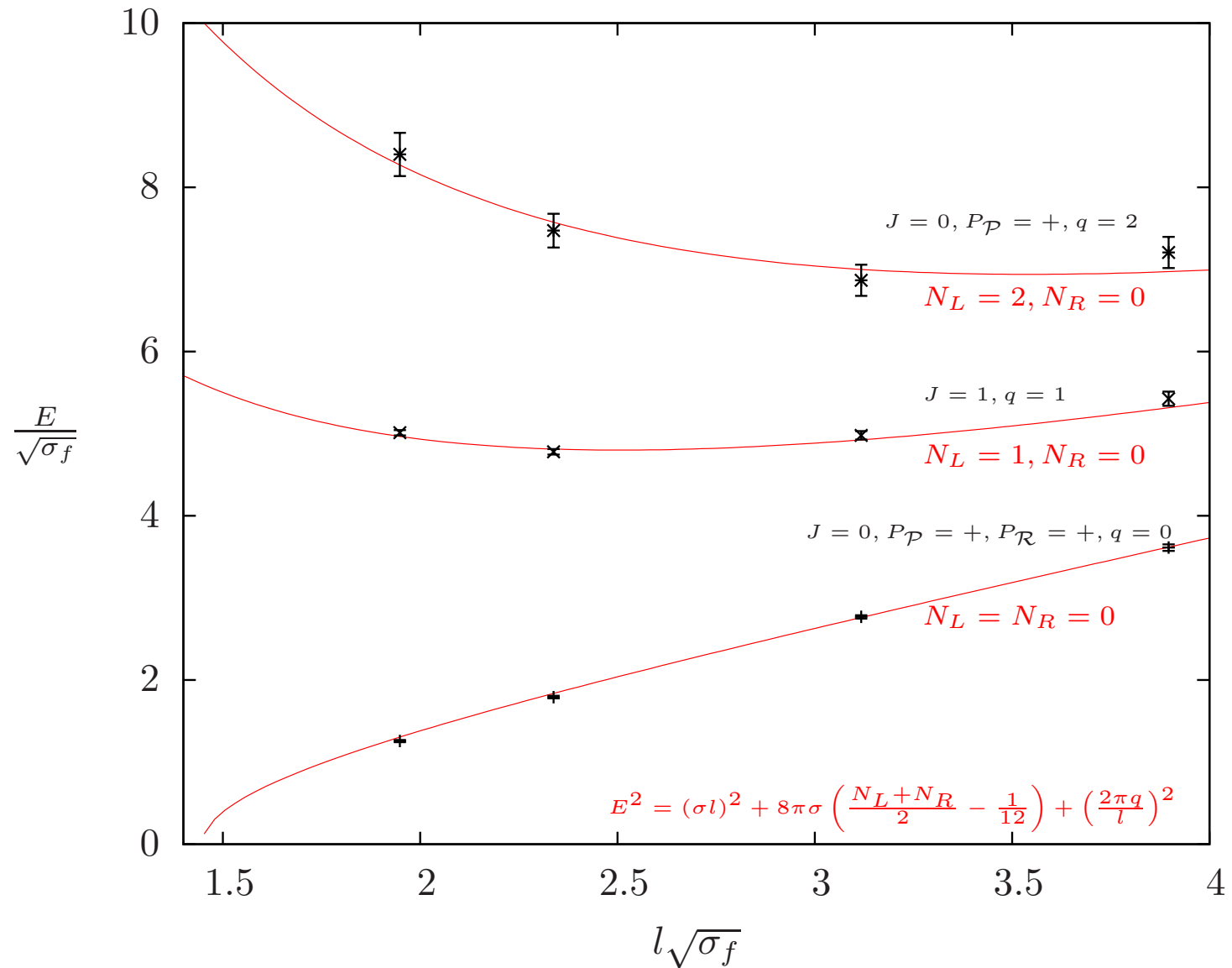
5. D=3+1 Results: $N_L = 2, N_R = 0$

- $N_L = 2, N_R = 0$ corresponds to the $q = 2$ ground state
- String theoretically:
 - $\alpha_{-1}^+ \alpha_{-1}^- | 0 \rangle \sim \text{g.s } J = 0, P_{\mathcal{P}} = +, q = 2$
 - $(\alpha_{-2}^+ + \alpha_{-2}^-, \alpha_{-2}^+ - \alpha_{-2}^-) | 0 \rangle \sim \text{g.s } J = 1, q = 2$
 - $(\alpha_{-1}^+ \alpha_{-1}^+ + \alpha_{-1}^- \alpha_{-1}^-) | 0 \rangle \sim \text{g.s } J = 2, P_{\mathcal{P}} = +, q = 2$
 - $(\alpha_{-1}^+ \alpha_{-1}^+ - \alpha_{-1}^- \alpha_{-1}^-) | 0 \rangle \sim \text{g.s } J = 2, P_{\mathcal{P}} = -, q = 2$
- One phonon $\rightarrow J = 1$, Two phonons $\rightarrow J = 0, 2$.
- One state expected to appear as $\{A_1, q = 2\}$ ground state
- One state expected to appear as $\{E, q = 2\}$ ground state
- One state expected to appear as $\{B_1, q = 2\}$ ground state
- One state expected to appear as $\{B_2, q = 2\}$ ground state
- Apply variational technique to the correlation matrices.

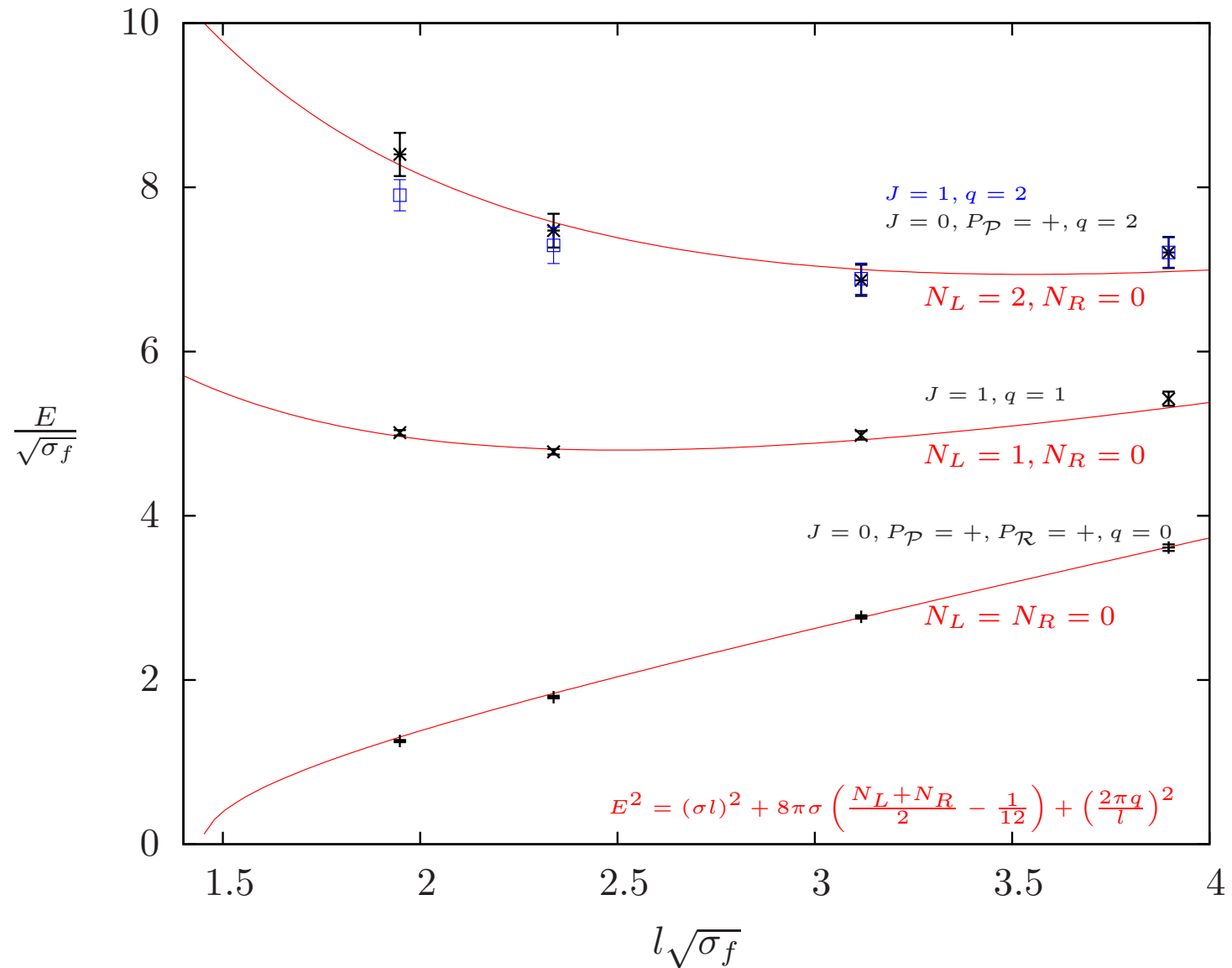
5. D=3+1 Results: $N_L = 2, N_R = 0$



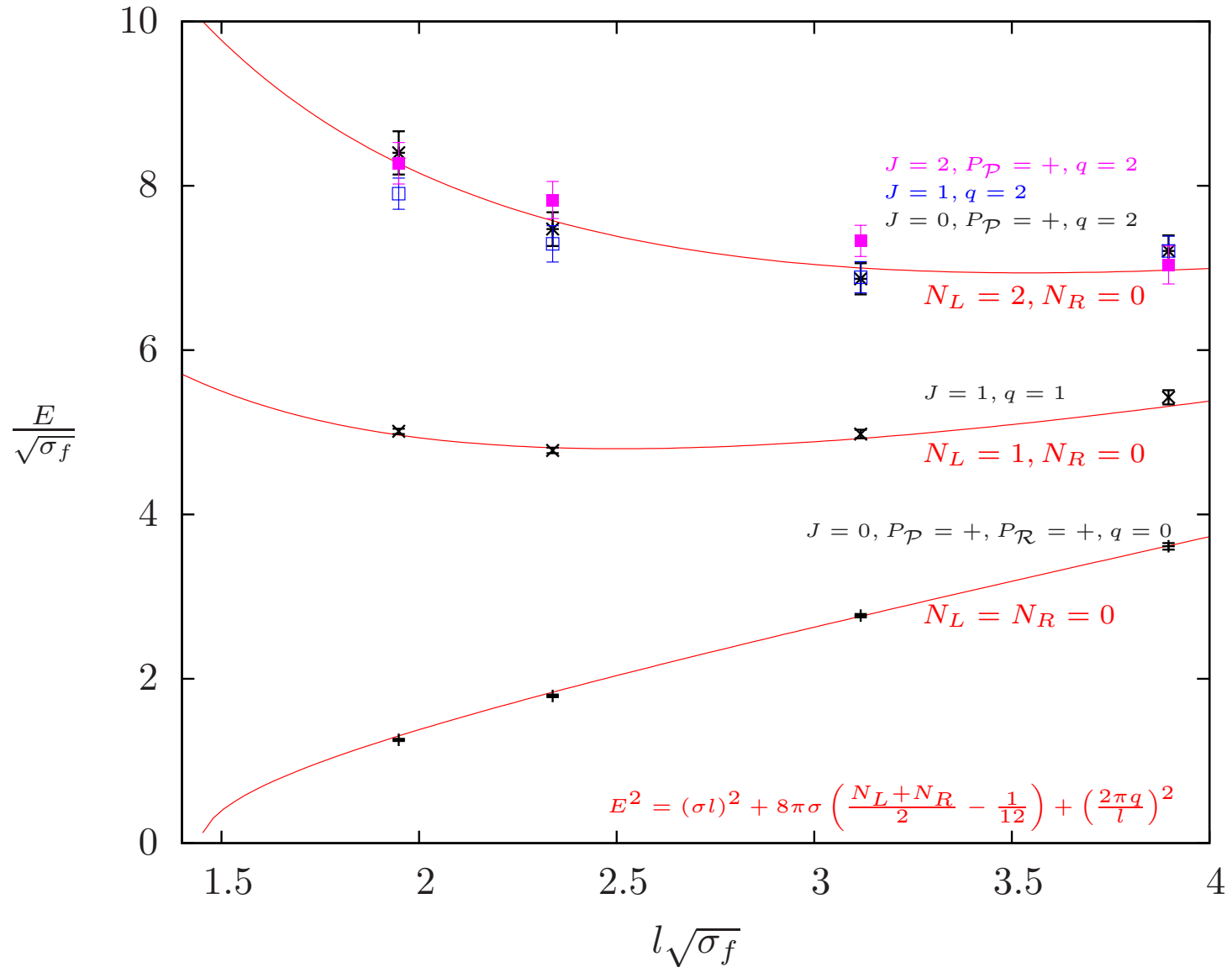
5. D=3+1 Results: $N_L = 2, N_R = 0$



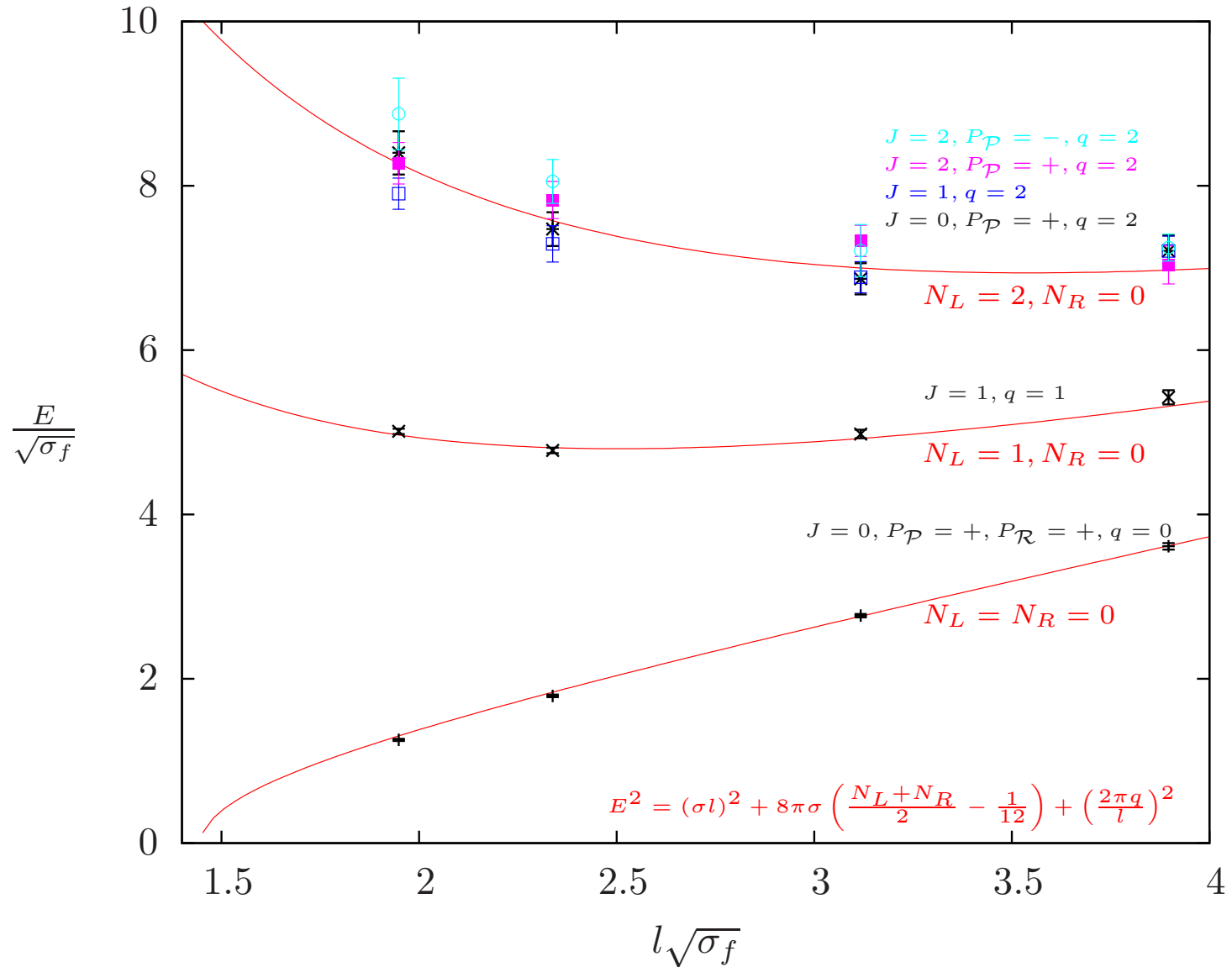
5. D=3+1 Results: $N_L = 2, N_R = 0$



5. D=3+1 Results: $N_L = 2, N_R = 0$



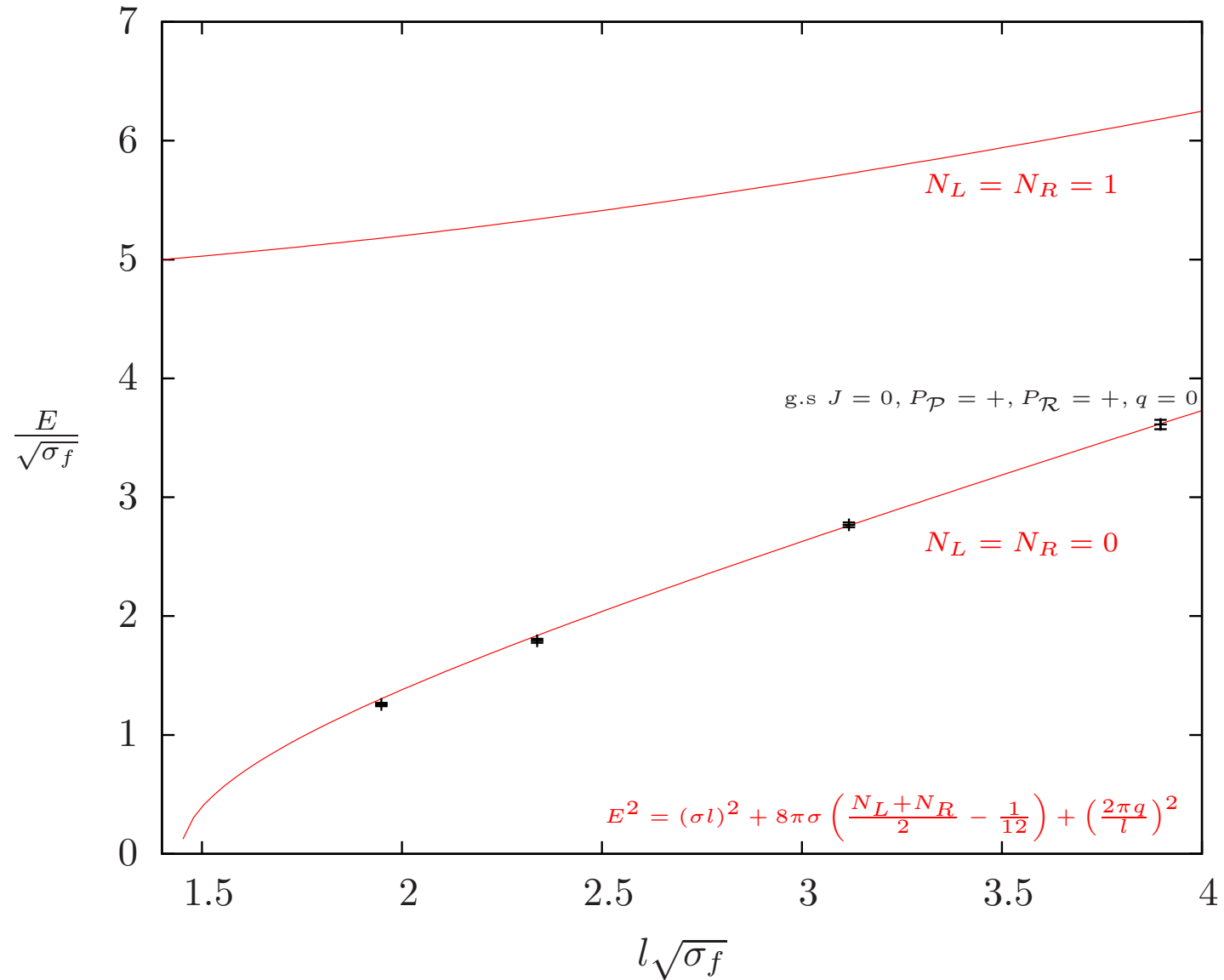
5. D=3+1 Results: $N_L = 2, N_R = 0$



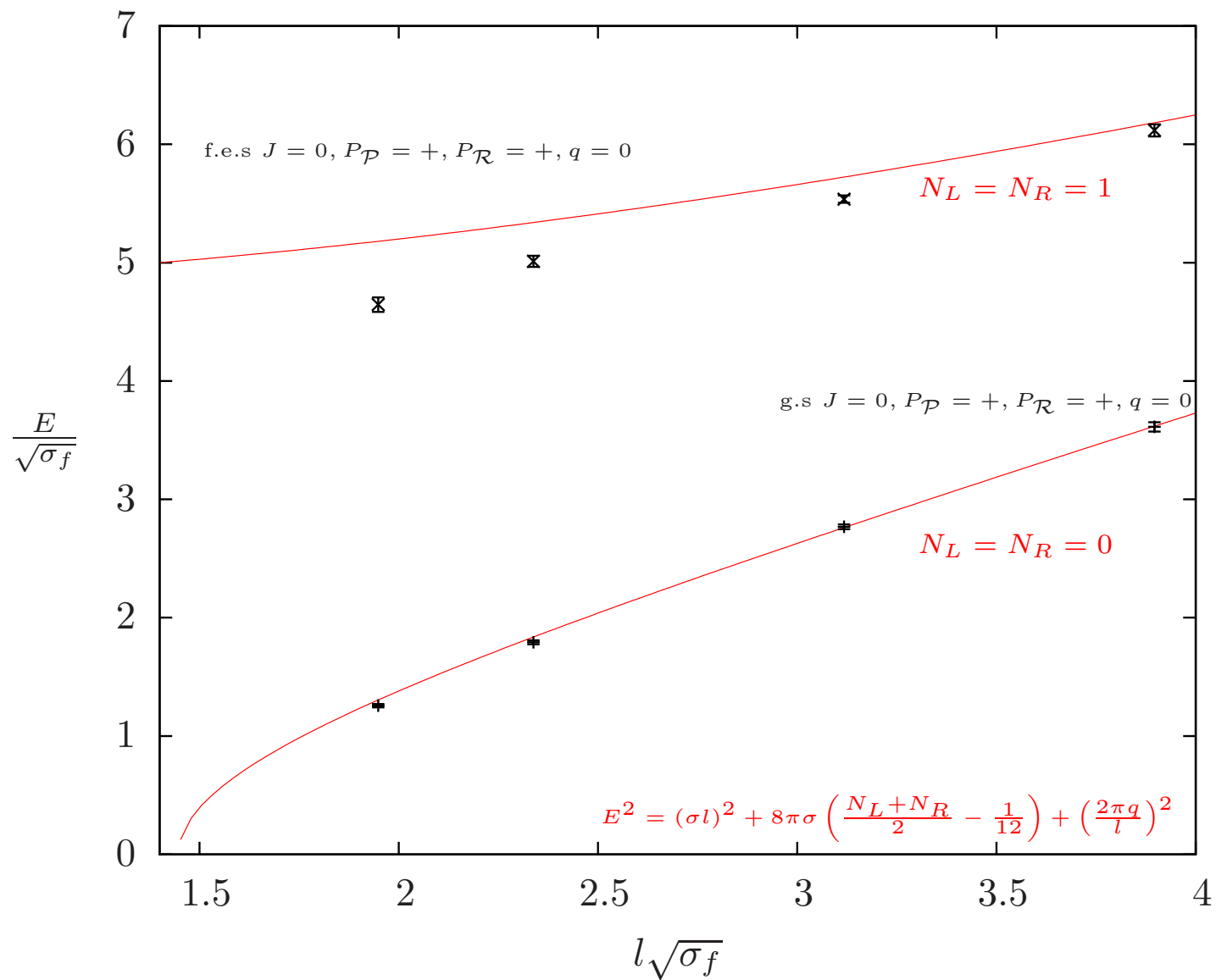
5. D=3+1 Results: $N_L = N_R = 1$

- $N_L = N_R = 1$ corresponds to the $q = 0$ first excited state.
- String theoretically:
 - $(\alpha_{-1}^+ \bar{\alpha}_{-1}^- + \alpha_{-1}^- \bar{\alpha}_{-1}^+) | 0 \rangle \sim$ f.e.s $J = 0, P_{\mathcal{P}} = +, P_{\mathcal{R}} = +, q = 0$
 - $(\alpha_{-1}^+ \bar{\alpha}_{-1}^- - \alpha_{-1}^- \bar{\alpha}_{-1}^+) | 0 \rangle \sim$ g.s $J = 0, P_{\mathcal{P}} = -, P_{\mathcal{R}} = -, q = 0$
 - $(\alpha_{-1}^+ \bar{\alpha}_{-1}^+ + \alpha_{-1}^- \bar{\alpha}_{-1}^-) | 0 \rangle \sim$ g.s $J = 2, P_{\mathcal{P}} = +, P_{\mathcal{R}} = +, q = 0$
 - $(\alpha_{-1}^+ \bar{\alpha}_{-1}^+ - \alpha_{-1}^- \bar{\alpha}_{-1}^-) | 0 \rangle \sim$ g.s $J = 2, P_{\mathcal{P}} = -, P_{\mathcal{R}} = +, q = 0$
- Two phonons $\rightarrow J = 0, 2$.
- One state expected to appear as $\{A_1, P_{\mathcal{R}} = +, q = 0\}$ first excited state
- One state expected to appear as $\{A_2, P_{\mathcal{R}} = -, q = 0\}$ ground state
- One state expected to appear as $\{B_1, P_{\mathcal{R}} = +, q = 0\}$ ground state
- One state expected to appear as $\{B_2, P_{\mathcal{R}} = +, q = 0\}$ ground state
- Apply variational technique to the correlation matrices.

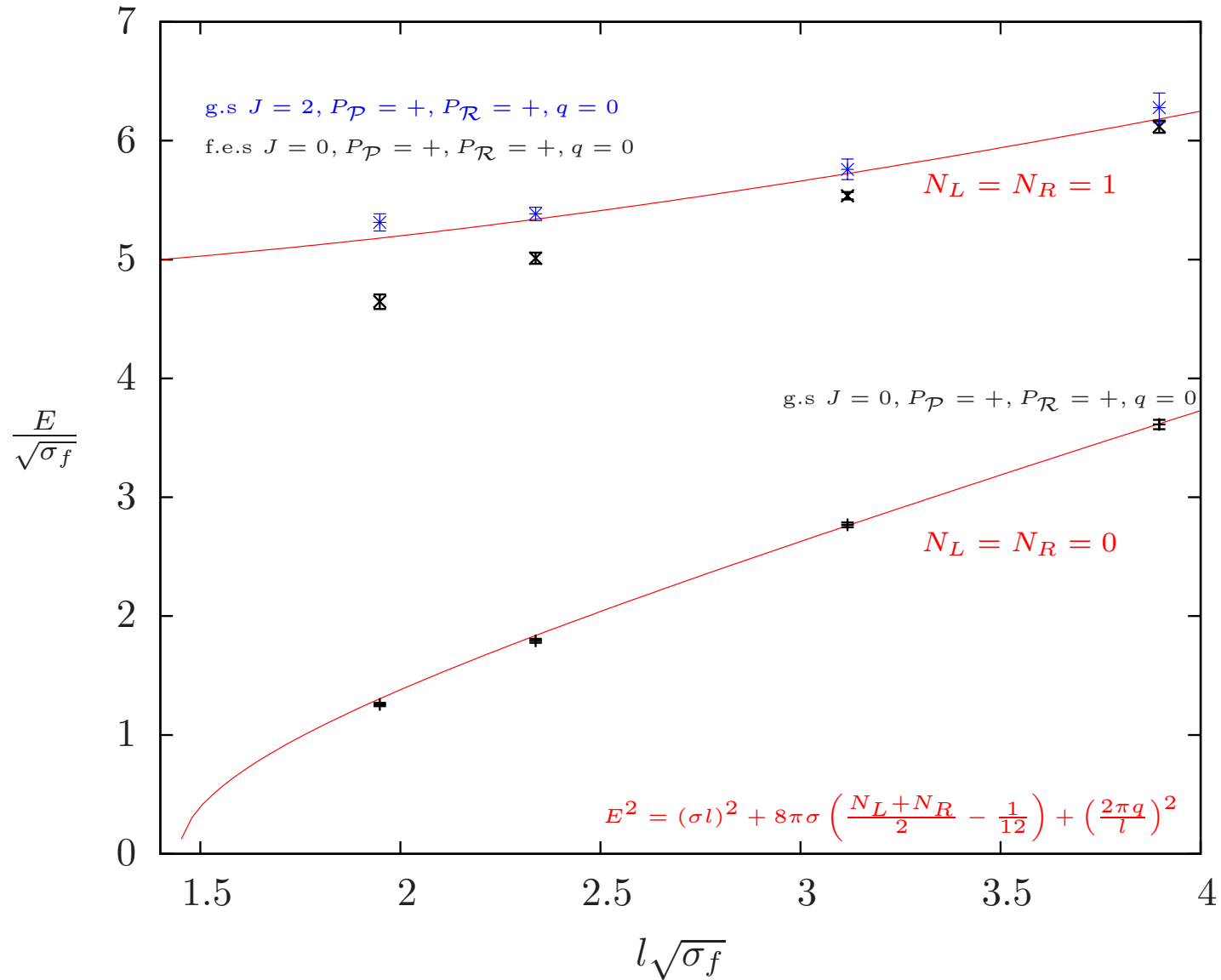
5. D=3+1 Results: $N_L = N_R = 1$



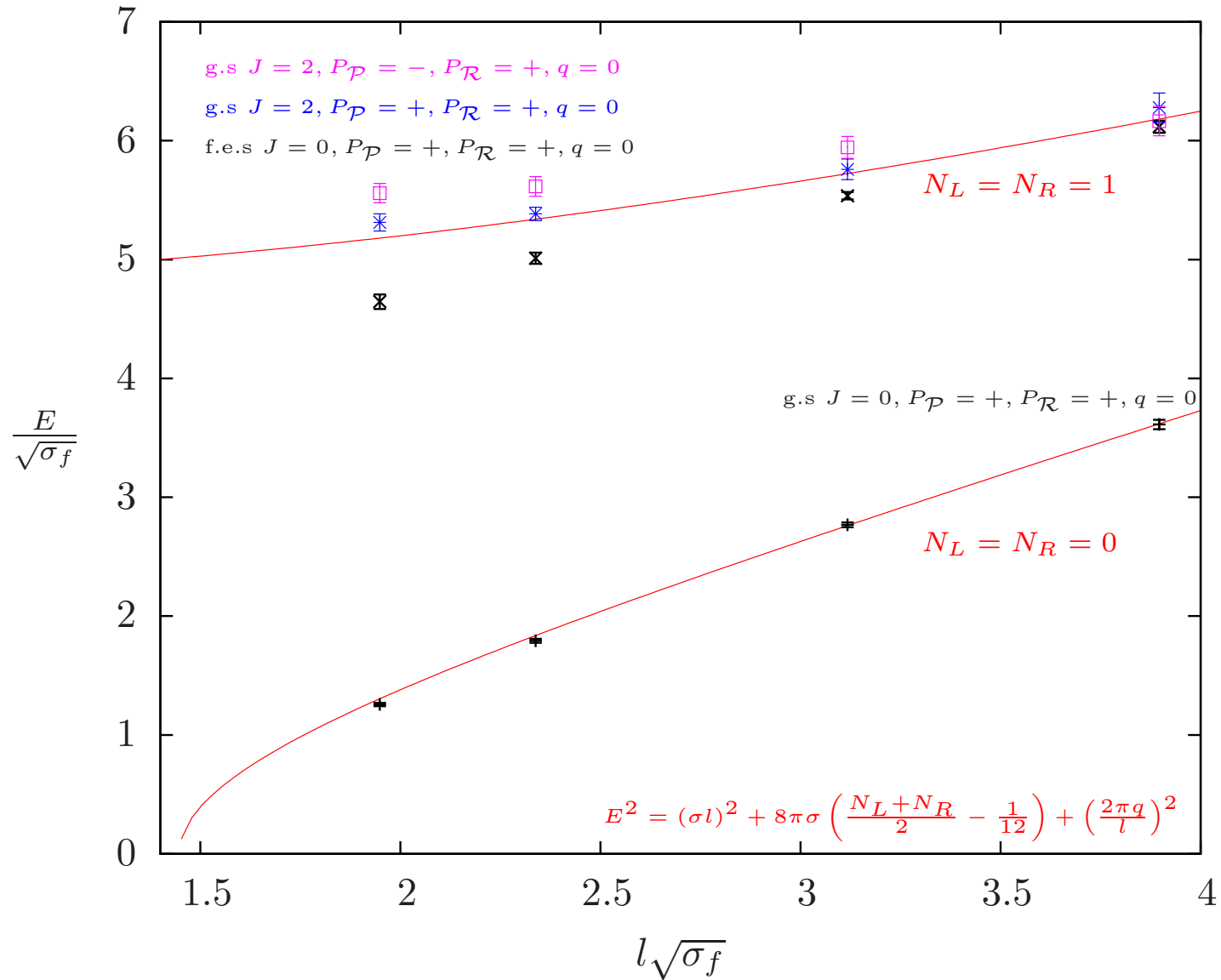
5. D=3+1 Results: $N_L = N_R = 1$



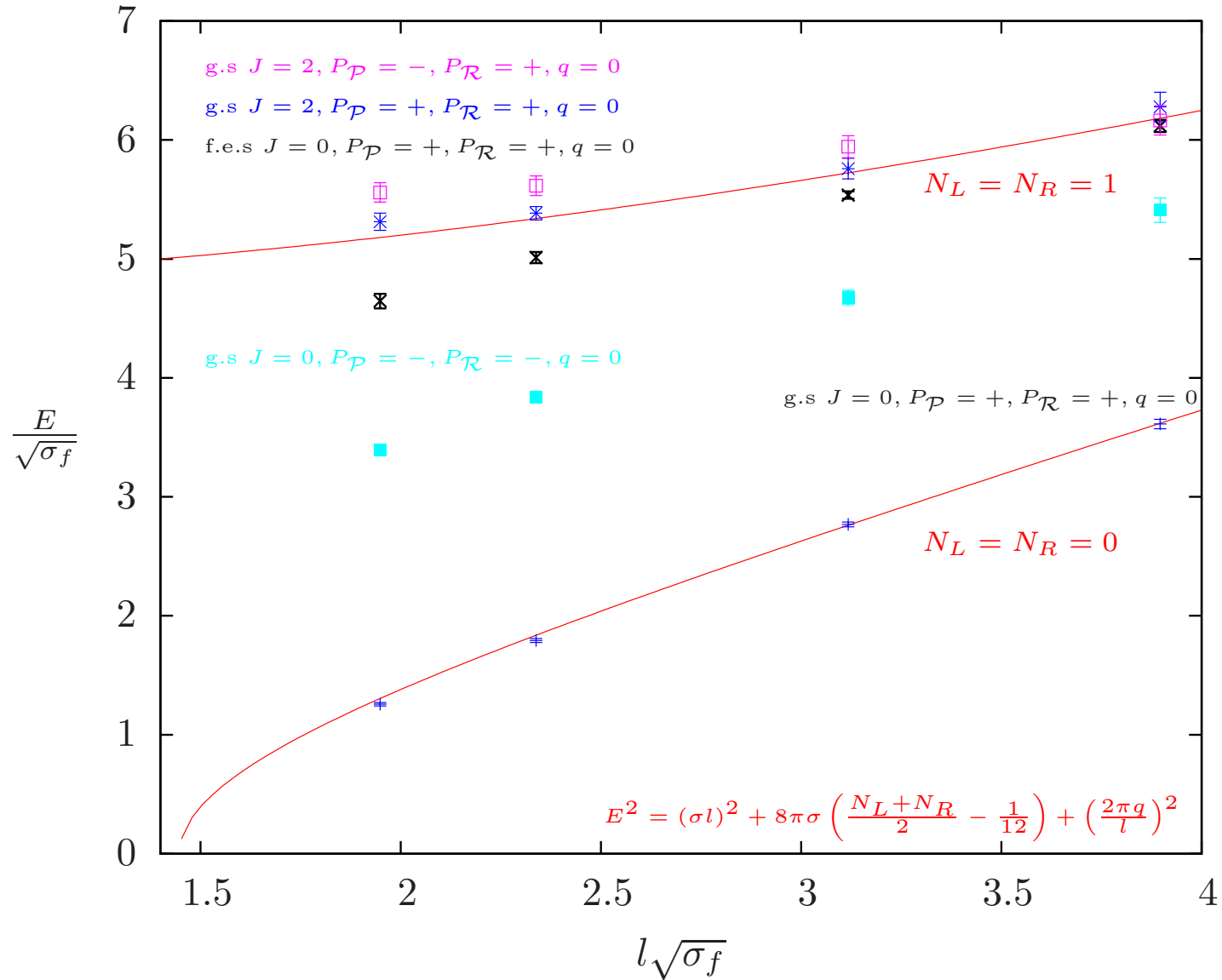
5. D=3+1 Results: $N_L = N_R = 1$



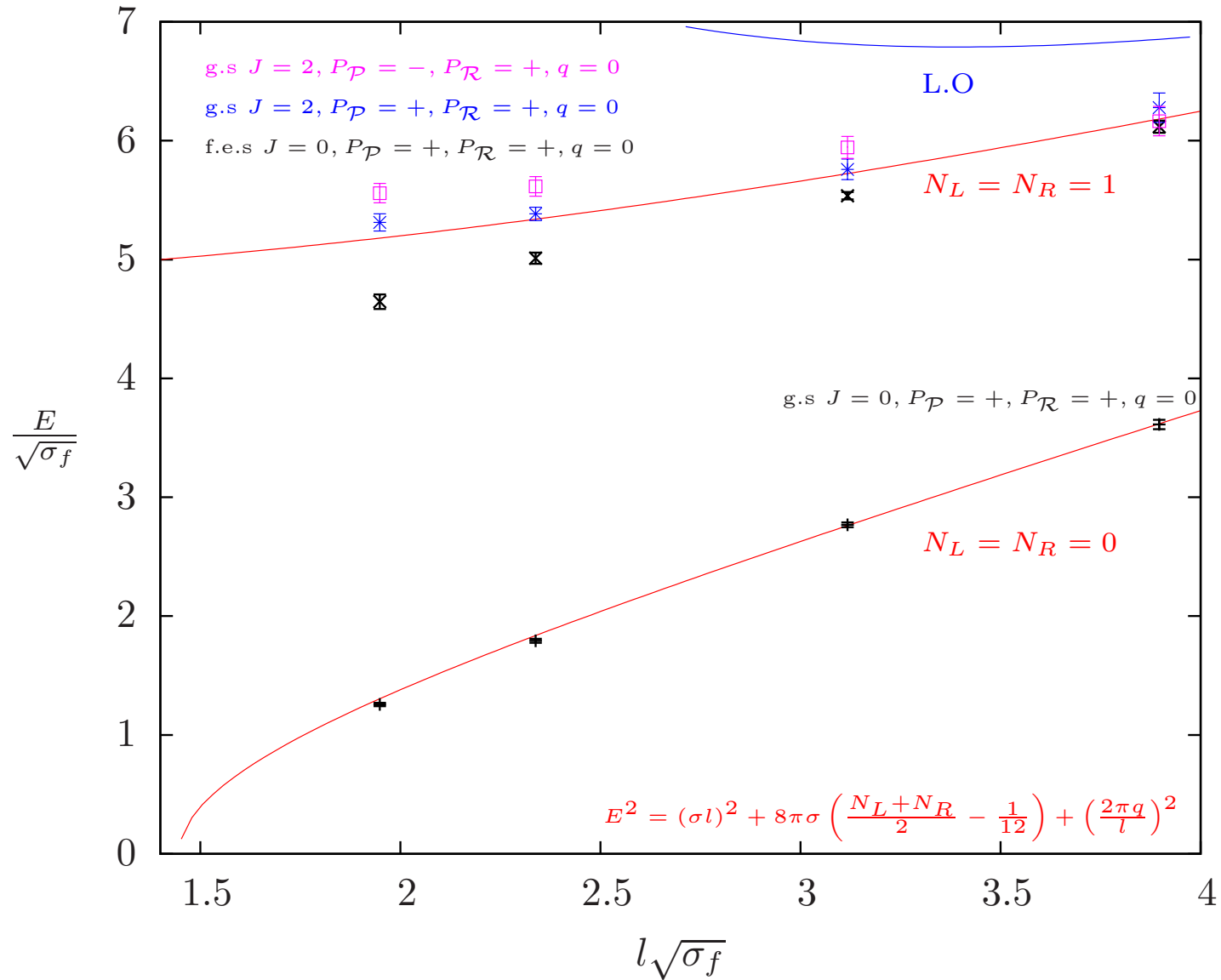
5. D=3+1 Results: $N_L = N_R = 1$



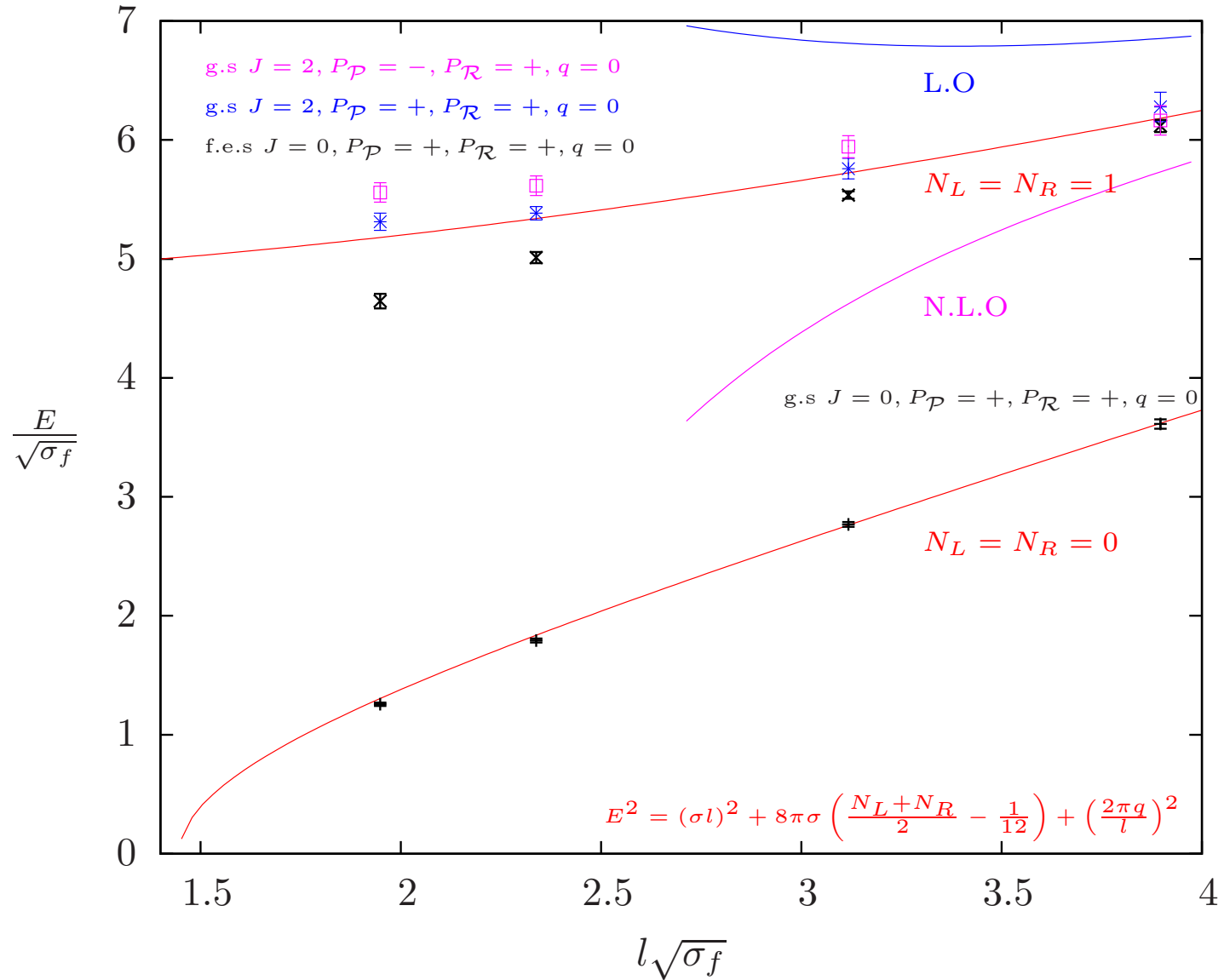
5. D=3+1 Results: $N_L = N_R = 1$



5. D=3+1 Results: $N_L = N_R = 1$



5. D=3+1 Results: $N_L = N_R = 1$



5. D=3+1 Results: $N_L = 2, N_R = 1$

- $N_L = 2, N_R = 1$ corresponds to the $q = 1$ first excited state.

- String theoretically:

$$- (\alpha_{-2}^+ \bar{\alpha}_{-1}^- + \alpha_{-2}^- \bar{\alpha}_{-1}^+) | 0 \rangle \rightarrow \text{g.s } \{A_1, q = 1\}$$

$$- (\alpha_{-2}^+ \bar{\alpha}_{-1}^- - \alpha_{-2}^- \bar{\alpha}_{-1}^+) | 0 \rangle \rightarrow \text{g.s } \{A_2, q = 1\}$$

$$- (\alpha_{-2}^+ \bar{\alpha}_{-1}^+ + \alpha_{-2}^- \bar{\alpha}_{-1}^-) | 0 \rangle \rightarrow \text{g.s } \{B_1, q = 1\}$$

$$- (\alpha_{-2}^+ \bar{\alpha}_{-1}^+ - \alpha_{-2}^- \bar{\alpha}_{-1}^-) | 0 \rangle \rightarrow \text{g.s } \{B_2, q = 1\}$$

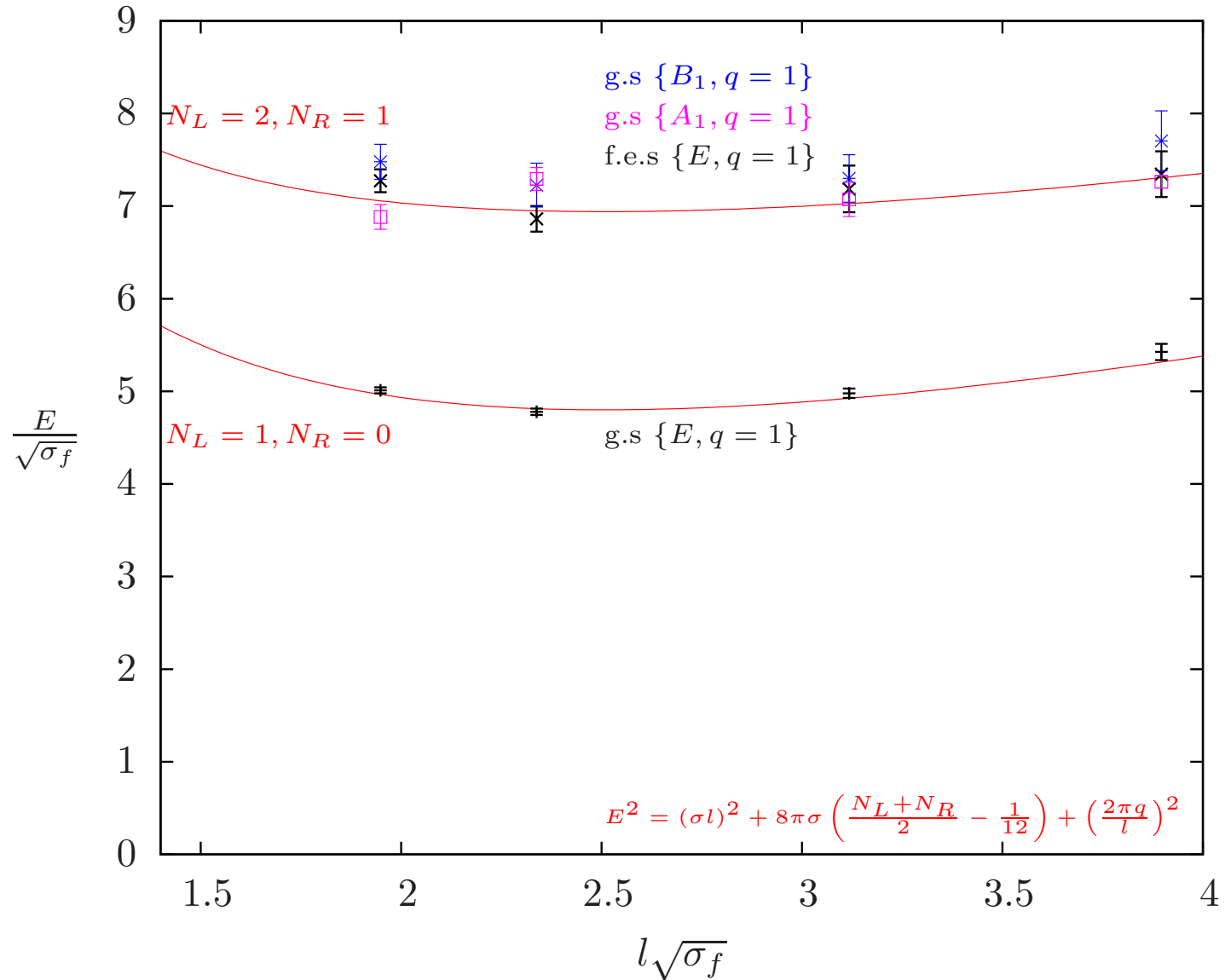
$$- (\alpha_{-1}^+ \alpha_{-1}^+ \bar{\alpha}_{-1}^- + \alpha_{-1}^- \alpha_{-1}^- \bar{\alpha}_{-1}^+, \alpha_{-1}^+ \alpha_{-1}^+ \bar{\alpha}_{-1}^- - \alpha_{-1}^- \alpha_{-1}^- \bar{\alpha}_{-1}^+) | 0 \rangle$$

$$(\alpha_{-1}^+ \alpha_{-1}^- \bar{\alpha}_{-1}^- + \alpha_{-1}^- \alpha_{-1}^+ \bar{\alpha}_{-1}^+, \alpha_{-1}^+ \alpha_{-1}^- \bar{\alpha}_{-1}^- - \alpha_{-1}^- \alpha_{-1}^+ \bar{\alpha}_{-1}^+) | 0 \rangle$$

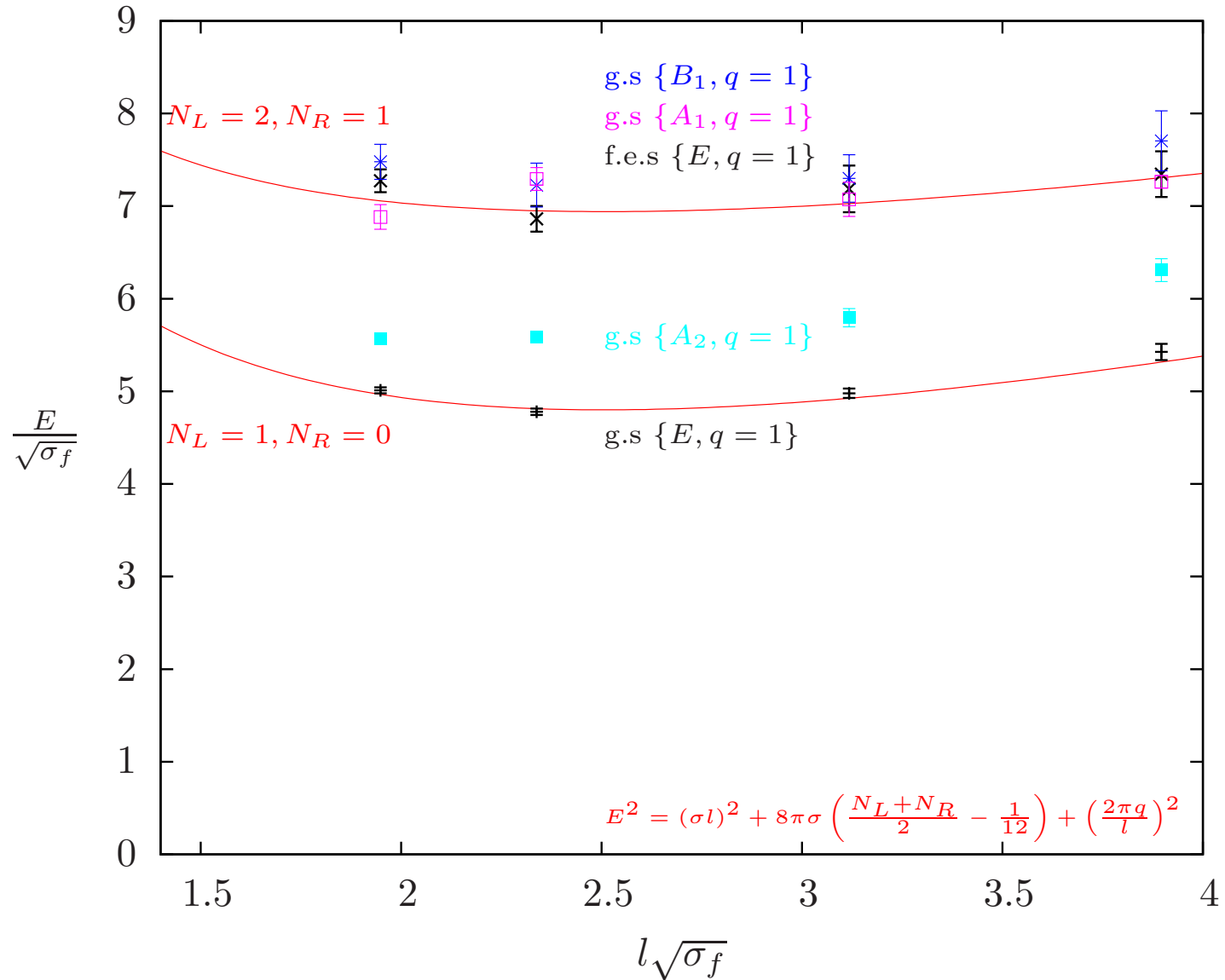
$$(\alpha_{-1}^+ \alpha_{-1}^+ \bar{\alpha}_{-1}^+ + \alpha_{-1}^- \alpha_{-1}^- \bar{\alpha}_{-1}^-, \alpha_{-1}^+ \alpha_{-1}^+ \bar{\alpha}_{-1}^+ - \alpha_{-1}^- \alpha_{-1}^- \bar{\alpha}_{-1}^-) | 0$$

$$\hookrightarrow \text{f.e.s, s.e.s and t.e.s } \{E, q = 1\}$$

5. D=3+1 Results: $N_L = 2, N_R = 1$



5. D=3+1 Results: $N_L = 2, N_R = 1$



6. Conclusions

- $D = 2 + 1$
 - Closed flux tube can be well described by Nambu-Goto even to small $l!$
 - Nambu-Goto is the best approach.
- $D = 3 + 1$
 - The spectrum is mostly closed to Nambu-Goto down to very small $l!$
 - Nambu-Goto is better than any other effective string theory model!
 - However, some states (A_2) are far from Nambu-Goto.
 - Non-Stringy dynamics of some kind?

7. Future Work

- Extract $k = 2$ -flux tube excitation spectrum for $SU(6)$, $\beta = 171$
- Investigation of states with large deviations in $D = 3 + 1$
- Extract more excited states in $D = 3 + 1$
- Increase projection onto massive states in $D = 3 + 1$
 1. Tuning Blocking/Smearing???
 2. Include more operators???
 3. More statistics???
 4. Any Suggestions???
- Investigation of $k = 2$ -flux tube excitation spectrum in $D = 3 + 1$.