

Logarithmic corrections to $O(a^2)$ lattice artifacts

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- Introduction
- $O(n)$ 2d sigma model

Numerical simulations of QFT are performed at bare couplings g_0 with finite correlation lengths $\xi(g_0) \rightarrow$ source of systematic error for physical quantities in the continuum limit $\xi \rightarrow \infty, g_0 \rightarrow g_{\text{crit}}$

it could be in the future that CPU so strong that *lattice artifacts* numerically irrelevant, but

the expected form is of theoretical interest

presently ξ attained not so large \rightarrow need extrapolations

consider e.g. mass spectrum: $m_k = 1/(a\xi_k)$, a =lattice spacing

near the continuum limit attempt fit:

$$[m_k/m_0](g_0) = [m_k/m_0](g_{\text{crit}}) + c_k \mathcal{O}((am_k)^p)$$

what is the value of p ?? integer??? log corrections ???

In a physical model often get an idea of the UV cutoff at g_0 by setting

$$a_0(g_0) \equiv 1/(\xi_0(g_0)m_{0\text{phys}})$$

Most of our knowledge concerning renormalization of quantum field theories stems from perturbation theory

no rigorous proofs in general: many results are *structural* and hence considered to carry over to non-perturbative formulations

supporting evidence from integrable models in 2d, and axiomatic work in 3d, and in framework of $1/n$ expansions

the same situation holds for **lattice artifacts**

such studies resulted in

Symanzik ('80) conjecture: leading artifacts of correlation functions of elementary fields are summarized in an effective action

Example: free scalar field theory on an infinite 4-dimensional hypercubic lattice with standard action:

$$S_0 = a^4 \sum_{x, \mu} \frac{1}{2} [\partial_\mu \phi(x) \partial_\mu \phi(x) + m^2 \phi(x)^2]$$

$$\partial_\mu f(x) = [f(x + a\hat{\mu}) - f(x)] / a$$

2-point function: $\tilde{G}(k) = [\hat{k}^2 + m^2]^{-1}$, $\hat{k}_\mu = \frac{2}{a} \sin \frac{ak_\mu}{2}$

for small a : $\hat{k}^2 = k^2 - \frac{1}{12}a^2 \sum_\mu k_\mu^4 + \mathcal{O}(a^4)$

$$\rightarrow \tilde{G}(k) = \tilde{G}_{\text{cont}}(k) - a^2 \langle S_1^{\text{eff}} \tilde{\phi}(k) \phi(0) \rangle_{\text{cont}} + \dots$$

with: $S_1^{\text{eff}} = - \int d^4x \sum_\mu \frac{1}{24} \partial_\mu^2 \phi(x) \partial_\mu^2 \phi(x)$

On-shell information from $G(x)$ when $x \sim$ physical distance

$$G(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{x}} R(\mathbf{k}, m) e^{-\epsilon(\mathbf{k}, a, m)x_0}$$

energy spectrum ϵ : $\cosh(a\epsilon(\mathbf{k}, a, m)) - 1 = \frac{1}{2}a^2 \left(\hat{\mathbf{k}}^2 + m^2 \right)$

pole mass: $m_p \equiv \epsilon(0, a, m)$

continuum limit: $a \rightarrow 0, m_p$ fixed

$$\epsilon(\mathbf{k}, a, m)^2 = m_p^2 + \mathbf{k}^2 - \frac{a^2}{12} T(\mathbf{k}, m_p) + O(a^4)$$

another lattice action: $S = S_0 + c_1 S_1 + c_2 S_2$ (with nnn interactions)

$$S_1 = a^4 \sum_{x,\mu} \frac{a^2}{2} \partial_\mu^* \partial_\mu \phi(x) \partial_\mu^* \partial_\mu \phi(x)$$

$$S_2 = a^4 \sum_{x,\mu,\nu} \frac{a^2}{2} \partial_\mu^* \partial_\mu \phi(x) \partial_\nu^* \partial_\nu \phi(x)$$

Effective Lagrangian:

$$\mathcal{L}_1^{\text{eff}} = \frac{1}{2} \left(c_1 - \frac{1}{12} \right) \sum_\mu \partial_\mu^2 \phi(x) \partial_\mu^2 \phi(x) + \frac{1}{2} c_2 \sum_{\mu\nu} \partial_\mu^2 \phi(x) \partial_\nu^2 \phi(x)$$

off-shell $O(a^2)$ improvement requires $c_1 = \frac{1}{12}$ and $c_2 = 0$

but *on-shell* improvement (energies improved) requires only $c_1 = \frac{1}{12}$

→ for on-shell improvement can drop terms which vanish using equations of motion

low order perturbative computations in various field theories \rightarrow

conjecture: In a large class of interacting lattice theories (in particular asymptotically free theories), Green function of products of a multiplicatively renormalizable lattice field φ at widely separated points x_i takes the form:

$$Z_\varphi^{r/2} \langle \varphi(x_1) \dots \varphi(x_r) \rangle_{\text{latt}} = \langle \varphi_0(x_1) \dots \varphi_0(x_r) \rangle_{\text{cont}} + a^p T_1 + a^p T_2 + \dots$$

$$T_1 = - \int d^d y \langle \mathcal{L}_1^{\text{eff}}(y) \varphi_0(x_1) \dots \varphi_0(x_r) \rangle_{\text{cont}}$$

Symanzik's effective Lagrangian: $\mathcal{L}_1^{\text{eff}} = \sum_j c_j \mathcal{O}_j$

\mathcal{O}_j : local operator dimension $d + p$, same symmetries as lattice action
coefficients c_j are a -dependent - but thought to be weak (log)

If φ is a composite field another term appears

$$T_2 = \sum_{k=1}^r \langle \varphi_0(x_1) \dots \varphi_1(x_k) \dots \varphi_0(x_r) \rangle_{\text{cont}}$$

φ_1 : sum of local operators of dimension $d_\varphi + p$, depending on the specific operator φ , having the same lattice quantum numbers as φ

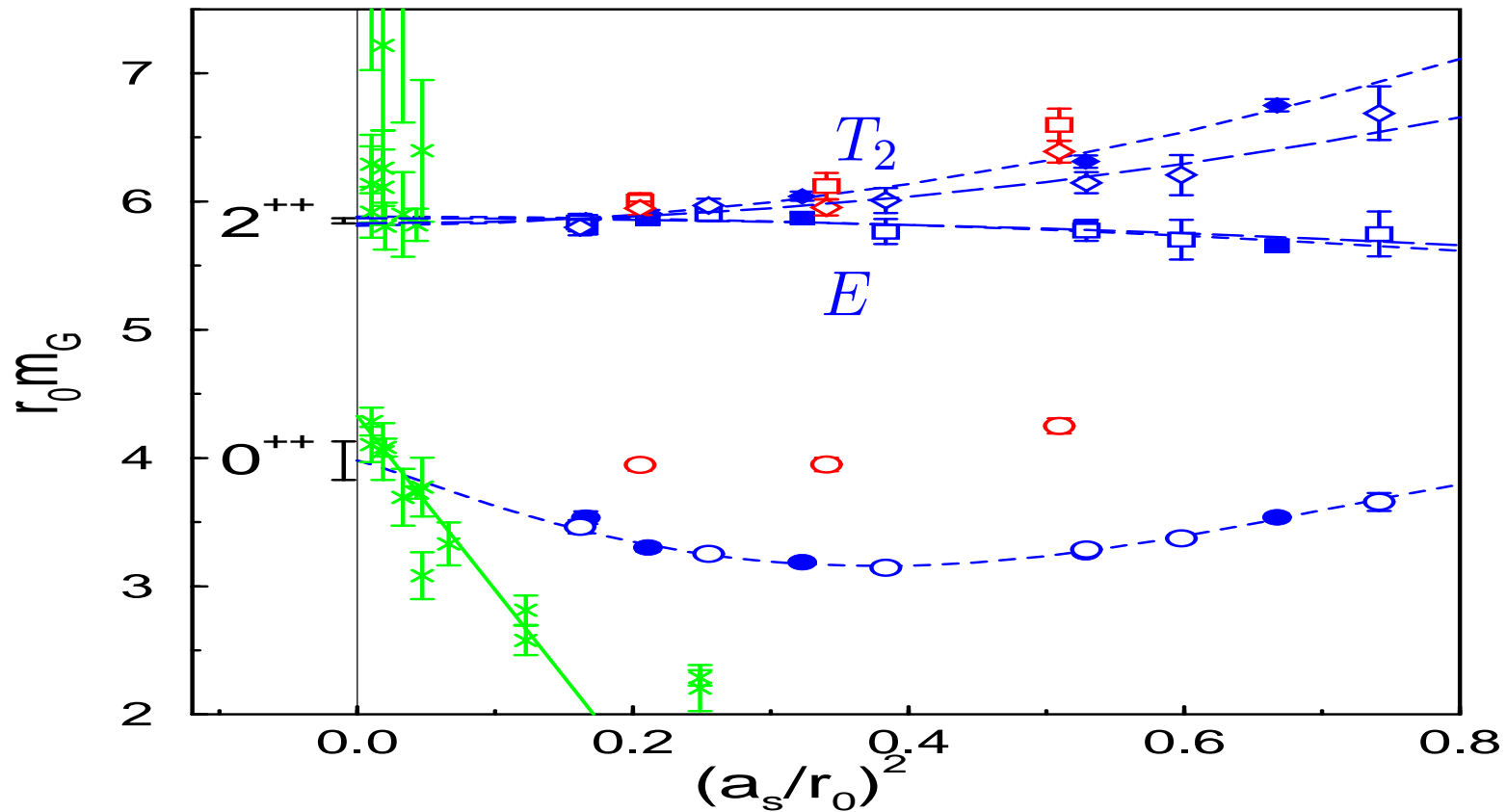
in T_1 general singularities at points $y = x_k \rightarrow$ need subtraction prescription, but arbitrariness in this \sim redefinition of φ_1

if conjecture is true expect

a) generic $O(a^2)$ artifacts in pure Yang–Mills' theory, since no gauge invariant scalar dim 5 operator, but $O(a)$ effects with pure Wilson fermions

b) possible to construct $O(a^2)$ -improved lattice actions for YM, $O(a)$ -improved Wilson fermions. Also improved operators....

Scaling of lowest glueballs (Peardon & Morningstar)



Data from Improved action (anisotropic lattice)

Wilson action and New action

indications of UNIVERSALITY

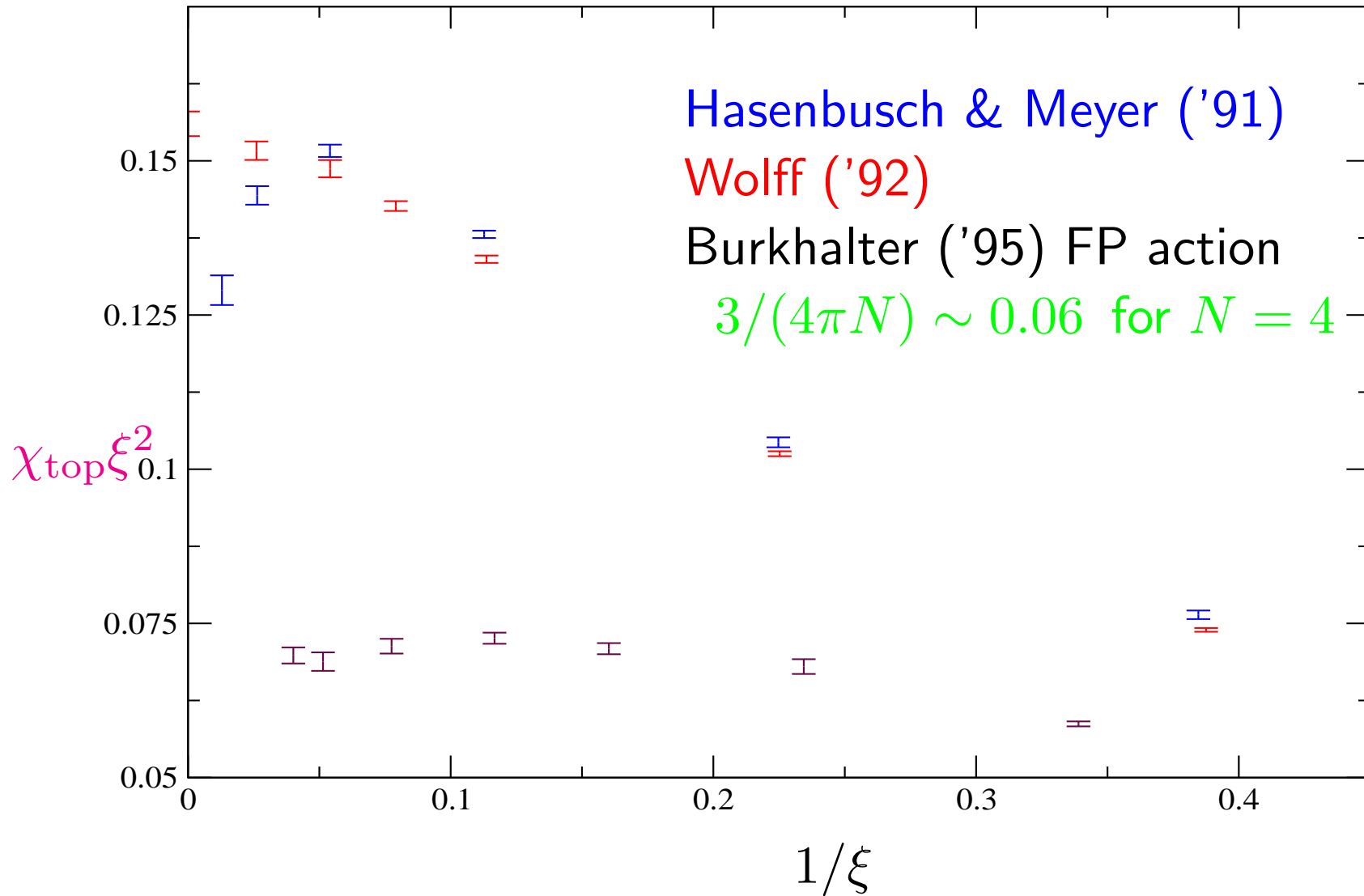
All present numerical data for QCD_4 seems consistent with Symanzik expectations, but until now only a small range of a is available

In 2-d theories one can simulate much larger correlation lengths, without large finite size effects

In various 2-d AF theories one expects $O(a^2)$ artifacts

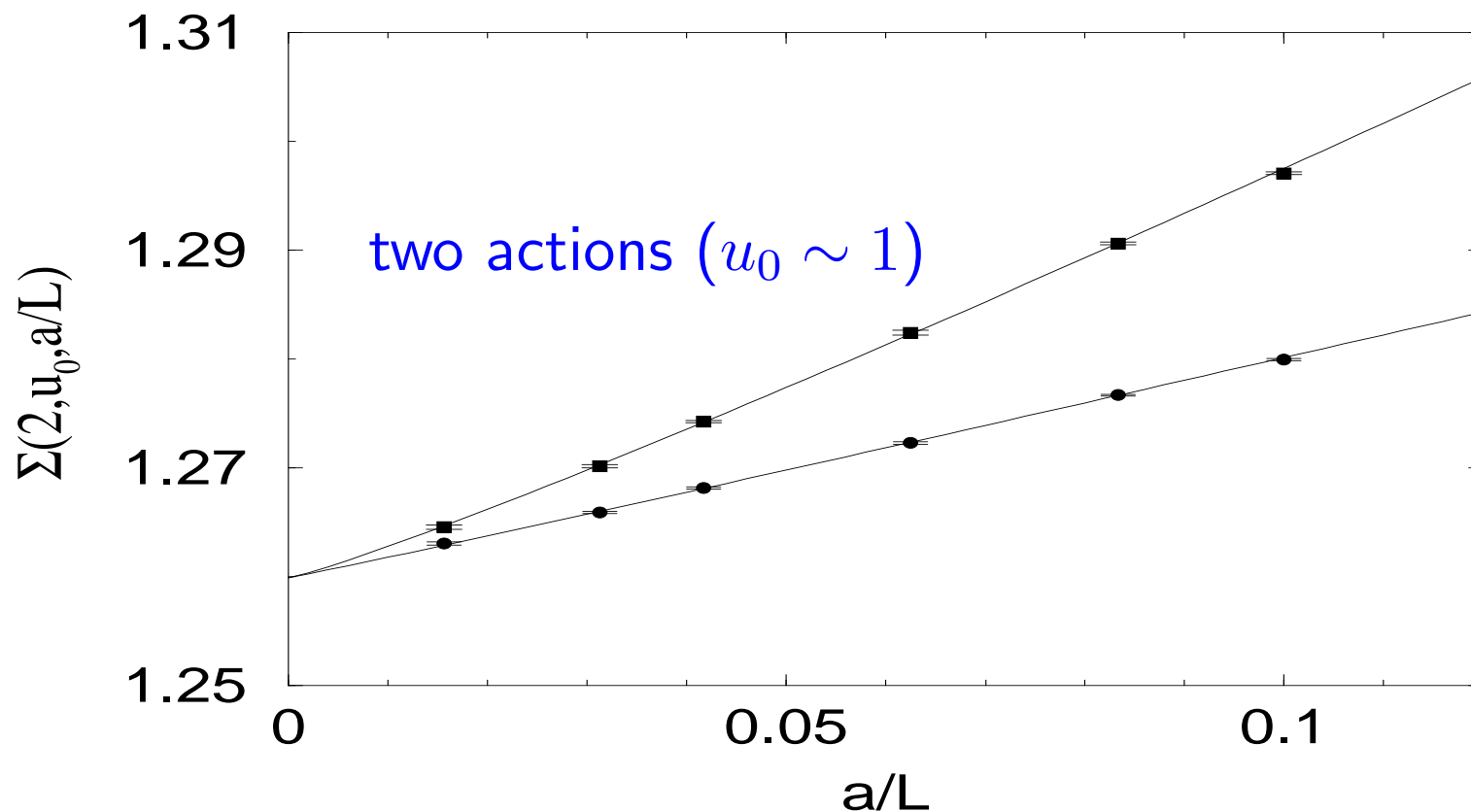
Often encounter unpleasant surprises!

$\chi_{\text{top}}\xi^2$ in the 2-d CP^3 model



Intrinsic lattice artifacts in definition of χ_{top} ?

Lattice artifacts in the step scaling function for 2-d $O(3)$ sigma model, apparently $\sim O(a)$ over a wide range of correlation lengths!
(Hasenfratz & Niedermayer 80')



mentioned by P. Hasenfratz as potential problem at LATT2000

but note indication of **universality**

Symanzik action framework generally accepted and useful, e.g. discussion of automatic $O(a)$ -improvement of TMQCD

but as yet no rigorous proof for any lattice theory to all orders PT

Keller has given an all order proof for ϕ_4^4 and QED_4 for a continuum regularization using flow equations

Subtle point: the continuum limit of lattice ϕ_4^4 is (probably) trivial - i.e. a free theory. The continuum limit is reached logarithmically!

Symanzik's action in this case describes $O(a^2)$ corrections to an effective (low energy) continuum theory

important ingredient of a lattice proof, often stated but not yet proven, is the small a expansion of an arbitrary ℓ -loop Feynman diagram:

$$F(p, a) \sim a^{-\omega} \sum_{n=0}^{\infty} a^n \sum_{r=0}^{\ell} (\ln am)^r F_{nr}(p)$$

Apparent $O(a)$ effects in $O(3)$ σ -model rather unsettling

investigate logarithmic corrections to the $O(a^2)$ effects in the framework of renormalized perturbation theory and $1/n$ expansion

find generic artifacts in the $O(n)$ σ -model are of the form $a^2 \ln^s(a^2)$ with $s = n/(n - 2)$, (Balog, Niedermayer & P.W. ('09))

for $n = 3$, $s = 3$: such strong logarithmic corrections to $O(a^2)$ effects can explain peculiar observed behavior

for $n = \infty$, $s = 1$: consistent with leading orders of the $1/n$ expansion (Knechtli, Leder, Balog, Wolff ('05))

$O(n)$ -sigma model

fields: S^a , $a = 1, \dots, n$ with constraint $S^2 = 1$

action including source terms, dimensional regularization $D = 2 - \varepsilon$:

$$\mathcal{A} = \frac{1}{g_0^2} \int d^D x \left\{ \frac{1}{2} \partial_\mu S \cdot \partial_\mu S - I \cdot S \right\}$$

bare correlation functions:

$$\mathcal{G}^{a_1 \dots a_r}(x_1, \dots, x_r) = g_0^{2r} \mathcal{Z}^{-1}[I_0] \left\{ \frac{\delta}{\delta I^{a_1}(x_1)} \cdots \frac{\delta}{\delta I^{a_r}(x_r)} \mathcal{Z}[I] \right\} \Big|_{I=I_0}$$

generating functional: $\mathcal{Z}[I] = \int (\mathcal{D}S) e^{-\mathcal{A}}$, $I_0^a(x) = m^2 \delta^{an}$

perturbative renormalizability: (Brezin, Zinn-Justin, Le Guillou ('76))

$$g_0^2 = \mu^\varepsilon g^2 Z_1, \quad m^2 = m_R^2 Z_1 Z^{-1/2}, \quad S_R = Z^{-1/2} S, \quad Z = 1 + g^2 \frac{z_1}{\varepsilon} + \dots$$

$O(n)$ -invariant corr. fns. IR finite for $m_R \rightarrow 0$ (David)

For PT parameterize the bare spin field:

$$S^i = g_0 \pi^i, \quad i = 1, 2, \dots, n-1; \quad S^n = \sigma = \sqrt{1 - g_0^2 \pi^2}$$

Ward identities: make infinitesimal change of variables

$\delta\pi^i(x) = \delta\omega Q^i(x)$ in the functional integral

Q^i : arbitrary local expression of the fields and sources

$$\delta\mathcal{A} = -\delta\omega \int d^D x \left(\square\pi^i - \alpha\pi^i + \frac{1}{g_0} I^i \right) Q^i$$

$$\alpha = \frac{\square\sigma + I^n(x)}{\sigma}$$

For DR measure $(\mathcal{D}S) \sim (\mathcal{D}\pi)$ invariant under arby. local transfn.

→ on-shell operator identity for the corresponding space integrals

$$\left(\square\pi^i - \alpha\pi^i + \frac{1}{g_0} I^i \right) Q^i \approx 0$$

Renormalized composite operators: $\mathcal{O}_{i(R)} = Z_{ij}(g, \varepsilon)\mathcal{O}_j$

correlation functions with operator insertion:

$$\widehat{\mathcal{G}}_{i(R)}^X(g, \mu, \varepsilon) = Z^{-r/2}(g, \varepsilon)Z_{ij}(g, \varepsilon)\mathcal{G}_j^X(g_0, \varepsilon)$$

$\mathcal{G}_{i(R)}^X = \widehat{\mathcal{G}}_{i(R)}^X|_{\varepsilon=0}$ satisfy the RG equation:

$$\left\{ \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \frac{r}{2} \gamma(g) \right\} \mathcal{G}_{i(R)}^X(g, \mu) + \nu_{ij}(g) \mathcal{G}_{j(R)}^X(g, \mu) = 0$$

$\beta(g) = -\beta_0 g^3 - \dots$, $\beta_0 = \frac{n-2}{4\pi}$, \rightarrow Asymptotic freedom ($n \geq 3$)

$\gamma(g) = \gamma_0 g^2 + \dots$, $\gamma_0 = \frac{n-1}{2\pi}$

anomalous dimension matrix:

$$\nu_{ij}(g) = Z_{is}(g, \varepsilon) \left(\beta(g) - \frac{\varepsilon g}{2} \right) \frac{\partial (Z^{-1})_{sj}(g, \varepsilon)}{\partial g} = -w_{ij} g^2 + p_{ij} g^4 + \dots$$

1. classify dimension 4 operators needed for Symanzik analysis

mass dimension four, $O(n)$ invariant, Lorentz scalar operators closed under renormalization (Brézin, Zinn-Justin, LeGuillou)

$$\mathcal{O}_1 = \frac{1}{8} (\partial_\mu S \cdot \partial_\mu S)^2, \quad \mathcal{O}_2 = \frac{1}{8} (\partial_\mu S \cdot \partial_\nu S) (\partial_\mu S \cdot \partial_\nu S)$$

$$\mathcal{O}_3 = \frac{1}{2} \square S \cdot \square S, \quad \mathcal{O}_4 = \frac{1}{2} \alpha \partial_\mu S \cdot \partial_\mu S, \quad \mathcal{O}_5 = \frac{1}{8} \alpha^2$$

WI \rightarrow insertions of the apparently $O(n)$ non-invariant operators \mathcal{O}_4 and \mathcal{O}_5 are (on-shell) equivalent to inserting manifestly invariant ones

Diagonalize: $\bar{U}_i = \mathcal{S}_{ij} \mathcal{O}_j$ renormalize multiplicatively at 1-loop (1-loop mixing matrix ω_{ij} computed by Brézin et al)

$$\text{WI} \rightarrow \bar{U}_3 \approx -\frac{1}{4} I \cdot \square S, \quad \bar{U}_4 \approx \frac{1}{8} I^a I^b (S^a S^b - \frac{1}{n} \delta^{ab}), \quad \bar{U}_5 \approx \frac{1}{2} I^2$$

$\rightarrow \bar{U}_j, j = 3, 4, 5$ renormalize diagonally all orders PT

$$Z_{\bar{U}_3} = Z_1^{-1}, \quad Z_{\bar{U}_4} = Z Z_\tau Z_1^{-2}, \quad Z_{\bar{U}_5} = Z Z_1^{-2}$$

also need to consider four-index symmetric tensor operators:

$$t_{\mu\nu\rho\sigma} = S \cdot \partial_\mu \partial_\nu \partial_\rho \partial_\sigma S ,$$

$$k_{\mu\nu\rho\sigma} = \frac{1}{3} \left\{ (\partial_\mu S \cdot \partial_\nu S) (\partial_\rho S \cdot \partial_\sigma S) + 2 \text{ perms} \right\}$$

totally traceless parts \hat{t}, \hat{k} of t, k :

$$A = \sum_{\mu=1}^D \hat{t}_{\mu\mu\mu\mu} , \quad B = \sum_{\mu=1}^D \hat{k}_{\mu\mu\mu\mu}$$

these operators invariant only under discrete (D -dimensional lattice) rotations & mixing diagonal to 1 loop

Lattice regularization

class of lattice actions quadratic in the spins:

$$\mathcal{A} = \frac{\beta}{2} a^4 \sum_{x,y} \sum_a S^a(x) K(x-y) S^a(y)$$

K short range, $\sum_x K(x) = 0$,

$K(z) = K(\mathcal{R}z)$, \mathcal{R} : lattice rotation or reflection

$$K_p = a^2 \sum_x e^{-ipx} K(x) = p^2 [1 + a^2 r(p) + O(a^4)]$$

$$r(p) = p^2 \left\{ \kappa_1 \frac{p^4}{(p^2)^2} + \kappa_2 \right\}, \quad p^r \equiv \sum_\mu p_\mu^r.$$

standard action (ST): $K_{\text{ST}}(z) \equiv \sum_\mu a^{-4} [2\delta_{z,0} - \delta_{z,a\hat{\mu}} - \delta_{z,-a\hat{\mu}}]$

$$K_{p;\text{ST}} = \hat{p}^2, \quad \hat{p}_\mu = \frac{2}{a} \sin\left(\frac{ap_\mu}{2}\right)$$

in this case $\kappa_1 = -1/12$, $\kappa_2 = 0$.

2. need lattice perturbation expansion 2-pt, 4-pt fns ($\beta = 1/\lambda_0^2$)

2-pt fn: $\langle S(x) \cdot S(y) \rangle = 1 + (n - 1)\lambda_0^2 \sum_{r=0} \lambda_0^{2r} C_r(x, y)$

$$\tilde{C}_0(p) = \frac{1}{K_p} = \frac{1}{p^2} [1 - a^2 r(p) + \dots]$$

$$\tilde{C}_1(p) = K_p^{-1} \left[\frac{1}{2} F(p) + \frac{k_p}{K_p} \right]$$

$$F(p) = \int_{-\pi/a}^{\pi/a} \frac{d^2_s}{(2\pi)^2} \frac{K_p - K_s - K_{p+s}}{K_{p+s} K_s}, \quad k_p = \int_{-\pi/a}^{\pi/a} \frac{d^2_s}{(2\pi)^2} \frac{K_p + K_s - K_{p+s}}{K_s}$$

4-pt fn: $\langle S(x_1) \cdot S(y_1) S(x_2) \cdot S(y_2) \rangle - \langle S(x_1) \cdot S(y_1) \rangle \langle S(x_2) \cdot S(y_2) \rangle$

$$= (n - 1)\lambda_0^4 \sum_{r=0} \lambda_0^{2r} C_r(x_1, y_1; x_2, y_2)$$

here C_0 has only a disconnected part

Relation between the renormalization schemes

Renormalized lattice correlation functions with finite continuum limit ($a \rightarrow 0$) got by applying wave function and coupling renormalization

$$\text{e.g. } \tilde{C}^{\text{MS}}(p_1, q_1, \dots, p_k, q_k) = Z_{\text{R}}^{-k} \tilde{C}(p_1, q_1, \dots, p_k, q_k) + \mathcal{O}(a^2)$$

provided relate couplings: $g^2 = Z_{1\text{R}}^{-1} \lambda_0^2$

$$Z_{\text{R}} = 1 + Z_{\text{R}}^{(1)} g^2 + Z_{\text{R}}^{(2)} g^4 + \dots, \quad Z_{1\text{R}} = 1 + Z_{1\text{R}}^{(1)} g^2 + Z_{1\text{R}}^{(2)} g^4 + \dots$$

$$\text{At 1-loop: } Z_{\text{R}}^{(1)} = (n-1) \frac{1}{2\pi} \left[\ln(a\mu) + \frac{1}{2}(c_1 + \gamma) \right]$$

$$Z_{1\text{R}}^{(1)} = (n-2) \frac{1}{2\pi} \left[\ln(a\mu) + \frac{1}{2}(c_1 + \gamma) \right] - A_0$$

$$\Lambda_{\text{latt}} = a^{-1} (2\beta_0 \lambda_0^2)^{-\chi} e^{-\frac{1}{2\beta_0 \lambda_0^2}} \left[1 + \mathcal{O}(\lambda_0^2) \right]$$

$$\rightarrow \frac{\Lambda_{\text{latt}}}{\Lambda_{\text{MS}}} = \exp \left[\frac{1}{2}(c_1 + \gamma) - \frac{2\pi A_0}{n-2} \right]$$

standard action: $c_1 = -5 \ln 2$, $A_0 = 1/4$, (Parisi ('80))

separation of the full lattice correlation function into a scaling piece and cutoff corrections is unambiguous and straightforward in PT

$$\mathcal{G}_{\text{latt}}^X(\lambda_0, a) = \mathcal{G}^{X(0)}(\lambda_0, a) + a^2 \mathcal{G}^{X(1)}(\lambda_0, a) + O(a^4)$$

where scaling functions $\mathcal{G}^{X(0)}$ and the leading cutoff corrections $\mathcal{G}^{X(1)}$ are still weakly (logarithmically) depending on a

$$\text{e.g. } \tilde{C}_1(p) = \frac{1}{4\pi p^2} \left[\ln(a^2 p^2) + c_1 + 4\pi A_0 \right. \\ \left. + a^2 p^2 \left\{ \kappa_1 \left(\frac{3}{4} - \frac{p^4}{(p^2)^2} \right) \ln(a^2 p^2) + c_3 \frac{p^4}{(p^2)^2} + c_4 \right\} + O(a^4 p^4) \right]$$

main assumption: expansion makes sense also beyond PT

usual renormalization theory deals with $\mathcal{G}^{X(0)}$,

analyze $\mathcal{G}^{X(1)}$ using Symanzik's effective Lagrangian

Symanzik's local effective Lagrangian \mathcal{L}_{eff} is defined in the continuum s.t. corresponding generating functional reproduces the lattice correlation functions up to terms of order a^4

includes all $\text{dim} \leq 4$ operators invariant under lattice symmetries

write either as:

$$\begin{aligned}
 -\mathcal{L}_{\text{eff}} = & -\frac{1}{2g_0^2} (\partial_\mu S \cdot \partial_\mu S) + \frac{1}{g_0^2} I \cdot S \\
 & + \frac{a^2}{g_0^2} \left\{ \tilde{Y}_1 \mathcal{O}_1 + \tilde{Y}_2 \mathcal{O}_2 + \tilde{Y}_3 \mathcal{O}_3 + \tilde{Y}_A A + \tilde{Y}_B B \right\} + \\
 & \frac{a^2}{g_0^2} \left\{ \tilde{W}_1 I \cdot \square S + \tilde{W}_2 I \cdot S (\partial_\mu S \cdot \partial_\mu S) \right\} + \frac{a^2}{g_0^2} \left\{ \tilde{X}_1 I^2 + \tilde{X}_2 (I \cdot S)^2 \right\}
 \end{aligned}$$

or eliminate the terms quadratic in the sources by using identities:

$$-\mathcal{L}_{\text{eff}} = -\frac{1}{2g_0^2} (\partial_\mu S \cdot \partial_\mu S) + \frac{1}{g_0^2} I \cdot S + \frac{a^2}{g_0^2} \left\{ \bar{Y}_A A + \bar{Y}_B B + \sum_{i=1}^5 \bar{Y}_i \mathcal{O}_i \right\}$$

introduce the operator basis:

$$U_i = \frac{1}{g_0^2} \bar{U}_i, \quad i = 1, \dots, 5, \quad U_6 = \frac{1}{g_0^2} A, \quad U_7 = \frac{1}{g_0^2} B$$

$$-\mathcal{L}_{\text{eff}} = -\frac{1}{2g_0^2} (\partial_\mu S \cdot \partial_\mu S) + \frac{1}{g_0^2} I \cdot S + a^2 \sum_{i=1}^7 c_i(g) U_{iR}$$

$$c_i(g) = \sum_{\ell=0}^{\infty} c_i^{(\ell)} g^{2\ell}$$

for our class of lattice actions obtain **tree level effective Lagrangian**:

$$-\mathcal{L}_{\text{eff}}^{(0)} = -\frac{1}{2\lambda_0^2} (\partial_\mu S \cdot \partial_\mu S) + \frac{a^2}{\lambda_0^2} \left\{ \frac{e_4}{24} A + \frac{e_0}{16} \mathcal{O}_3 \right\} + \mathcal{O}(a^4)$$

$$\text{with } e_0 = -12\kappa_1, \quad e_4 = -12\kappa_1 - 16\kappa_2$$

corresponds to:

$$c_1^{(0)} = \frac{e_0}{4(n-1)}, \quad c_2^{(0)} = -\frac{e_0}{4(n-1)}, \quad c_3^{(0)} = \frac{e_0(n^2-2n-4)}{4n(n-2)}$$

$$c_4^{(0)} = \frac{9e_0}{4(2n-1)}, \quad c_5^{(0)} = \frac{(3-n)(3+n)e_0}{16n(n-1)}, \quad c_6^{(0)} = \frac{e_4}{24}, \quad c_7^{(0)} = 0$$

effective action st lattice correlation functions (set $a\mu = 1$)

$$\mathcal{G}^{X(0)}(\lambda_0, a) = y^r(g) \mathcal{G}_{(R)}^X(g, a^{-1})$$

$$\mathcal{G}^{X(1)}(\lambda_0, a) = y^r(g) \sum_{i=1}^7 c_i(g) \mathcal{G}_{i(R)}^X(g, a^{-1})$$

$y(g)$: renormalization constant relating lattice and DR sources

$$\mathcal{G}_{\text{latt}}^X(\lambda_0, a) = \mathcal{G}^{X(0)}(\lambda_0, a) \{1 + a^2 \delta^X(\lambda_0, a)\} + \mathcal{O}(a^4)$$

with $\delta^X(\lambda_0, a) = \sum_{i=1}^7 c_i(g) \delta_i^X(g, a)$

$$\delta_i^X(g, a) = \frac{\mathcal{G}_{i(R)}^X(g, a^{-1})}{\mathcal{G}_{(R)}^X(g, a^{-1})} \text{ satisfy RG equation } \star:$$

$$\left\{ -a \frac{\partial}{\partial a} + \beta(g) \frac{\partial}{\partial g} \right\} \delta_i^X(g, a) = -\nu_{ij}(g) \delta_j^X(g, a)$$

anomalous dimension matrix: $\nu_{ij}(g) = -2\beta_0 \Delta_i \delta_{ij} g^2 + \nu_{ij}^{(2)} g^4 + \dots$

eigenvalues $\Delta_i = \left\{ \frac{n}{n-2}; -1; 0; \frac{1-n}{n-2}; \frac{1}{n-2}; 0; -1 \right\}$

introduce $U_{ij}(g)$ which solves the ordinary differential equation

$$U'_{ij}(g) = -\rho_{is}(g) U_{sj}(g), \quad \rho_{ij}(g) = \frac{\nu_{ij}(g)}{\beta(g)}$$

solution of \star : $\delta_i^X(g, a) = U_{ij}(g) D_j^X(\Lambda)$

lattice artifacts: $\delta^X(\lambda_0, a) = \sum_{i=1}^7 \hat{v}_i(g) D_i^X(\Lambda)$

with $\hat{v}_i(g) = \sum_{s=1}^7 c_s(g) U_{si}(g)$

functions D_j^X are non-perturbative and depend on (X)

$\hat{v}_i(g)$: perturbative and same for all physical quantities,
but depend on the lattice action used

$$U_{ij}(g) = \{\delta_{ij} + Q_{ij}(g)\} g^{-2\Delta_j}, \quad Q_{ij}(g) = \sum_{\ell=2}^{\infty} k_{ij}^{(\ell)} g^{2\ell-2}$$

→ lattice artifacts: $\delta^X(\lambda_0, a) = \sum_{i=1}^7 v_i(g) g^{-2\Delta_i} D_i^X(\Lambda)$

$$v_i(g) = c_i(g) + \sum_s c_s(g) Q_{si}(g) = \sum_{\ell=0}^{\infty} v_i^{(\ell)} g^{2\ell}$$

leading term corresponds to $\Delta_1 = \frac{n}{n-2}$

& sub-leading one to $\Delta_5 = \frac{1}{n-2}$

note U_5 doesn't contribute on-shell whereas U_1 does

$$\rightarrow \delta^X(\lambda_0, a) = v_1(g) D_1^X (g^{-2})^{\Delta_1} + v_5(g) D_5^X (g^{-2})^{\Delta_5} + \dots$$

$$= D_1^X \left\{ (g^{-2})^{\Delta_1} v_1^{(0)} + v_1^{(1)} (g^{-2})^{\Delta_1-1} \right\} + \dots$$

$$v_1^{(0)} = c_1^{(0)} = \frac{e_0}{4(n-1)}, v_1^{(1)} = c_1^{(1)} + \sum_{s=1}^2 c_s^{(0)} k_{s1}^{(2)}$$

generic leading artifact: $a^2 [\ln(a^2)]^{\frac{n}{n-2}}$

consistent with behavior $a^2 [\ln(a^2)]$ found in leading orders $1/n$ expansion (Balog, Knechtli, Leder, & Wolff 07)

write **final result**

$$\delta^X(\lambda_0, a) = \frac{e_0}{4(n-1)} \beta^{n/(n-2)} D_1^X(\Lambda) \left\{ 1 + \frac{r^{(2)}}{2\pi\beta} \right\} + \dots$$

where $D_1^X(\Lambda) = \Lambda^2 f^X(\Lambda L_1, \Lambda L_2, \dots)$

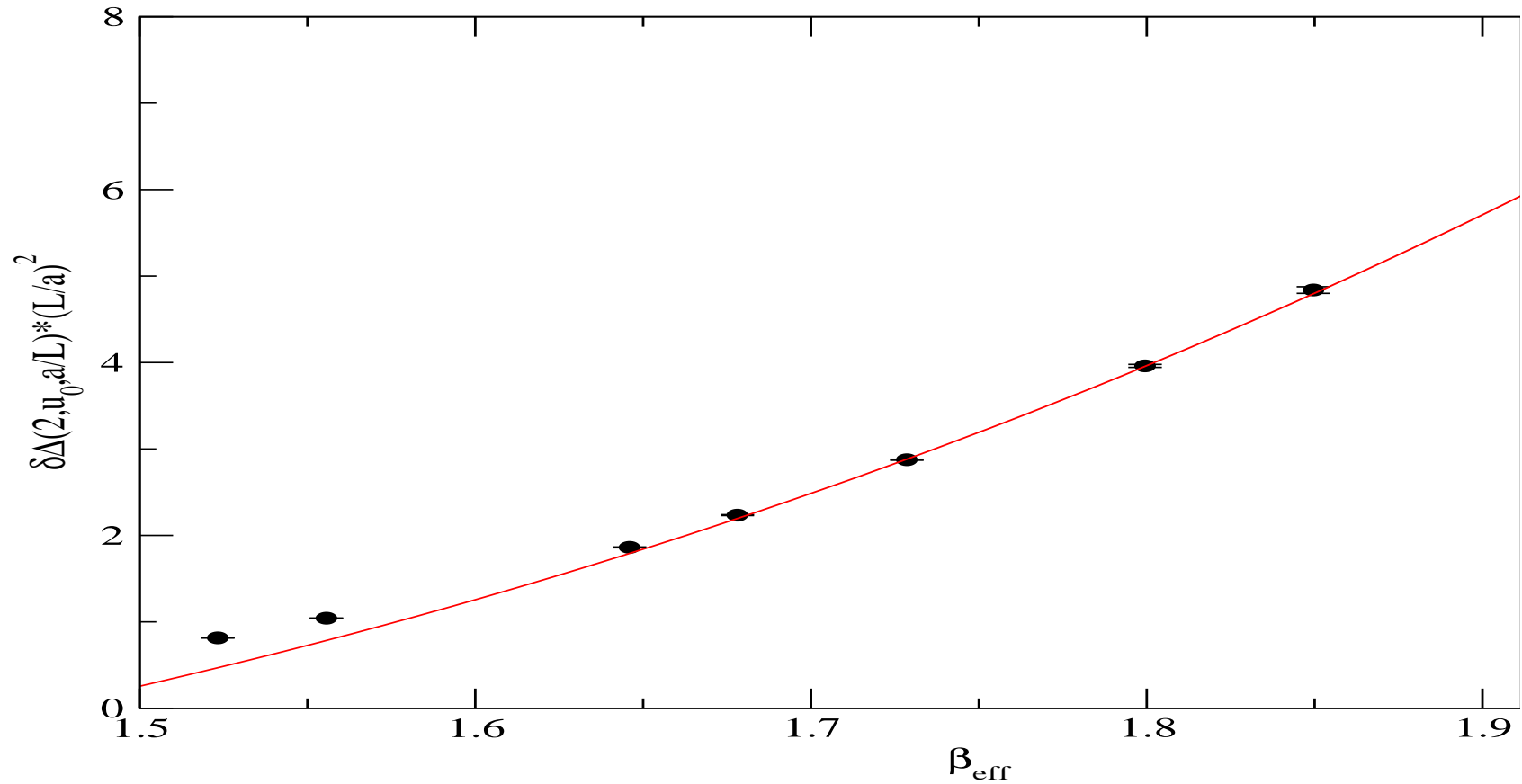
computation of $r^{(2)} = r_I^{(2)} + r_{II}^{(2)} + r_{III}^{(2)}$

$r_I^{(2)} = \frac{8\pi(n-1)}{e_0} c_1^{(1)}$: involves 1-loop coefficients of the effective action

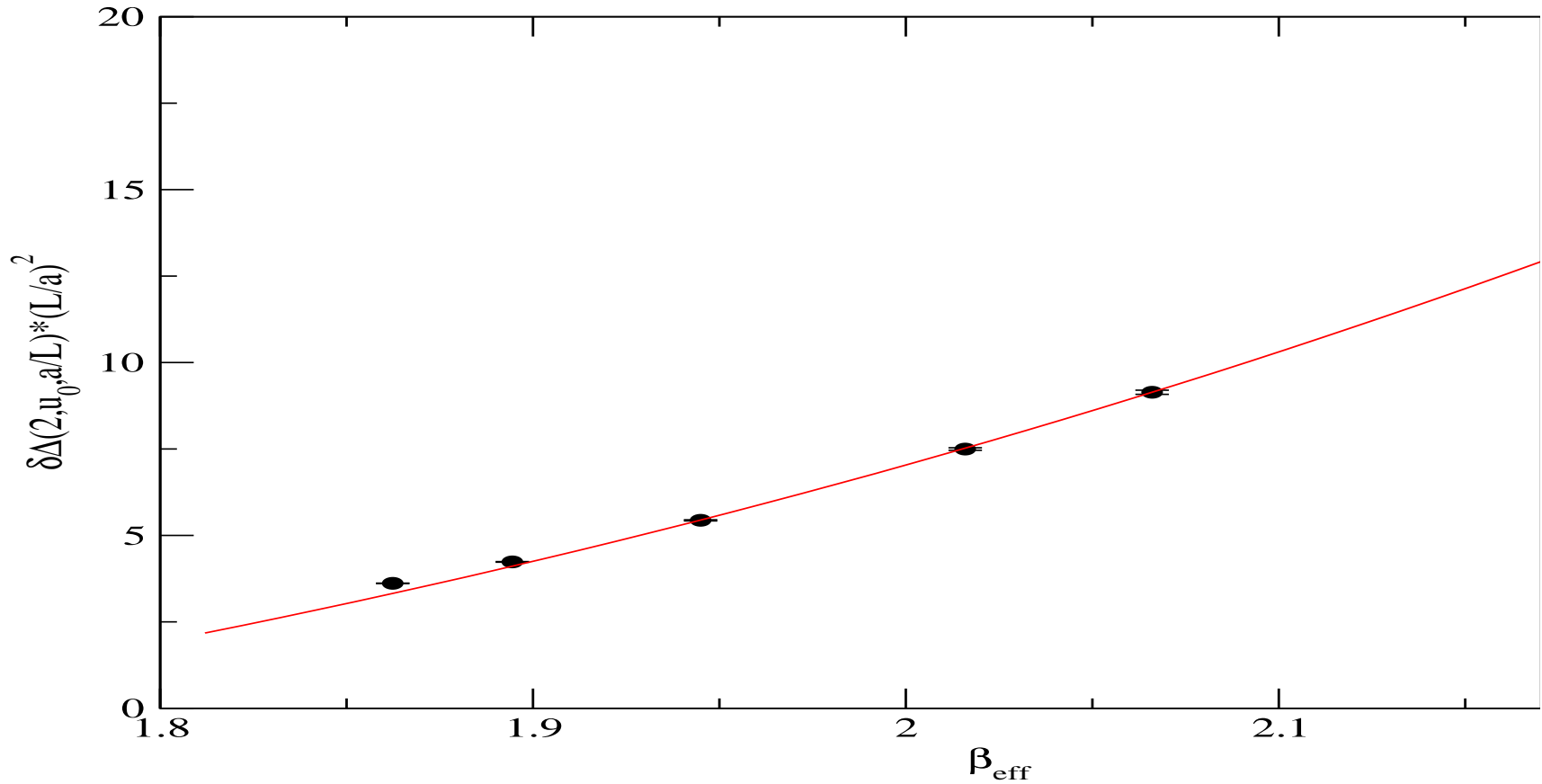
$r_{II}^{(2)} = \frac{n}{n-2}(1 - \psi)$: involves relation between MS and lattice couplings $g^2 = \lambda_0^2 + \frac{\psi}{2\pi} \lambda_0^4 + \dots$

$r_{III}^{(2)} = (2\pi)^2 \left[\frac{1}{n-2} \nu_{11}^{(2)} + \frac{1}{n} \nu_{21}^{(2)} \right]$: needs 2-loop anomalous dimensions

I,III are lengthy computations

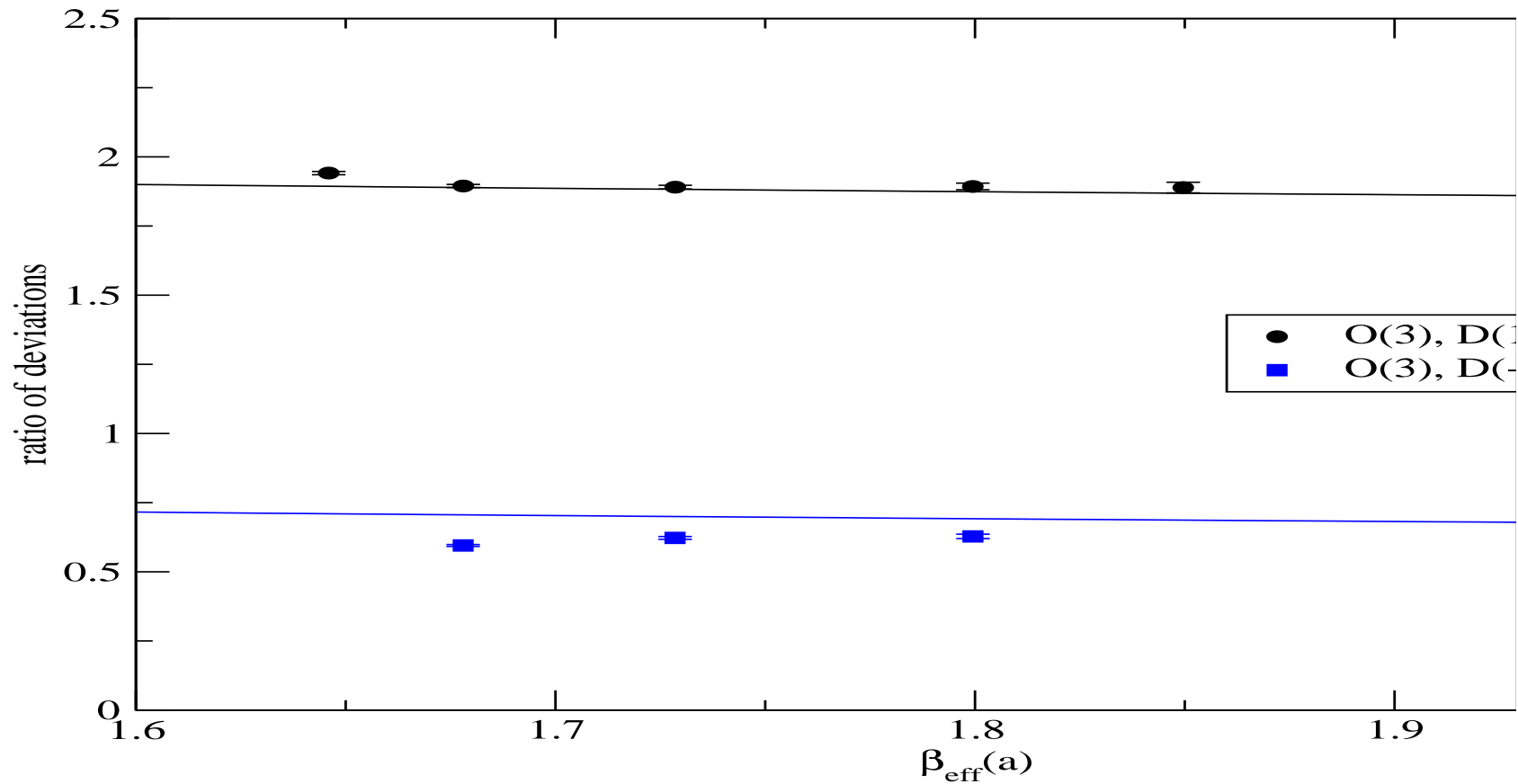


$\delta\Sigma(2, u_0, a/L)(L/a)^2$ for $O(3)$, ST action. The solid curve is the fit $3.1320(\beta^3 + c\beta^2 - 0.4883\beta)$, with $c = r^{(2)}/(2\pi) = -1.1386$ fixed.



$\delta\Sigma(2, u_0, a/L)(L/a)^2$ for $O(3)$, $D(1/3)$ action. The solid curve is the fit $5.4803(\beta^3 + c\beta^2 - 0.2309\beta)$, with $c = r^{(2)}/(2\pi) = -1.5641$ fixed.

$D(\alpha)$ action has $K_p = \hat{p}^2 + \frac{1}{2}a^2\alpha [\hat{p}^4 - (\hat{p}^2)^2]$



Ratio of deviations $\delta\Sigma(2, u_0, a/L)$ for the D(1/3)/ST and D(-1/4)/ST actions vs. $\beta_{\text{eff}}^{\text{ST}}(a)$. The solid curves are the predictions $5/3(1 + 0.224/\beta_{\text{eff}}^{\text{ST}})$, and $1/2(1 + 0.691/\beta_{\text{eff}}^{\text{ST}})$.

Conclusions

- lattice artifacts normally decrease very rapidly as a^2 , but in $O(3)$ partially compensated by the cubic logarithmic increase
- Symanzik effective action description is supported by its parameter-free prediction for ratios of artifacts from different actions
- in the large n limit PT RG formula reproduces previous results
- similar computations should, in our opinion, when technically possible accompany precision lattice studies of QCD in order to control and better estimate systematic errors arising from lattice artifacts for extrapolations to the continuum limit.