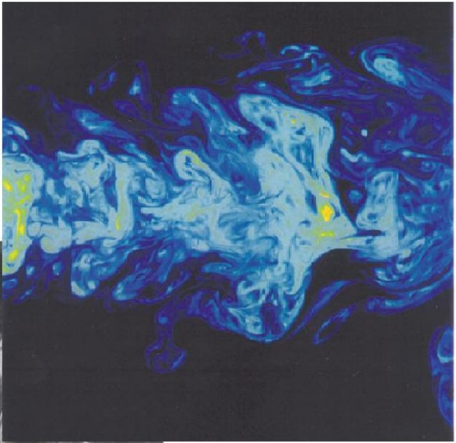
Turbulence on the Lattice

David Mesterhazy

Humboldt University Berlin
Department of Physics

Work done together with K. Jansen (NIC, DESY Zeuthen)

Nature **405**, 639 (2000)



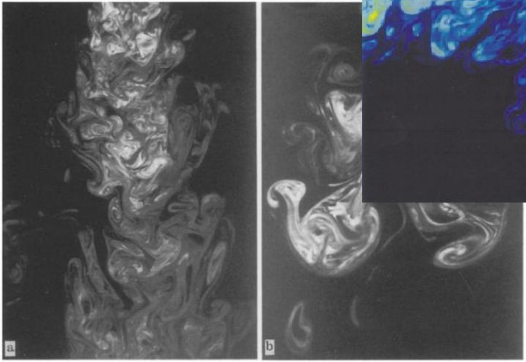


Fig. 1 a and b. Two dimensional sections of an axisymmetric water jet visualized by laser induced fluorescence: **a** axial and **b** orthogonal, $x/D = 21$; the two pictures do not correspond to the same instant

Injection scale L with a characteristic velocity u_L

Energy dissipation via kinematic viscosity ν

Dissipation length

$$L_d = \left(\nu^3/\epsilon\right)^{\frac{1}{4}}$$

where ϵ is the mean energy dissipation

Hydrodynamic flow described by Navier-Stokes equations

$$\partial_t u_i + u_j \partial_j u_i - \nu \nabla^2 u_i = -\frac{1}{\rho} \partial_i p$$
$$\partial_t \rho + \partial_i (\rho u_i) = 0$$

Plus initial and boundary conditions

Equations of motion for viscous flow and mass conservation

Often one assumes incompressibility, i.e. constant ρ

Thereby the equation of continuity takes the form

$$\partial_i u_i = 0$$

Scale invariance

$$u_i(\mathbf{r}, t) \mapsto \lambda^n u_i(\lambda^{-1}\mathbf{r}, \lambda^{n-1} t)\,,\ \forall \lambda > 0$$

For finite viscosity, only $n = -1$ permitted

Define the Reynolds number

$$Re = u_L L/\nu$$

as a scale invariant dimensionless quantity

Highly erratic, turbulent flow at high values of the Reynolds number

$$Re = u_L L/\nu$$

Requires small ν, if L and u_L fixed as system parameters

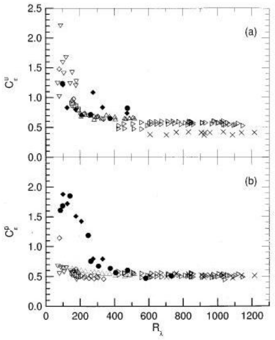
But the *energy dissipation* ϵ remains *finite!*

Leads to a small dissipation scale, i.e. $L_d/L \ll 1$

Constant transfer of energy from large to small scales in the inertial range

$$L_d \ll r \ll L$$

$\rightarrow$ *Richardson energy cascade*



Phys. Fluids, **14**, 1289 (2002)

Frisch, U., *Turbulence: The Legacy of A.N. Kolmogorov* (Cambridge University Press, 1995)

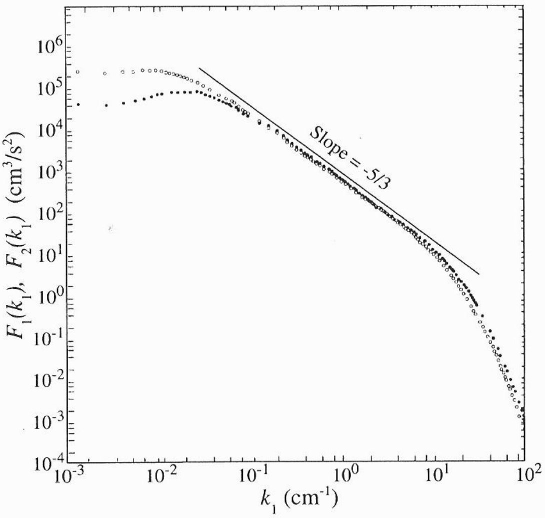


Fig. 5.7. log–log plot of the energy spectra of the streamwise component (white circles) and lateral component (black circles) of the velocity fluctuations in the time domain in a jet with $R_\lambda = 626$ (Champagne 1978).

To maintain a turbulent state one introduces a forcing term in the Navier-Stokes equations

$$\partial_t u_i + u_j \partial_j u_i - \nu \nabla^2 u_i = -\partial_i p + f$$
$$\partial_i u_i = 0$$

effective at large scales

Large-scale motions u_L in turbulent flow certainly affected by forcing and boundary conditions of flow

Expect small-scale structures to be independent of forcing characteristics

However at some scale L_d dissipative effects set in

Interested in quantities independent of forcing characteristics and dissipation
$\rightarrow$ *Universality?*

Functions of velocity differences (local fluctuations)

$$\Delta u_i(\mathbf{x}+\mathbf{r},\mathbf{x}) = u_i(\mathbf{x}+\mathbf{r}) - u_i(\mathbf{x})$$

with the seperation in the inertial range $L_d \ll r \ll L$

Structure functions

$$S_p(\mathbf{r}) = \overline{\left(\Delta u_i(\mathbf{x}+\mathbf{r},\mathbf{x})\, r_i/|\mathbf{r}|\right)^p}\,,\ p > 0$$

as moments of the veloctiy field fluctuations

$\longrightarrow$ *Statistical theory of turbulence*

Frisch, U., *Turbulence: The Legacy of A.N. Kolmogorov* (Cambridge University Press, 1995)

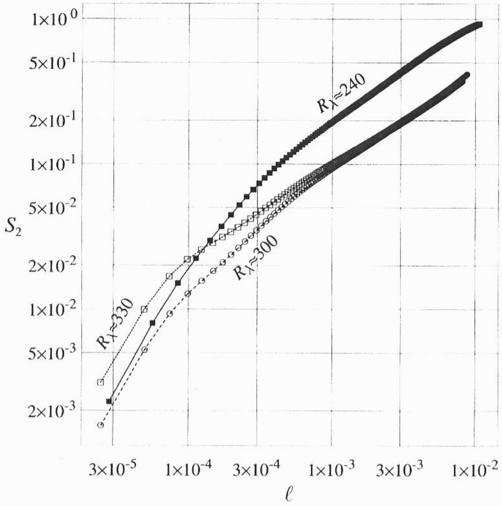


Fig. 5.3. log–log plot of the second order transverse structure function measured in the space domain by the RELIEF flow tagging technique in a turbulent jet at various R_λs, as labeled (Noullez, Wallace, Lempert, Miles and Frisch 1996).

K41 theory (Kolmogorov, 1941)

Formulation of several hypothesis that lead to a scaling theory

Fully developed turbulence

$$Re \gg 1$$

Statistical symmetries, i.e. homogeneity, isotropy and scale invariance

Small-scale statistical properties uniquely determined by scale r and mean energy dissipation rate ϵ :

$$S_p(r) \sim r^{\zeta_p} \ , \quad \text{where } \zeta_p = p/3 \ ,$$

$\rightarrow$ Universality of scaling exponents ζ_p

Experimentally K41 scaling behavior is very well verified for low order structure functions $S_p(r)$

Higher order moments indicate failure of simple scaling

Scaling exponents

$$\zeta_p = \frac{d \log S_p(r)}{d \log r}$$

concave function of the order p, where only $\zeta_3 = 1$ (*constant energy flux*)

However still indications of *universal scaling behavior*

Frisch, U., *Turbulence: The Legacy of A.N. Kolmogorov* (Cambridge University Press, 1995)

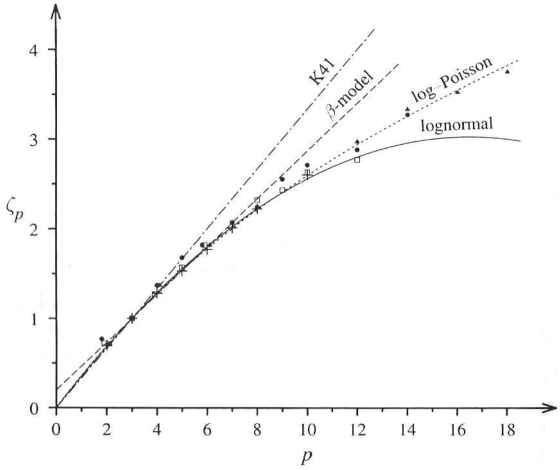


Fig. 8.8. Exponent ζ_p of structure functions in the time domain of order p vs p. Inverted white triangles: data from Van Atta and Park (1972); black circles, white squares and black triangles: data from Anselmet, Gagne, Hopfinger and Antonia (1984) with $R_\lambda = 515, 536, 852$, respectively; + signs: S1 data processed by 'ESS' (see p. 131). Straight chain line: $\zeta_p = p/3$ (K41); dashed line: β-model (eq. (8.31)) with $D = 2.8$; solid line: lognormal model (eq. (8.122)) with $\mu = 0.2$; dotted line: log–Poisson model (eq. (8.141)).

Intermittent bursts of dissipative activity observed in turbulent flows

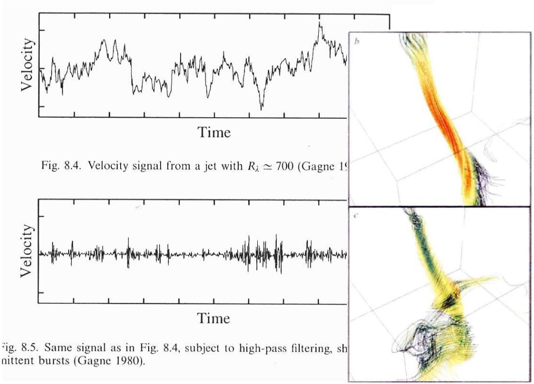


Fig. 8.4. Velocity signal from a jet with $R_\lambda \simeq 700$ (Gagne 19

'ig. 8.5. Same signal as in Fig. 8.4, subject to high-pass filtering, sh
nittent bursts (Gagne 1980).

Nature, **344**, 226 (1990)

Coherent dissipative structures (vortex filaments) in turbulent flow from direct numerical simulations

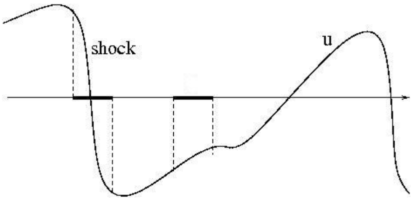
$\rightarrow$ Statistical properties of these structures?

One-dimensional Burgers equation

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = 0$$

Simple model to understand issues related to intermittency and anomalous scaling

Intermittent structures well known – shocks with large negative velocity gradient



Same type of mathematical structure as Navier-Stokes equation in terms of nonlinear and viscous terms

Further many invariance and conservation laws in common

Burgers equation exactly integrable via the Cole-Hopf transformation

$$u = -2\nu\,\partial_x \log\psi$$

(imaginary-time) Schrödinger equation

$$\partial_t\psi + \nu\partial_x^2\psi = \frac{1}{2\nu}F\psi$$

Feynman-Kac

$$\psi(x,t) = \left\langle \exp\left[\frac{1}{2\nu}\psi_0(W_{t_0}) - \frac{1}{2\nu}\int_{t_0}^{t} F(W_s,s)ds\right]\right\rangle_W$$

No spontaneous arise of randomness as in Navier-Navier Stokes equations

Artificially generate a turbulent state by a *random* forcing term

$$\partial_t u + u\partial_x u - \nu\partial_x^2 u = f$$

taken to be Gaussian, white-in-time, characterized by

$$\langle \hat{f}(k,t)\hat{f}(k',t') \rangle = D(k)\delta(k+k')\,\delta(t-t')$$

$\rightarrow$ Different possibilities for $D(k)$, e.g. large-scale *or* self-similar $\sim |k|^\beta$ type forcing

Expect a formation of shocks at random positions

Shocks at random positions lead to highly intermittent behavior
$\rightarrow$ Statistical properties of dissipative shocks?

Structure functions of order $p > 0$ for small r

$$S_p(r) = \langle \, |\Delta u(r)|^p \, \rangle \sim C_p |r|^p + C'_p |r|$$

First term comes from the smooth parts of the velocity field between shocks

Second term from the $O(r)$ probability to observe a shock somewhere in the interval r

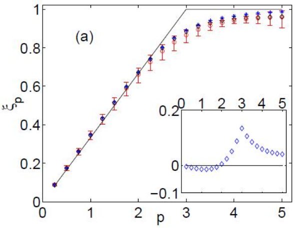
Bifractal scaling behavior

$$S_p(r) \sim r^{\zeta_p} \, , \quad \zeta_p = \min{(p, 1)}$$

→ *Multifractality?*

range of different values for the scaling index corresponding to different regions of the measure

→ *Corrections to scaling laws?*



Phys. Rev. Lett. **94**, 194501 (2005)

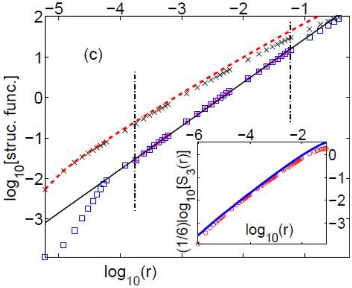
A lot of work has already been done . . .

Nearly all of these methods exploit integrability in some way or the other:

Fast Legendre transform algorithm for the velocity potential ($u = -\partial_x \phi$)

$$\phi(x,t) = \max_a \left[\phi(y,t') - \frac{(x-y)^2}{2(t-t')} \right], \; t > t'$$

J. Sci. Comp. **9**, 259 (1994)



Phys. Rev. Lett. **94**, 194501 (2005)

Analytic description of intermittency via special field-force configurations (*instantons*) corresponding to the rare fluctuations that give the main contribution into the high-order moments

Phys. Rev. Lett. **78**, 1452 (1997)

Path integral formulation via Martin-Siggia-Rose formalism

Introduce conjugate field variable p that absorbes the dissipated energy

$$Z = \int \mathcal{D}u \, \mathcal{D}p \exp\left(\imath S[u, p]\right)$$

with the Lagrangian density

$$\mathcal{L} = p\big(\partial_t u + u\partial_x u - \nu\partial_x^2 u\big) + \frac{\imath}{2} p \left(\chi * p\right)$$

Gaussian integration over conjugate field gives effective theory

$$\mathcal{L} = \frac{1}{2}\left(\partial_t u + u\partial_x u - \nu\partial_x^2 u\right)\Big(\chi^{-1} * \left(\partial_t u + u\partial_x u - \nu\partial_x^2 u\right)\Big)$$

Defines a *Gibbs measure*

$$P = \exp\big(- S[u]\big)$$

Path integral to calculate ensemble averages numerically with the theory defined on a *finite-size lattice*

Integration over grid variables $u_i(n)$, where i denotes the discretized space, and n discretized time

Ensemble average of observable $\mathcal{O}$ calculated in the discretized theory

$$\langle \mathcal{O} \rangle = \int \prod_{i,n} du_i(n)\, \mathcal{O}[\{u_i(n)\}] \exp\left[- S[\{u_i(n)\}]\right]$$

Single-site action $S(u_i(n))$ is quadratic (under certain conditions)

$$S\big(u_i(n)\big) = A_i(n)\big[u_i(n) - C_i(n)\big]^2 + B_i(n)$$

where $A_i(n) > 0$

$A_i(n)$, $C_i(n)$ and $B_i(n)$ will in general depend on different nodes on the lattice

Heatbath Monte Carlo algorithm to update single-field variables

Writing the single-site action $S(u_i(n))$ in a quadratic form imposes restrictions on the possible discretizations of the action

Nonlinear term discretized as

$$u\partial_x u \to u_i(n)u_i'(n)$$

where

$$u_i'(n) = \frac{u_{i+1}(n) - u_{i-1}(n)}{2\Delta x}$$

Time derivative and viscous term as usual, i.e.

$$\partial_t u \to \frac{u_i(n+1) - u_i(n)}{\Delta t}\,, \qquad \partial_x^2 u \to \frac{u_{i+1}(n) - 2u_i(n) + u_{i-1}(n)}{\Delta x^2}$$

To ensure correct continuum behavior one has to set the physical parameters accordingly

Viscosity

$$\nu = \tilde{\nu}\ \Delta x^2/\Delta t$$

Reynolds number

$$Re = \epsilon^{1/3} L^{4/3}/\nu$$

Specify $\tilde{\nu}$, ϵ, L in simulations, and keep the *physical parameters* ν*, Re, L fixed* in the continuum extrapolation

Periodic boundary conditions in space

Fixed boundaries in time

Several *constraints* have to be imposed:

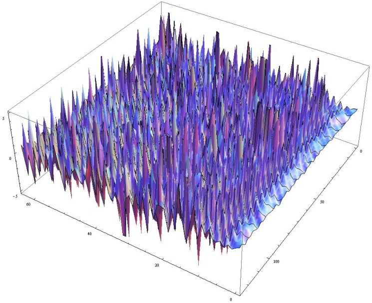
Inertial range has to fit entirely onto the lattice, i.e.

$$L_d = L\,Re^{-\frac{3}{4}} > \Delta x \;, \quad \text{and} \;\; L < 1$$

To ensure *correct dynamics*

$$\tilde{\nu} < 1/2$$

$\rightarrow$ *Mapping to lattice spin model?*



$\longrightarrow$ Leads to *high computational requirements*, e.g.

Simulation at $Re = 10^3$ requires *at least* $\sim 10^3$ lattice points in the space direction
$\longrightarrow$ Navier-Stokes

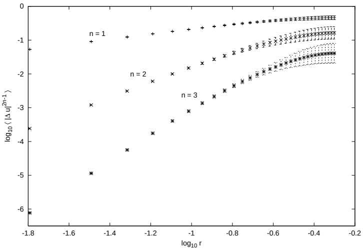
At the moment simulations feasible up to $Re \sim 1024$

Parallelization is an issue $\longrightarrow$ *Hybrid Monte Carlo?*

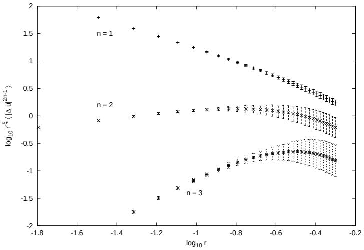
Algorithmic enhacements to reduce autocorrelation, and thermalization effects

Heat bath algorithm with successive over-relaxation

Large sets of independent measurements $O(10^5)$ necessary

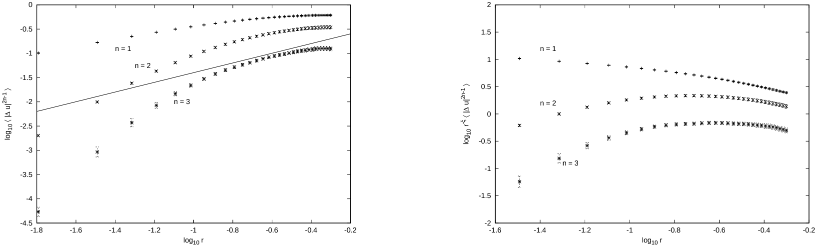


(a)



(b)

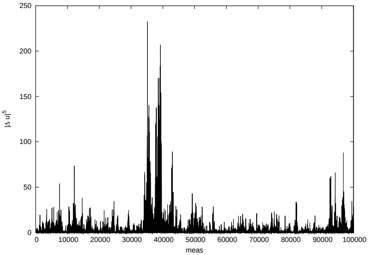
(a) Velocity differences $\langle|\Delta u|^{2n-1}\rangle$ for $n = 1, 2, 3$. (b) Compensated moments $r^{-\xi}\langle|\Delta u|^{2n-1}\rangle$, with $\xi = 1.9$.
The data was obtained for the parameters $\mathrm{Re} = 4$, $\nu = 0.07$ on a 62×1024 lattice.



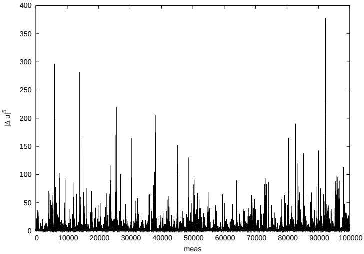
(a) (b)

(a) Velocity differences $\langle|\Delta u|^{2n-1}\rangle$ for $n = 1, 2, 3$. (b) Compensated moments $r^{-\xi}\langle|\Delta u|^{2n-1}\rangle$, with $\xi = 1$.

The data was obtained for the parameters $\mathrm{Re} = 15$, $\nu = 0.02$ on a 62×256 lattice.

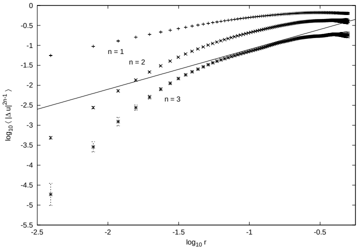


(a)

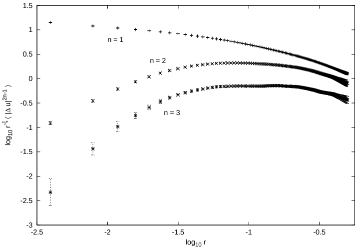


(b)

Velocity difference $|\Delta u|^5$ evaluated at a fixed seperation $r^* = 0.25$ in the inertial range. (a) Re $= 15$, $\nu = 0.02$ on a 62×1024 lattice. (b) Re $= 15$, $\nu = 0.02$ on a 62×256 lattice.

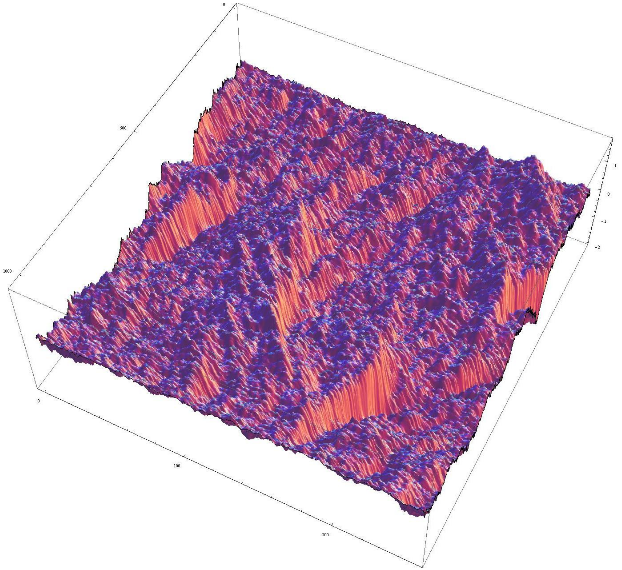


(a)



(b)

(a) Velocity differences $\langle|\Delta u|^{2n-1}\rangle$ for $n = 1, 2, 3$. (b) Compensated moments $r^{-\xi}\langle|\Delta u|^{2n-1}\rangle$, with $\xi = 1$.
The data was obtained for the parameters $\mathrm{Re} = 63$, $\nu = 4{\cdot}10^{-3}$ on a 254×1024 lattice.



Example configuration for Re $= 63$, $\nu = 4\cdot10^{-3}$ on a 254×1024 lattice.

Summary:

- Monte Carlo simulations in the path integral method as a new approach to obtain the statistical behavior of small-scale structures in turbulent flows

- Computation of structure functions and determination of scaling behavior

- Need to understand systematic effects at finite Reynolds number, corrections to the scaling behavior, and finite lattice size to extract structure function scaling exponents to high precision

A lot of interesting physics that can be explored . . .